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Continuation theorems for weakly inward Kakutani and Strongly inward acyclic maps

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Abstract

In this paper we begin by presenting a general Leray-Schauder alternative and a topological transversality theorem for Kakutani (upper semicontinuous maps with nonempty convex compact values) compact weakly inward maps. Then with some observations and extra assumptions we present a Leray-Schauder alternative and a topological transversality theorem for acyclic (upper semicontinuous maps with nonempty acyclic compact values) compact strongly inward maps.

Keywords: Essential maps, homotopy, inward maps, acyclic maps.

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1. Introduction

In the first part of this paper we discuss Kakutani maps, i.e., upper semicontinuous compact maps with nonempty convex compact values. We begin by defining the notion of essential maps and then the notion of homotopy. Then we present a simple result which will then generate a very general Leray-Schauder type result and a topological transversality theorem for Kakutani weakly inward maps. The topological transversality theorem simply states that if two maps F and G are homotopic then F is essential if and only if G is essential. The second part of the paper discusses upper semicontinuous compact strongly inward maps with nonempty acyclic compact values. Again a simple result will generate a Leray-Schauder type result. However to obtain a topological transversality theorem in this setting some observations and assumptions need to be considered. We refer the reader to [1, 2, 4, 8–10] for continuation type results in other settings.

Let E be a Banach space (or more generally a locally convex Hausdorff linear topological space) and let C be a closed convex subset of E. The set

$$I_C(x) = \{x + \lambda (y - x) : \ \lambda \geqslant 0, \ y \in C\}$$

is called the inward set of C at x. A mapping $F : C \to 2^E$ (here 2^E denotes the family of nonempty subsets of E) is said to be weakly inward with respect to C if

$$F(x) \cap \overline{I_C(x)} \neq \emptyset \ \text{ on } \ C.$$

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Recall (see [3]) if C is a closed convex subset of a Banach space then

$$\overline{I_C(x)} = \bar{I}_C(x) = x + \overline{\{\lambda\,(y-x) : \lambda \geqslant 1,\, y \in C\}}.$$

To establish our results recall the following fixed point result in [3].

Theorem 1.1. *Let E be (real) Banach space and C a closed bounded convex subset of E. Suppose $F : C \to 2^E$ is a upper semicontinuous compact (or more generally, condensing) map with closed convex values. If $F(x) \cap \overline{I_C(x)} \neq \emptyset$ on C, then F has a fixed point in C.*

Finally recall [6] a nonempty topological spaces is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial.

2. Topological transversality theorem

Let $E = (E, \|.\|)$ be a Banach space, C a closed convex subset of E, U_0 a bounded open subset of E, and $U = U_0 \cap C$.

Definition 2.1. We say $F \in WI(\overline{U}, E)$ if $F : \overline{U} \to K(E)$ is a upper semicontinuous compact weakly inward with respect to C (i.e., $F(x) \cap \overline{I_C(x)} \neq \emptyset$ on $\overline{U}$) map; here $\overline{U}$ denotes the closure of U in C and $K(E)$ denotes the family of nonempty convex compact subsets of E.

Definition 2.2. We say $F \in WI_{\partial U}(\overline{U}, E)$ if $F \in WI(\overline{U}, E)$ and $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in C.

Now we introduce our main notion, namely essential maps.

Definition 2.3. We say $F \in WI_{\partial U}(\overline{U}, E)$ is essential in $WI_{\partial U}(\overline{U}, E)$ if for every map $G \in WI_{\partial U}(\overline{U}, E)$ with $F|_{\partial U} = G|_{\partial U}$ there exists a $x \in U$ with $x \in G(x)$.

Remark 2.4. If F is essential in $WI_{\partial U}(\overline{U}, E)$, then there exists a $x \in U$ with $x \in F(x)$ (take $G = F$ in Definition 2.3).

Next we consider the notion of homotopy.

Definition 2.5. Let $\Phi, \Psi \in WI_{\partial U}(\overline{U}, E)$. We say $\Phi \cong \Psi$ in $WI_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous, compact map $H : \overline{U} \times [0,1] \to C(E)$ with $H_t : \overline{U} \to K(E)$ belonging to $WI_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$, $H_0 = \Phi$ and $H_1 = \Psi$; here $C(E)$ denotes the family of nonempty compact subsets of E and $H_t(x) = H(x,t)$.

Remark 2.6. A standard argument guarantees that $\cong$ in $WI_{\partial U}(\overline{U}, E)$ is an equivalence relation.

Next we present a result which will, then with a few observations generate a Leray-Schauder alternative and a topological transversality theorem.

Theorem 2.7. *Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open subset of E, $U = U_0 \cap C$ and $F \in WI_{\partial U}(\overline{U}, E)$. Suppose $G \in WI_{\partial U}(\overline{U}, E)$ is essential in $WI_{\partial U}(\overline{U}, E)$ and assume the following holds:*

$$\text{for any } \ \theta \in WI_{\partial U}(\overline{U}, E) \ \text{ with } \ \theta|_{\partial U} = F|_{\partial U} \text{ we have } \ G \cong \theta \ \text{ in } \ WI_{\partial U}(\overline{U}, E). \tag{2.1}$$

Then F is essential in $WI_{\partial U}(\overline{U}, E)$.

Proof. Consider any map $\theta \in WI_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$. We must show there exists a $x \in U$ with $x \in \theta(x)$. Now (2.1) guarantees a upper semicontinuous, compact map $H : \overline{U} \times [0,1] \to C(E)$ with $H_t \in WI_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$ (here $H_t(.) = H(., t)$), $H_0 = G$ and $H_1 = \theta$. Let

$$\Omega = \left\{x \in \overline{U} : \ x \in H(x,t) \ \text{ for some } \ t \in [0,1]\right\}.$$

Now $\Omega \neq \emptyset$ since G is essential in $WI_{\partial U}(\overline{U}, E)$ (see Remark 2.4 with $t = 0$). Also Ω is closed since H is upper semicontinuous (in fact Ω is compact since H is compact). Also note $\Omega \cap \partial U = \emptyset$ since $H_t \in WI_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$. Then there exists a Urysohn continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map $R : \overline{U} \to K(E)$ by $R(x) = H(x, \mu(x)) = H_{\mu(x)}(x) = H \circ g(x)$ where $g : \overline{U} \to \overline{U} \times [0,1]$ is given by $g(x) = (x, \mu(x))$. Note R is a upper semicontinuous compact map with $R|_{\partial U} = G|_{\partial U}$ since if $x \in \partial U$, then $R(x) = H(x,0) = G(x)$. Next we will show R is weakly inward with respect to C. First notice for each fixed $s \in [0,1]$ we have $H_s(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in \overline{U}$. Now for a fixed $x \in \overline{U}$ let $\mu(x) = s$ so $R(x) = H_{\mu(x)}(x) = H_s(x)$ and as a result $R(x) \cap \overline{I_C(x)} = H_s(x) \cap \overline{I_C(x)} \neq \emptyset$, i.e., R is weakly inward with respect to C. Consequently $R \in WI_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$. Now since G is essential in $WI_{\partial U}(\overline{U}, E)$, then there exists a $x \in U$ with $x \in R(x)$, i.e., $x \in H_{\mu(x)}(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$ and as a result $x \in H_1(x) = \theta(x)$. □

We are now in a position to present a very general Leray-Schauder type result.

Theorem 2.8. *Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open subset of E, $U = U_0 \cap C$ and $F \in WI_{\partial U}(\overline{U}, E)$. Suppose $G \in WI_{\partial U}(\overline{U}, E)$ is essential in $WI_{\partial U}(\overline{U}, E)$ and $x \notin t\,F(x) + (1-t)\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then F is essential in $WI_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Proof. Let $\theta \in WI_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$ and let $H(x,t) = t\theta(x) + (1-t)G(x)$. Note $H : \overline{U} \times [0,1] \to C(E)$ (in fact $K(E)$) is a upper semicontinuous compact map and $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$ (note if $x \in \partial U$ and $t \in [0,1]$, then since $\theta|_{\partial U} = F|_{\partial U}$ we have $H_t(x) = t\,\theta(x) + (1-t)\,G(x) = t\,F(x) + (1-t)\,G(x)$), $H_0 = G$ and $H_1 = \theta$. Also for each $t \in [0,1]$ note $H_t \in WI_{\partial U}(\overline{U}, E)$ since for $x \in \overline{U}$ we have $\theta(x) \cap \overline{I_C(x)} \neq \emptyset$ and $G(x) \cap \overline{I_C(x)} \neq \emptyset$ and so since $H_t(x) = t\,\theta(x) + (1-t)\,G(x)$ we have $H_t(x) \cap \overline{I_C(x)} \neq \emptyset$ (note $\overline{I_C(x)}$ is a convex subset of E). Thus $G \cong \theta$ in $WI_{\partial U}(\overline{U}, E)$ so (2.1) holds and our result follows from Theorem 2.7. □

Theorem 2.9. *Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open subset of E, $U = U_0 \cap C$ and $u_0 \in U$. Then the constant map $G : \overline{U} \to \{u_0\}$ is essential in $WI_{\partial U}(\overline{U}, E)$.*

Proof. Let $\theta \in WI_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = G|_{\partial U} = \{u_0\}$. We must show there exists a $x \in U$ with $x \in \theta(x)$. Now U_0 is bounded so we can choose a constant $R > 0$ with

$$U_0 \subset \{x \in E : \ \|x\| < R\} \ \text{ and } \ \theta(\overline{U}) \subset \{x \in E : \ \|x\| < R\}.$$

Let

$$D = C \cap \{x \in E : \ \|x\| < R+1\}$$

and consider

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U}, \\ \{u_0\}, & x \in \overline{D} \backslash \overline{U}. \end{cases}$$

Note $J : \overline{D} \to K(E)$ is a upper semicontinuous compact map. We claim J is weakly inward with respect to $\overline{D}$ (i.e., $J(x) \cap \overline{I_{\overline{D}}(x)} \neq \emptyset$ on $\overline{D}$). If the claim is true, then we are finished since Theorem 1.1 guarantees a $y \in \overline{D}$ with $y \in J(y)$ and note immediately that $y \in U$ since $u_0 \in U$ and thus $y \in \theta(y)$.

To prove the claim note if $x \in \overline{D} \backslash \overline{U}$, then $J(x) = \{u_0\} \subseteq \overline{I_{\overline{D}}(x)}$ since $u_0 \in U_0 \cap C$ so $u_0 \in \overline{D}$. Now let $x \in \overline{U}$ and take $y \in J(x) = \theta(x)$ with $y \in \overline{I_C(x)}$ (recall $\theta(x) \cap \overline{I_C(x)} \neq \emptyset$). There exists $\{\lambda_n\}_{n \in \mathbf{N}}$ with $\lambda_n \geqslant 1$ for $n \in \mathbf{N}$ and $\{z_{\lambda_n}\}_{n \in \mathbf{N}} \subseteq C$ with

$$\|y - [x + \lambda_n(z_{\lambda_n} - x)]\| \to 0 \ \text{ as } \ n \to \infty.$$

Let $v_{\lambda_n} = x + \lambda_n(z_{\lambda_n} - x)$. Then $v_{\lambda_n} \to y$ as $n \to \infty$ so $v_{\lambda_n} \in \{x \in E : \ \|x\| < R+1\}$ for n sufficiently large. Let $\mu_n = \frac{1}{\lambda_n}$. Then $z_{\lambda_n} = (1-\mu_n)\,x + \mu_n\,v_{\lambda_n}$ so $z_{\lambda_n} \in \{x \in E : \ \|x\| < R+1\}$ for n sufficiently large. In addition since $\{z_{\lambda_n}\}_{n \in \mathbf{N}} \subseteq C$ we have $z_{\lambda_n} \in \overline{D}$ for n sufficiently large and $\|y - [x + \lambda_n(z_{\lambda_n} - x)]\| \to 0$ as $n \to \infty$. Thus $y \in \overline{I_{\overline{D}}(x)}$ so $y \in J(x) \cap \overline{I_{\overline{D}}(x)}$. □

Combining Theorems 2.8 and 2.9 gives the following result.

Theorem 2.10. *Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open subset of E, $U = U_0 \cap C$, $u_0 \in U$ and $F \in WI_{\partial U}(\overline{U}, E)$. Suppose $x \notin t\,F(x) + (1-t)\{u_0\}$ for $x \in \partial U$ and $t \in (0,1)$. Then F is essential in $WI_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Next we establish the topological transversality theorem and in the proof we use the following:

$$\text{if } \Phi, \Psi \in WI_{\partial U}(\overline{U}, E) \text{ with } \Phi|_{\partial U} = \Psi|_{\partial U}, \text{ then } \Phi \cong \Psi \text{ in } WI_{\partial U}(\overline{U}, E). \tag{2.2}$$

To see this let $H(x,t) = (1-t)\,\Phi(x) + t\,\Psi(x)$ and note $H : \overline{U} \times [0,1] \to C(E)$ (in fact $K(E)$) is a upper semicontinuous compact map with $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (note if $x \in \partial U$ and $t \in (0,1)$, then $H_t(x) = (1-t)\Phi(x) + t\Psi(x) = \Phi(x)$ since $\Phi|_{\partial U} = \Psi|_{\partial U}$ and note $x \notin \Phi(x)$ since $\Phi \in WI_{\partial U}(\overline{U}, E)$) and for each $t \in [0,1]$ note $H_t \in WI_{\partial U}(\overline{U}, E)$ since for $x \in \overline{U}$ we have $\Phi(x) \cap \overline{I_C(x)} \neq \emptyset$ and $\Psi(x) \cap \overline{I_C(x)} \neq \emptyset$ and so since $H_t(x) = (1-t)\Phi(x) + t\Psi(x)$ we have $H_t(x) \cap \overline{I_C(x)} \neq \emptyset$ (note $\overline{I_C(x)}$ is a convex subset of E).

Remark 2.11. From (2.2) note in (2.1) since $\theta \in WI_{\partial U}(\overline{U}, E)$ and $\theta|_{\partial U} = F|_{\partial U}$, then $\theta \cong F$ in $WI_{\partial U}(\overline{U}, E)$.

Now we state and prove the topological transversality theorem.

Theorem 2.12. *Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open subset of E, $U = U_0 \cap C$, $F \in WI_{\partial U}(\overline{U}, E)$, $G \in WI_{\partial U}(\overline{U}, E)$ and $F \cong G$ in $WI_{\partial U}(\overline{U}, E)$. Now F is essential in $WI_{\partial U}(\overline{U}, E)$ if and only if G is essential in $WI_{\partial U}(\overline{U}, E)$.*

Proof. Assume G is essential in $WI_{\partial U}(\overline{U}, E)$. We will use Theorem 2.7 to show F is essential in $WI_{\partial U}(\overline{U}, E)$. Let $\theta \in WI_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$. Now (2.2) guarantees that $\theta \cong F$ in $WI_{\partial U}(\overline{U}, E)$ and this together with $F \cong G$ in $WI_{\partial U}(\overline{U}, E)$ implies $\theta \cong G$ in $WI_{\partial U}(\overline{U}, E)$ (recall $\cong$ in $WI_{\partial U}(\overline{U}, E)$ is an equivalence relation). Thus (2.1) holds so Theorem 2.7 guarantees that F is essential in $WI_{\partial U}(\overline{U}, E)$. A similar argument shows if F is essential in $WI_{\partial U}(\overline{U}, E)$, then G is essential in $WI_{\partial U}(\overline{U}, E)$. □

Remark 2.13. In the theory above (see Definition 2.1 etc.) we assumed the maps are compact. However slight adjustments in the above proofs guarantee that one could replace compact maps with α-condensing maps [3].

Remark 2.14. It is easy to extend the above theory when E is a Banach space is replaced by E is a Fréchet space (metric d) and here C is a closed convex subset of E and U_0 a d-bounded (i.e., there exists a constant $R > 0$ with $U_0 \subset \{x \in E : \ d(0,x) < R\}$) open subset of E. In this situation we say $F \in I(\overline{U}, E)$ if $F : \overline{U} \to K(E)$ is a upper semicontinuous compact inward with respect to C (i.e., $F(x) \cap I_C(x) \neq \emptyset$ on $\overline{U}$) map. Also we have the analogue of Definitions 2.2, 2.3, 2.5, and trivial adjustments in the proofs above will establish the analogue of Theorems 2.7, 2.8, 2.9, 2.10, and 2.12 (note instead of applying Theorem 1.1 in Theorem 2.9 we will apply Theorem 6 in [7]). We also refer the reader to [1, 9].

Next we will consider acyclic maps instead of Kakutani maps. Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open subset of E, and $U = U_0 \cap C$.

Definition 2.15. We say $F \in AI(\overline{U}, E)$ if $F : \overline{U} \to A(E)$ is a upper semicontinuous compact strongly inward with respect to C (i.e., $F(x) \subseteq I_C(x)$ on $\overline{U}$) map; here $A(E)$ denotes the family of nonempty acyclic compact subsets of E.

Definition 2.16. We say $F \in AI_{\partial U}(\overline{U}, E)$ if $F \in AI(\overline{U}, E)$ and $x \notin F(x)$ for $x \in \partial U$.

Definition 2.17. We say $F \in AI_{\partial U}(\overline{U}, E)$ is essential in $AI_{\partial U}(\overline{U}, E)$ if for every map $G \in AI_{\partial U}(\overline{U}, E)$ with $F|_{\partial U} = G|_{\partial U}$ there exists a $x \in U$ with $x \in G(x)$.

Definition 2.18. Let $\Phi, \Psi \in AI_{\partial U}(\overline{U}, E)$. We say $\Phi \cong \Psi$ in $AI_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous, compact map $H : \overline{U} \times [0,1] \to C(E)$ with $H_t : \overline{U} \to A(E)$ belonging to $AI_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$, $H_0 = \Phi$, and $H_1 = \Psi$ (here $H_t(x) = H(x,t)$ and we note a standard argument guarantees that $\cong$ in $AI_{\partial U}(\overline{U}, E)$ is an equivalence relation).

The same argument as in Theorem 2.7 (with very slight adjustments) guarantees the following result.

Theorem 2.19. *Let* E *be a Banach space,* C *a closed convex subset of* E, U_0 *a bounded open subset of* E, $U = U_0 \cap C$ *and* $F \in AI_{\partial U}(\overline{U}, E)$. *Suppose* $G \in AI_{\partial U}(\overline{U}, E)$ *is essential in* $AI_{\partial U}(\overline{U}, E)$ *and assume the following holds:*

$$\text{for any } \theta \in AI_{\partial U}(\overline{U}, E) \text{ with } \theta|_{\partial U} = F|_{\partial U} \text{ we have } G \cong \theta \text{ in } AI_{\partial U}(\overline{U}, E). \tag{2.3}$$

Then F *is essential in* $AI_{\partial U}(\overline{U}, E)$.

Theorem 2.20. *Let* E *be a Banach space,* C *a closed convex subset of* E, U_0 *a bounded open subset of* E, $U = U_0 \cap C$, *and* $0 \in U$. *Then the zero map* $G : \overline{U} \to \{0\}$ *is essential in* $AI_{\partial U}(\overline{U}, E)$.

Proof. Let $\theta \in AI_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = G|_{\partial U} = \{0\}$. There exists a constant $R > 0$ with

$$U_0 \subset \{x \in E : \|x\| < R\} \text{ and } \theta(\overline{U}) \subset \{x \in E : \|x\| < R\}.$$

Let

$$D = C \cap \{x \in E : \|x\| < R + 1\}$$

and consider

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U}, \\ \{0\}, & x \in \overline{D} \backslash \overline{U}. \end{cases}$$

Note $J : \overline{D} \to A(E)$ is a upper semicontinuous compact map and a slight adjustment of the argument in Theorem 2.9 guarantees that J is strongly inward with respect to $\overline{D}$ (i.e., $J(x) \subseteq I_{\overline{D}}(x)$ on $\overline{D}$). Now apply Theorem 2.7 in [1] so there exists a $y \in \overline{D}$ with $y \in J(y)$ and since $0 \in U$, then $y \in U$ and thus $y \in \theta(y)$. □

Now we present a Leray-Schauder type result.

Theorem 2.21. *Let* E *be a Banach space,* C *a closed convex subset of* E, U_0 *a bounded open subset of* E, $U = U_0 \cap C$, $0 \in U$, *and* $F \in AI_{\partial U}(\overline{U}, E)$. *Suppose* $x \notin t\,F(x)$ *for* $x \in \partial U$ *and* $t \in (0, 1)$. *Then* F *is essential in* $AI_{\partial U}(\overline{U}, E)$ *(in particular* F *has a fixed point in* U*).*

Proof. Let G be the zero map which is essential in $AI_{\partial U}(\overline{U}, E)$ from Theorem 2.20. Now let $\theta \in AI_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$ and let $H(x, t) = t\,\theta(x)$. Note $H : \overline{U} \times [0, 1] \to C(E)$ is a upper semicontinuous compact map and $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in [0, 1]$ (note if $x \in \partial U$ and $t \in [0, 1]$, then since $\theta|_{\partial U} = F|_{\partial U}$ we have $H_t(x) = t\,\theta(x) = t\,F(x)$), $H_0 = G$, and $H_1 = \theta$. Also for each fixed $t \in [0, 1]$ note $H_t : \overline{U} \to A(E)$ (recall homeomorphic spaces have isomorphic homology groups so H_t is acyclic valued). Also for each $t \in [0, 1]$ note $H_t \in AI_{\partial U}(\overline{U}, E)$ since for $x \in \overline{U}$ we have $\theta(x) \subseteq I_C(x)$ and $0 \in I_C(x)$ (recall $0 \in U = U_0 \cap C$) so we have $t\,\theta(x) = t\,\theta(x) + (1 - t)\,0 \in I_C(x)$ since $I_C(x)$ is a convex subset of E. Thus $\theta \cong \{0\} = G$ in $AI_{\partial U}(\overline{U}, E)$ so (2.3) holds and our result follows from Theorem 2.19. □

To obtain a topological transversality theorem for acyclic strongly inward maps we need to add some extra assumptions. Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open convex subset of E, and $U = U_0 \cap C$. For the remainder of this paper assume

$$\text{there exists a continuous retraction } r \text{ from } \overline{U} \text{ onto } \partial U. \tag{2.4}$$

Remark 2.22. If E is a normed linear space, $C \subseteq E$ a cone and U_0 an open ball in E, then [5] guarantees that (2.4) is true.

We also assume

$$\text{for any } \Phi \in AI_{\partial U}(\overline{U}, E) \text{ we have } \Phi(\lambda\, r(x) + (1 - \lambda)\, x) \subseteq I_C(x) \text{ on } \overline{U} \text{ for any } \lambda \in (0, 1]. \tag{2.5}$$

Now we prove the following:

$$\text{if } G, F \in AI_{\partial U}(\overline{U}, E) \text{ with } G|_{\partial U} = F|_{\partial U}, \text{ then } G \cong F \text{ in } AI_{\partial U}(\overline{U}, E). \tag{2.6}$$

Let $F^\star(x) = F(r(x))$ for $x \in \overline{U}$ and note $F^\star(x) = G(r(x))$ for $x \in \overline{U}$ since $G|_{\partial U} = F|_{\partial U}$. Let

$$H(x,t) = G(2t\,r(x) + (1-2t)\,x) \quad \text{for} \quad (x,t) \in \overline{U} \times \left[0, \frac{1}{2}\right].$$

Note $H : \overline{U} \times [0, \frac{1}{2}] \to C(E)$ is a upper semicontinuous compact map and for fixed $x \in \overline{U}$ and $t \in [0, \frac{1}{2}]$ note $H_t(x)$ has acyclic values and also note if $x \in \partial U$ and $t \in [0, \frac{1}{2}]$ with $x \in H_t(x)$, then $x \in G(2t\,r(x) + (1-2t)\,x) = G(x)$, a contradiction so as a result $x \notin H_t(x)$ for $x \in \overline{U}$ and $t \in [0, \frac{1}{2}]$. Finally note for fixed $t \in [0, \frac{1}{2}]$ that $H_t(x) \subseteq I_C(x)$ on $\overline{U}$ from (2.5). Thus $G \cong F^\star$ in $AI_{\partial U}(\overline{U}, E)$. Similarly with

$$\Lambda(x,t) = F((2-2t)\,r(x) + (2t-1)\,x) \quad \text{for} \quad (x,t) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

we have $F^\star \cong F$ in $AI_{\partial U}(\overline{U}, E)$. Combining gives $G \cong F$ in $AI_{\partial U}(\overline{U}, E)$ so (2.6) is true.

Theorem 2.23. *Let E be a Banach space, C a closed convex subset of E, U_0 a bounded open convex subset of E, $U = U_0 \cap C$, $F \in AI_{\partial U}(\overline{U}, E)$, $G \in AI_{\partial U}(\overline{U}, E)$, and $F \cong G$ in $AI_{\partial U}(\overline{U}, E)$. Assume (2.4) and (2.5) hold. Now F is essential in $AI_{\partial U}(\overline{U}, E)$ if and only if G is essential in $AI_{\partial U}(\overline{U}, E)$.*

Proof. Assume G is essential in $AI_{\partial U}(\overline{U}, E)$. We will use Theorem 2.19 to show F is essential in $AI_{\partial U}(\overline{U}, E)$. Let $\theta \in AI_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$. Now (2.6) guarantees that $\theta \cong F$ in $AI_{\partial U}(\overline{U}, E)$ and this together with $F \cong G$ in $AI_{\partial U}(\overline{U}, E)$ implies $\theta \cong G$ in $AI_{\partial U}(\overline{U}, E)$. Thus (2.3) holds so Theorem 2.19 guarantees that F is essential in $AI_{\partial U}(\overline{U}, E)$. A similar argument shows if F is essential in $AI_{\partial U}(\overline{U}, E)$, then G is essential in $AI_{\partial U}(\overline{U}, E)$. □

Remark 2.24. One could replace strongly inward with respect to C with $F(x) \subseteq \overline{I_C(x)}$ on $\overline{U}$ in Definitions 2.15, 2.16, 2.17, 2.18, and trivial adjustments in the proofs above will establish Theorems 2.19 and 2.20 (note instead of [1] one could for example use Theorem 3 and Corollary 1 in [4]). Similarly we can obtain an analogue of Theorem 2.23.

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