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An inertia-based algorithm for pseudomonotone variational inequality and fixed point problems in real Hilbert space

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Abstract

The aim of this work is to study a pseudomonotone variational inequality and a fixed point problem involving pseudocontractive mappings in real Hilbert spaces. We introduce an inertia-based iterative algorithm for finding a common solution to this problem. The strong convergence of the proposed algorithm is proved. Finally, numerical examples are provided and also meaningful comparisons of these results with those in [Y. Yao, M. Postolache, J. C. Yao, Mathematics, **7** (2019), 14 pages], proving that at our proposed numerical schemes are more efficient.

Keywords: Pseudomonotone variational inequality, pseudocontractive mapping, fixed point problem, Hilbert space.

2020 MSC: 47H09, 47H10, 49M05, 54H25.

1. Introduction

Let H be a real Hilbert space. We denote by $\langle \cdot, \cdot \rangle$ the inner product on H and $\| \cdot \|$ the norm induced by the inner product. Let C be a nonempty, closed, and convex subset of H. In this paper, we study a problem related to the well-known variational inequality problem *VIP* defined as follows: find $x^* \in C$ such that

$$\langle f(x^*), y - x^* \rangle \geqslant 0, \tag{1.1}$$

where $f\colon H \to H$ is a nonlinear operator. Consider that $VI(C, f)$ is the solution set to problem (1.1) and make the following assumptions:

(i) $VI(C, f)$ is nonempty;

(ii) f is pseudomonotone on H, that is

$$\langle f(\bar{x}), y - \bar{x} \rangle \geqslant 0 \Rightarrow \langle f(x), x - \bar{x} \rangle \geqslant 0, \text{ for all } \bar{x}, x \in H;$$

(iii) f is $L-$Lipschitz continuous on H (for some $L > 0$), that is

$$\|f(v) - f(w)\| \leqslant L\|v - w\|, \text{ for all } v, w \in H.$$

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Variational inequalities represent an efficient tool in the study of problems arising from optimization, economics, differential equations, engineering etc. For details, please refer to [2, 4, 12] and the references therein.

A well known iterative method to solve the VIP (1.1) when f is of monotone-type is the projected gradient rule defined by:

$$x_{n+1} = P_C(I - \lambda f)x_n, \ n \in \mathbb{N}, \tag{1.2}$$

where P_C is the metric projection map from H onto C and $\lambda > 0$ is the step size. The projected gradient algorithm has been studied extensively by many authors (see, e.g., [13, 15] and the references contained therein).

The following facts on the numerical scheme (1.2) are considered known (see, e.g., [19]).

(i) If the operator f is η-strongly monotone and L-Lipschitz continuous, with $0 < \lambda < \frac{\eta}{L^2}$, then there exist a unique point in $VI(C, f)$, and the sequence $\{x_n\}$ generated by process (1.2) converges strongly to this point.

(ii) If f is η-inverse strongly monotone, the solution to the VIP(1.1) does not always exist. In the situation in which $VI(C, f)$ is nonempty and $0 < \lambda < 2\eta$, then it is a closed and convex subset of H. In this case, the sequence $\{x_n\}$ generated by (1.2) converges weakly to one of the points in $VI(C, f)$.

(iii) The hypotheses that f is monotone and Lipschitz continuous do not ensure the convergence of the sequence $\{x_n\}$ generated by scheme (1.2) to an element of $VI(C, f)$. To overcome this deficiency of the projected gradient rule (1.2), in 1976, Korpelevich [14] introduced the so-called extragradient method for solving the VIP (1.1) when f is monotone and L-Lipschitz continuous in the finite dimensional Euclidean space $\mathbb{R}^n$, as follows:

$$\begin{cases} y_n = P_C(I - \lambda f)x_n, \\ x_{n+1} = P_C(I - \lambda f)y_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where the step size is $\lambda \in \left(0, \frac{1}{L}\right)$. It has been proved that the sequence $\{x_n\}$ converges to a point in $VI(C, f)$.

The Korpelevich iterative sequence (1.3) has been developed in different forms. For more results involving (1.3), please consult [6, 14, 20, 21].

Next, we consider the following fixed point problem

$$\text{Find } x \in C : Tx = x, \tag{1.4}$$

where $T \colon C \to C$ is a pseudocontractive mapping. Solving fixed point problems involving different nonlinear operators has been studied for many years in literature, under different settings (see, e.g., [5, 7, 10, 18] and the references therein). More precisely, algorithmic approximation theories and experiments of pseudocontractive operators have been considered, see for instance [1, 25]. In 2019, Y. Yao et al. [24] studied the problem of finding a fixed point of pseudocontractive operator and a solution point to variational inequality problems in real Hilbert spaces. They introduced an iterative algorithm and proved that the sequence strongly converges to a common solution to the pseudomonotone variational inequalities and a fixed point of pseudocontractive operator. The construction of fast convergent iterative algorithms is of interest for practical applications. In this direction, an inertial-type extrapolation method was first proposed by Polyak [16] as an acceleration process. In recent years, some authors have constructed various fast iterative algorithms by inertial extrapolation techniques, such as the inertial Mann algorithms [15], the inertial forward-backward splitting algorithms [14] and so on.

Motivated by the work of Yao et al. [24] and the ongoing research in this direction, it is our purpose in this article to propose an inertia-based iterative scheme and prove that the sequence strongly converges to a common solution to a pseudomonotone variational inequality and a fixed point of a Lipschitz pseudocontractive operator in real Hilbert spaces. Next, we compare the performance of our proposed schemes with those of Yao et al. [24]. We provide numerical examples and use them to realize the second objective. The examples show that our proposed algorithms are more efficient than those of Yao et al. [24].

2. Preliminaries

We start by recollecting some important classes of operators, which will be used in the sequel.

Definition 2.1. Let $T\colon H \to H$ be an operator on the Hilbert space H.

(i) T is said to be a contraction of coefficient β if there exists $\beta \in (0,1)$ such that

$$\|Tx - Ty\| \leqslant \beta\|x - y\|, \text{ for all } x, y \in H.$$

(ii) T is called θ-strongly monotone if there exists $\theta > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geqslant \theta\|x - y\|^2, \text{ for all } x, y \in H.$$

(iii) The mapping T is θ-inverse strongly monotone if there exists $\theta > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geqslant \theta\|T(x) - T(y)\|^2, \text{ for all } x, y \in H.$$

(iv) T is a pseudocontractive operator if

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in H.$$

(v) T is called weakly sequentially continuous if for any given sequence $\{x_n\} \subset H$ which satisfies the relation $x_n \rightharpoonup x^*$, necessarily we have $T(x_n) \to T(x^*)$.

The relations in the next lemma are features of real Hilbert spaces.

Lemma 2.2. *Let H be a real Hilbert space. The next relations are accomplished.*

(i) *For all x, $y \in H$, and $\delta \in [0,1]$ ([3]),*

$$\|\delta u + (1-\delta)y\|^2 = \delta\|x\|^2 + (1-\delta)\|v\|^2 - \delta(1-\delta)\|x - y\|^2.$$

(ii) *For all x, $y \in H$,*

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2.$$

(iii) *If x, $y \in H$,*

$$\|x - y\|^2 \leqslant \|x\|^2 + 2\langle y, x - y\rangle.$$

We recall now a property which characterizes Lipschitz pseudocontractive operators.

Lemma 2.3 ([23]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. If $T\colon C \to C$ is an L-Lipschitz pseudocontractive operator, and $0 < \eta < \frac{1}{\sqrt{1+L^2}+1}$, then, for any fixed point p of T, and for any $u \in C$, the next inequality is accomplished*

$$\|p - T((1-\eta)u + \eta Tu)\|^2 \leqslant \|u - p\|^2 + (1-\eta)\|u - T((1-\eta)u + \eta Tu)\|^2.$$

The following lemma provides conditions in which a sequence endowed with suitable properties is convergent to zero.

Lemma 2.4 ([22]). *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers such that $a_{n+1} \leqslant (1 - \delta_n)a_n + b_n + c_n$, for all $n \geqslant 1$, where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a sequence of real numbers. Furthermore, suppose that $\sum\limits_{n=1}^{\infty} c_n < \infty$. Then the following results hold.*

(i) *If $b_n \leqslant \delta_n M$ for some $M \geqslant 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum\limits_{n=1}^{\infty} \delta_n = \infty$ and $\lim\limits_{n\to\infty} \sup \frac{b_n}{\delta_n} \leqslant 0$, then the sequence $\{a_n\}$ is convergent to zero.*

Lemma 2.5 ([25]). *Let H be a real Hilbert space, C a nonempty, closed, and convex subset of H and $T\colon C \to C$ be a continuous pseudocontractive operator. Then*

(i) *The set of the fixed pints of f T is a closed, and convex subset of C.*

(ii) *The operator T is demi-closed, i.e, $u_n \rightharpoonup \bar{u}$, and $T(u_n) \to u^*$ implies that $T(\bar{u}) = u^*$.*

3. Main Results

In this section, we introduce an inertia-based iterative algorithm in order to determine the solutions to both problems (1.1) and (1.4), and provide a convergence analysis of the iterative sequence.

In the next iterative process, H is a Hilbert space, and C is a nonempty, closed, and convex subset of H. f is a Lipschitz operator on C, of constant κ, $T\colon C \to C$, and $h\colon C \to C$. Denote by $W(x_n) = \{x | \text{there exists} \{x_{n_j}\} \subseteq \{x_n\}, x_{n_j} \rightharpoonup x\}$.

Algorithm 3.1 (Inertia-based iterative procedures for the variational inequality (1.1) and the fixed point problem (1.4))**.** Let x_0, $x_1 \in C$ be fixed, and $\{\alpha_n\}$, $\{\sigma_n\}$ $\{\delta_n\}$ be sequences in $(0,1)$. Consider $\gamma \in (0,1)$, $\mu \in (0,1)$, $\theta \in (0,1)$, and $\tau \in (0,2)$ respectively.

Step 1. Let $w_0 \in C$ be an initial value. Set $n = 0$.

Step 2. Assume that the sequence $\{w_n\}$ in C has been constructed and then calculate $P_C[w_n - f(w_n)]$.

Step 3.

Case 1. If $P_C[w_n - f(w_n)] \neq w_n$, then compute the sequence $\{y_n\}$ as follows:

$$y_n = P_C[w_n - \mu\gamma^{m(w_n)} f(w_n)], \tag{3.1}$$

where $m(w_n) = \min\{0,1,2,3,\cdots\}$ and checks

$$\mu\gamma^{m(w_n)}\|f(w_n) - f(y_n)\| \leqslant \theta\|w_n - y_n\|.$$

Define the sequences $\{w_n\}$, $\{u_n\}$, $\{\xi_n\}$, and $\{x_n\}$ by the following rules:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = P_C\left[w_n - \tau(1-\theta)\|w_n - y_n\|^2 \frac{w_n - y_n + \mu\gamma^{m(w_n)} f(y_n)}{\|w_n - y_n + \mu\gamma^{m(w_n)} f(y_n)\|^2}\right], \\ \xi_n = (1-\sigma_n)u_n + \sigma_n T[(1-\delta_n)u_n + \delta_n T u_n], \\ x_{n+1} = \alpha_n h(\xi_n) + (1-\alpha_n)\xi_n, \ \text{for all } n \geqslant 1. \end{cases} \tag{3.2}$$

Case 2. If $P_C[w_n - f(w_n)] = w_n$, then calculate $\{x_{n+1}\}$ in the following form:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ \xi_n = (1-\sigma_n)w_n + \sigma_n T[(1-\delta_n)u_n + \delta_n T w_n], \\ x_{n+1} = \alpha_n h(\xi_n) + (1-\alpha_n)\xi_n, \ \text{for all, } n \geqslant 1. \end{cases}$$

Step 4. Set $n := n+1$, and return to Step 2.

Following Remarks 1 and 2 from [24], we consider the next observations.

Remark 3.2. The orthogonal metric projection $P_C : H \to C$ possesses the following characteristic: for any given $x \in H$,

$$\langle x - P_C(x), y - P_C(x)\rangle \leqslant 0, \text{ for all } y \in C.$$

Therefore, $x \in VI(C,f)$ if and only if $x = P_C[x - \tau f(x)]$, for all $\tau > 0$. Presume that, at some iterative step, $w_n = P_C[w_n - f(w_n)]$. Then w_n is a solution to the variational inequality problem, and hence $\omega_n(w_n) \subset VI(C,f)$. Consequently, $W_n(x_n) \subset VI(C,f)$.

Remark 3.3. Following [24, Remark 2], for given w_n, there can be found $m(w_n)$ such that (3.1) holds.

The next proposition, whose proof follows the footsteps of that of Proposition 1 in [24], will help us show that our proposed iterative scheme, and mainly states that Algorithm 3.1 is well defined.

Proposition 3.4 ([24])**.** *Suppose that* H *is a Hilbert space and* C *is a nonempty, closed, and convex subset of* H*, and that* $x^* = P_\Gamma h(x^*)$ *(see Theorem 3.6 below for definition of* Γ*).*

1.) *In the context of Algorithm 3.1, if* $w_n \neq P_C[w_n - f(w_n)]$*, then* $w_n - y_n + \mu\gamma^{m(w_n)} f(y_n) \neq 0$.

2.) *For any* $n \geqslant 1$,
$$\langle w_n - y_n + \mu\gamma^{m(w_n)} f(y_n), w_n - x^* \rangle \geqslant (1-\theta)\|w_n - y_n\|^2.$$

3.) *For any* $n \geqslant 1$,
$$\|u_n - x^*\|^2 \leqslant \|w_n - x^*\|^2 - \frac{(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\|w_n - y_n + \mu\gamma^{m(w_n)} f(y_n)\|^2}. \tag{3.3}$$

4.) *For any* $n \geqslant 1$,
$$\|\xi_n - x^*\|^2 \leqslant \|u_n - x^*\|^2 - \sigma_n(\delta_n - \sigma_n)\|u_n - T[(1-\delta_n)u_n + \delta_n T u_n]\|^2. \tag{3.4}$$

Remark 3.5. In Case 1 from above, $f(w_n) \neq 0$, by Remark 3.2, and $\mu\gamma^{m(w_n)} f(y_n) \neq 0$, by Remark 3.3, for all $n \geqslant 0$. Using Proposition 3.4, we deduce that the sequence $\{u_n\}$ is well-defined, and hence the sequence $\{x_n\}$ is well-defined. Furthermore, if $P_C[w_n - f(w_n)] \neq w_n$, we can determine the sequence $\{y_n\}$ as follows:
$$y_n = P_C[w_n - \mu\gamma^{m(w_n)} f(w_n)],$$
where $m(w_n) = \min\{0, 1, 2, 3, \ldots\}$ satisfies the inequality
$$\mu\gamma^{m(w_n)}\|f(w_n) - f(y_n)\| \leqslant \theta\|w_n - y_n\|. \tag{3.5}$$

Theorem 3.6. *Let* $C \neq \emptyset$ *be a convex and closed subset of a real Hilbert Space* H*, and* f *be pseudomonotone operator on* H *which is weakly sequentially continuous, and Lipschitz continuous on* C *with a Lipschitz constant* $\kappa > 0$*. Let* $T\colon C \to C$ *be an L-Lipschitz pseudocontractive operator with* $L \geqslant 1$ *and* $h\colon C \to C$ *be a* β *contractive map. Assume that* $\Gamma := VI(C, f) \cap F(T) \neq \emptyset$ *and let the iterative parameters* $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\theta_n\}$*, and* $\{\delta_n\}$ *satisfy the following conditions:*

(C1): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2): $0 \leqslant \theta_n \leqslant \bar{\theta}_n$, *with* $\bar{\theta}_n$ *defined by*
$$\bar{\theta}_n := \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}, \end{cases}$$
where $\sum_{n=1}^{\infty} \epsilon_n < \infty$*, and* $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$;

(C3): $0 < \underline{\sigma} < \sigma_n < \bar{\sigma} < \delta_n < \bar{\delta} < \frac{1}{\sqrt{1+L^2}+1}$, *for all* $n \geqslant 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to* $x^* \in \Gamma$*, with* $x^* = P_\Gamma h(x^*)$.

Proof. We divide the proof into three different steps.

Step 1. We show that the sequence $\{x_n\}$ is bounded.

Having in view (3.2), and the inequalities (3.3) and (3.4) from Proposition 3.4, we get
$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(h(\xi_n) - x^*) + (1-\alpha_n)(\xi_n - x^*)\| \\
&\leqslant (1-\alpha_n)\|\xi_n - x^*\| + \alpha_n\|h(\xi_n) - x^*\| \\
&\leqslant (1-\alpha_n)\|\xi_n - x^*\| + \alpha_n\|h(\xi_n) - h(x^*)\| + \alpha_n\|h(x^*) - x^*\| \\
&\leqslant (1-\alpha_n)\|\xi_n - x^*\| + \alpha_n\beta\|\xi_n - x^*\| + \alpha_n\|h(x^*) - x^*\| \\
&\leqslant (1-\alpha_n+\alpha_n\beta)\|\xi_n - x^*\| + \alpha_n\|h(x^*) - x^*\| \\
&\leqslant (1-\alpha_n+\alpha_n\beta)\|w_n - x^*\| + \alpha_n\|h(x^*) - x^*\|, \ n \geqslant 1.
\end{aligned}$$

Therefore, it can be noticed that
$$\|x_{n+1} - x^*\| \leqslant [1 - \alpha_n(1-\beta)]\,\|x_n + \theta_n(x_n - x_{n+1}) - x^*\| + \alpha_n\|h(x^*) - x^*\|$$

$$\begin{aligned}
&\leqslant (1-\alpha_n(1-\beta))[\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|] + \alpha_n\|h(x^*) - x^*\| \\
&= (1-\alpha_n(1-\beta))\|x_n - x^*\| + (1-\alpha_n(1-\beta))\theta_n\|x_n - x_{n-1}\| + \alpha_n\|h(x^*) - x^*\| \qquad (3.6)\\
&\leqslant (1-\alpha_n(1-\beta))\|x_n - x^*\| + \alpha_n(1-\beta)\left[\frac{\|h(x^*) - x^*\|}{1-\beta} + \frac{(1-\alpha_n(1-\beta)\theta\|x_n - x_{n-1}\|}{\alpha_n(1-\beta)}\right]\\
&\leqslant (1-\alpha_n(1-\beta))\|x_n - x^*\| + \alpha_n(1-\beta)\left[\frac{\|h(x^*) - x^*\|}{1-\beta} + \frac{\epsilon_n}{\alpha_n(1-\beta)}\right].
\end{aligned}$$

Applying Lemma 2.4 (i), and condition (C2) from Theorem 3.6, inequality (3.6) implies the boundedness of $\{x_n\}$. Hence, $\{\xi_n\}, \{h(\xi_n)\}, \{u_n\}, \{Tu_n\}$, and $\{w_n\}$ are also bounded. Let $M > 0$ so that $\|x_n\| \leqslant M$, for any $n \geqslant 1$.

Step 2: We aim to prove that $W(w_n) \subset \Gamma$.

Taking advantage of inequality (3.2) and Lemma 2.2 (iii), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(h(\xi_n) - x^*) + (1-\alpha_n)(\xi_n - x^*)\|^2 \\
&\leqslant (1-\alpha_n)^2\|\xi_n - x^*\|^2 + 2\alpha_n\langle h(\xi_n) - x^*, x_{n+1} - x^*\rangle.
\end{aligned} \qquad (3.7)$$

Virtue of (3.3) and (3.4) imply that (3.7) becomes

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leqslant (1-\alpha)^2\left[\|u_n - x^*\|^2 - \sigma_n(\delta_n - \sigma_n)\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|^2\right]\\
&\leqslant (1-\alpha_n)^2\|w_n - x^*\|^2 - (1-\alpha_n)^2\sigma_n(\delta_n - \sigma_n)\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|^2\\
&\quad - \frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2} + 2\alpha_n\langle h(\xi_n) - x^*, x_{n+1} - x^*\rangle \qquad (3.8)\\
&\leqslant (1-\alpha_n)^2\|w_n - x^*\|^2 + \alpha_n\left[-(1-\alpha_n)^2\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n\|^2}{\alpha_n}\right.\\
&\quad\left. - \frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\alpha_n\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2} + 2\langle h(\xi_n) - x^*, x_{n+1} - x^*\rangle\right].
\end{aligned}$$

Keeping in mind Condition (C2), it can be observed that

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2\\
&= \|x_n - x^*\|^2 + 2\theta_n\|x_n - x^*\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2\\
&= \|x_n - x^*\|^2 + \theta_n\|x_n - x_{n-1}\|\left[2\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|\right] \qquad (3.9)\\
&\leqslant \|x_n - x^*\|^2 + 3\theta_n\|x_n - x_{n-1}\|M \text{ for some } M > 0\\
&\leqslant \|x_n - x^*\|^2 + 3M\epsilon_n.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
2\alpha_n\langle h(\xi_n) - x^*, x_{n+1} - x^*\rangle &= 2\alpha_n\langle h(\xi_n) - h(x^*) + h(x^*) - x^*, x_{n+1} - x^*\rangle\\
&\leqslant 2\alpha_n\|h(\xi_n) - h(x^*)\|\|x_{n+1} - x^*\| + 2\alpha_n\langle h(x^*) - x^*, x_{n+1} - x^*\rangle\\
&\leqslant 2\alpha_n\beta\|\xi_n - x^*\|\|x_{n+1} - x^*\| + 2\alpha_n\langle h(x^*) - x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

From inequalities (3.3), (3.4), (3.9), it follows that

$$\|\xi_n - x^*\| \leqslant \|w_n - x^*\| \leqslant \|x_n - x^*\| + \sqrt{3M\epsilon_n.}$$

Therefore,

$$\begin{aligned}
2\alpha_n\langle h(\xi_n) - x^*, x_{n+1} - x^*\rangle &\leqslant 2\alpha_n\beta\left[\|x_n - x^*\| + \sqrt{3M\epsilon_n.}\right]\|x_{n+1} - x^*\|\\
&\quad + 2\alpha_n\langle h(x^*) - x^*, x_{n+1} - x^*\rangle.
\end{aligned} \qquad (3.10)$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n(h(\xi_n)-x^*)+(1-\alpha_n)(\xi_n-x^*)\| \\ &\leqslant \alpha_n\|h(\xi_n)-x^*\|+(1-\alpha_n)\|\xi_n-x^*\| \\ &\leqslant \|x_n-x^*\|+\sqrt{3M\epsilon_n}+\alpha_n\|h(\xi_n)-x^*\|.\end{aligned} \tag{3.11}$$

Having in mind relations (3.10) and (3.11), it follows that

$$\begin{aligned}2\alpha_n\langle h(\xi_n)-x^*,x_{n+1}-x^*\rangle \leqslant 2\alpha_n\beta\left[\|x_n-x^*\|+\sqrt{3M\epsilon_n}\right]\cdot\left[\|x_n-x^*\|+\alpha_n\|h(\xi_n)-x^*\|\right.\\ \left.+\sqrt{3M\epsilon_n}\right]+2\alpha_n\langle h(x^*)-x^*,x_{n+1}-x^*\rangle.\end{aligned} \tag{3.12}$$

Using relations (3.9) and (3.12) in (3.8), we are led to

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leqslant (1-\alpha_n)^2\left[\|x_n-x^*\|^2+3M\epsilon_n\right] \\ &\quad+\alpha_n[-(1-\alpha_n)^2\sigma_n(\delta_n-\sigma_n)\frac{\|u_n-T[(1-\delta_n)u_n+\delta_nTu_n\|^2}{\alpha_n} \\ &\quad-\frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n-y_n\|^4}{\alpha_n\|w_n-y_n+\mu\gamma^{m(w_n)}f(y_n)\|^2} \\ &\quad+2\alpha_n\beta\left[\|x_n-x^*\|+\sqrt{3M\epsilon_n.}\right]\cdot\left[\|x_n-x^*\|+\alpha_n\|h(\xi_n)-x^*\|+\sqrt{3M\epsilon_n}\right] \\ &\quad+2\alpha_n\langle h(x^*)-x^*,x_{n+1}-x^*\rangle] \\ &= (1-\alpha_n)^2\|x_n-x^*\|^2+(1-\alpha_n)^2 3M\epsilon_n \\ &\quad+2\alpha_n\beta\left[\|x_n-x^*\|^2+\alpha_n\|x_n-x^*\|\|h(\xi_n)-x^*\|+\sqrt{3M\epsilon_n}\|x_n-x^*\|\right. \\ &\quad\left.+\sqrt{3M\epsilon_n}\|x_n-x^*\|+\alpha_n\sqrt{3M\epsilon_n}\|h(\xi_n)-x^*\|+\sqrt{3M\epsilon_n.}\sqrt{3M\epsilon_n.}\right] \\ &\quad+2\alpha_n\langle h(x^*)-x^*,x_{n+1}-x^*\rangle \\ &\quad+\alpha_n\left[-(1-\alpha_n)^2\sigma_n(\delta_n-\sigma_n)\frac{\|u_n-T[(1-\delta_n)u_n+\delta_nTu_n\|^2}{\alpha_n}\right. \\ &\quad\left.-\frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n-y_n\|^4}{\alpha_n\|w_n-y_n+\mu\gamma^{m(w_n)}f(y_n)\|^2}\right]. \\ &= (1-\alpha_n)^2\|x_n-x^*\|^2+(1-\alpha_n)^2 3M\epsilon_n \\ &\quad+2\alpha_n\beta\left[\|x_n-x^*\|^2+\alpha_n\|x_n-x^*\|\|h(\xi_n)-x^*\|+2\sqrt{3M\epsilon_n}\|x_n-x^*\|\right. \\ &\quad\left.+\alpha_n\sqrt{3M\epsilon_n}\|h(\xi_n)-x^*\|+3M\epsilon_n\right]+2\alpha_n\langle h(x^*)-x^*,x_{n+1}-x^*\rangle \\ &\quad+\alpha_n\left[-(1-\alpha_n)^2\sigma_n(\delta_n-\sigma_n)\frac{\|u_n-T[(1-\delta_n)u_n+\delta_nTu_n\|^2}{\alpha_n}\right. \\ &\quad\left.-\frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n-y_n\|^4}{\alpha_n\|w_n-y_n+\mu\gamma^{m(w_n)}f(y_n)\|^2}\right].\end{aligned} \tag{3.13}$$

Boundedness of $\{\|x_n-x^*\|\},\{\|x_n-x^*\|\|\|h(\xi_n)-x^*\|\}$ and $\|h(\xi_n)-x^*\|\}\}$ gives that there exist $K_0>0, K_1>0, K_2>0,$ and $K>0$ such that

$$\|x_n-x^*\|\|\|h(\xi_n)-x^*\|\leqslant K,\ \|x_n-x^*\|\leqslant K_0,\ \|x_n-x^*\|^2\leqslant K_1\text{and } \|h(\xi_n)-x^*\|\leqslant K_2.$$

Consequently, from (3.13), we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \\ &\leqslant \left(1-2\alpha_n(1-\beta)\right)\|x_n-x^*\|^2+\alpha_n[-(1-\alpha_n)^2\sigma_n(\delta_n-\sigma_n)\frac{\|u_n-T[(1-\delta_n)u_n+\delta_nTu_n\|^2}{\alpha_n}\end{aligned}$$

$$- \frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\alpha_n\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2}$$
$$+ 2\alpha_n^2\beta K + 4\alpha_n\beta\sqrt{3M\epsilon_n}K_0 + 2\alpha_n^2\beta K_2 + 6\alpha_n\beta M\epsilon_n + \alpha_n^2 K_1 + 3M\epsilon_n + 2\alpha_n\langle h(x^*) - x^*, x_{n+1} - x^*\rangle]$$

$$\begin{aligned} &= \big(1 - 2\alpha_n(1-\beta)\big)\|x_n - x^*\|^2 + \alpha_n[-(1-\alpha_n)^2\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n\|^2}{\alpha_n} \\ &\quad - \frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\alpha_n\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2} \\ &\quad + \alpha_n\big(2\alpha_n\beta K + 4\beta\sqrt{3M\epsilon_n}K_0 + 2\alpha_n\beta K_2 + 6\beta M\epsilon_n + \alpha_n K_1 + 3M\frac{\epsilon_n}{\alpha_n}\big) + 2\langle h(x^*) - x^*, x_{n+1} - x^*\rangle] \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\leqslant \big(1 - 2\alpha_n(1-\beta)\big)\|x_n - x^*\|^2 + \alpha_n[-(1-\alpha_n)^2\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n\|^2}{\alpha_n} \\ &\quad - \frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\alpha_n\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2} \\ &\quad + \alpha_n\big(2\alpha_n^2\beta K' + 4\alpha_n\beta\sqrt{3M\epsilon_n}K' + 2\alpha_n^2\beta K' + 6\alpha_n\beta M\epsilon_n + \alpha_n^2 K' + 3M\epsilon_n + 3M\frac{\epsilon_n}{\alpha_n}\big) \\ &\quad + 2\alpha_n\langle h(x^*) - x^*, x_{n+1} - x^*\rangle] \\ &= \big(1 - 2\alpha_n(1-\beta)\big)\|x_n - x^*\|^2 + \alpha_n\Big[-(1-\alpha_n)\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n\|^2}{\alpha_n} \\ &\quad - \frac{(1-\alpha_n)(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\alpha_n\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2} d + \beta_n + 2\langle h(x^*) - x^*, x_{n+1} - x^*\rangle\Big], \end{aligned}$$

where $\beta_n := 2\alpha_n\beta K' + 4\beta\sqrt{3M\epsilon_n}K' + 2\alpha_n\beta K' + 6\beta M\epsilon_n + \alpha_n K' + 3M\frac{\epsilon_n}{\alpha_n}$, and $K' = \max\{K, K_0, K_1, K_2\}$. Notice that $\beta_n \to 0$ as $n \to \infty$. Set $S_n = \|x_n - x^*\|$ and

$$\begin{aligned} t_n = &-\frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\alpha_n\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2} \\ &- (1-\alpha_n)^2\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n\|^2}{\alpha_n} + \beta_n + 2\langle h(x^*) - x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{3.15}$$

Inequality (3.14) leads to

$$S_{n+1} \leqslant (1 - 2\alpha_n(1-\beta))S_n + \alpha_n t_n, \; n \in \mathbb{N}. \tag{3.16}$$

We show that $\limsup_{n\to\infty} t_n$ is finite. From (3.15), we deduce that

$$t_n \leqslant \beta_n + 2\langle h(x^*) - x^*, x_{n+1} - x^*\rangle \leqslant \beta_n + 2\|h(x^*) - x^*\|\|x_{n+1} - x^*\|.$$

This inequality, jointly with the boundedness of $\{x_n\}$, implies that $\limsup_{n\longrightarrow\infty} t_n$ has an upper bound.

Next, we show that $\limsup_{n\longrightarrow\infty} t_n$ has a lower bound. More precisely, we prove that $-1 \leqslant \limsup_{n\longrightarrow\infty} t_n$. Presume this inequality does not hold true. It follows that $\limsup_{n\longrightarrow\infty} t_n < -1$. So, there exists $N_0 \in \mathbb{N}$ such that $t_n < -1$ when $n \geqslant N_0$.

From (3.16), we have

$$\begin{aligned} S_{n+1} &\leqslant (1 - 2\alpha_n(1-\beta))S_n + \alpha_n t_n \\ &\leqslant (1 - 2\alpha_n(1-\beta))S_n - \alpha_n \\ &= S_n - \alpha_n(1 + 2S_n) + 2\alpha_n\beta S_n \leqslant S_n - \alpha_n(1 + 2S_n) + 2\alpha_n S_n \\ &\leqslant S_n - \alpha_n. \end{aligned}$$

Hence, $S_{n+1} \leqslant S_{N_0} - \sum_{k=N_0}^{n} \alpha_k$, and we get that $\limsup_{n\longrightarrow\infty} S_n \leqslant S_{N_0} - \limsup_{n\longrightarrow\infty} \sum_{k=N_0}^{n} \alpha_k = -\infty$. This is a contradiction, so $-1 \leqslant \limsup_{n\to\infty} t_n < \infty$.

Due to the fact that $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_i}\} \subset \{w_n\}$ such that $w_{n_i} \rightharpoonup p \in C$ and

$$\begin{aligned}\limsup_{n\to\infty} t_{n_i} = \lim_{i\to\infty} t_n = \lim_{i\to\infty} [&-\frac{(2-\tau)\tau(1-\theta)^2\|w_{n_i}-y_{n_i}\|^4}{\alpha_{n_i}\|w_{n_i}-y_{n_i}+\mu\gamma^{m(w_{n_i})}f(y_{n_i})\|^2} \\ &+\beta_{n_i}+2\langle h(x^*)-x^*, x_{n_i+1}-x^*\rangle \\ &-\sigma_{n_i}(\delta_{n_i}-\sigma_{n_i})\frac{\|u_{n_i}-T[(1-\delta_{n_i})u_{n_i}+\delta_{n_i}Tu_{n_i}]\|^2}{\alpha_{n_i}}].\end{aligned} \tag{3.17}$$

Based on the boundedness of the sequences $\{x_{n_i}\}$, without loss of generality, we may assume that $\lim_{i\to\infty} 2\langle h(x^*)-x^*, x_{n_i+1}-x^*\rangle$ exists. Also, using (3.17), we can show the existence of the following limit

$$\lim_{i\to\infty}\left[\frac{(2-\tau)\tau(1-\theta)^2\|w_{n_i}-y_{n_i}\|^4}{\alpha_{n_i}\|w_{n_i}-y_{n_i}+\mu\gamma^{m(w_{n_i})}f(y_{n_i})\|^2}+\sigma_{n_i}(\delta_{n_i}-\sigma_{n_i})\frac{\|u_{n_i}-T[(1-\delta_{n_i})u_{n_i}+\delta_{n_i}Tu_{n_i}]\|^2}{\alpha_{n_i}}\right]. \tag{3.18}$$

Since $\lim_{i\to\infty}\alpha_{n_i}=0$, and $\liminf_{i\to\infty}\sigma_{n_i}(\delta_{n_i}-\sigma_{n_i})>0$, from (3.18) it follows that

$$\lim_{i\to\infty}\frac{\|w_{n_i}-y_{n_i}\|^4}{\|w_{n_i}-y_{n_i}+\mu\gamma^{m(w_{n_i})}f(y_{n_i})\|^2}=0, \tag{3.19}$$

and

$$\lim_{i\to\infty}\|u_{n_i}-T[(1-\delta_{n_i})u_{n_i}+\delta_{n_i}Tu_{n_i}]\|^2=0. \tag{3.20}$$

Considering that $\|w_{n_i}-y_{n_i}+\mu\gamma^{m(w_{n_i})}f(y_{n_i})\|$ is bounded and having in mind relation (3.19), we obtain

$$\lim_{i\to\infty}\|w_{n_i}-y_{n_i}\|=0. \tag{3.21}$$

From (3.2), and assumption (C2), we have

$$\|x_{n_i}-y_{n_i}\| \leqslant \|x_{n_i}-w_{n_i}\|+\|w_{n_i}-y_{n_i}\| = \theta_{n_i}\|x_{n_i}-x_{n_{i-1}}\|+\|w_{n_i}-y_{n_i}\|,$$

and also

$$\lim_{i\to\infty}\|x_{n_i}-y_{n_i}\|=0. \tag{3.22}$$

Inequalities (3.5) and (3.21) compel

$$\lim_{i\to\infty}\|f(w_{n_i})-f(y_{n_i})\|=0.$$

Using (3.22) and the fact that f is Lipschitz continuous, we have

$$\lim_{i\to\infty}\|f(x_{n_i})-f(y_{n_i})\|=0.$$

From (3.2), we deduce that

$$\begin{aligned}\|u_n-w_n\| &= \left\|P_C\left[w_n-\tau(1-\theta)\|w_n-y_n\|^2\frac{w_n-y_n+\mu\gamma^{m(w_n)}f(y_n)}{\|w_n-y_n+\mu\gamma^{m(w_n)}f(y_n)\|^2}\right]-P_Cw_n\right\| \\ &\leqslant \frac{\tau(1-\theta)\|w_n-y_n\|^2}{\|w_n-y_n+\mu\gamma^{m(w_n)}f(y_n)\|}.\end{aligned}$$

Mixing this inequality with (3.19), we obtain

$$\lim_{i\to\infty}\|u_{n_i}-w_{n_i}\|=0. \tag{3.23}$$

Applying (3.2), assumption (2) and having in view (3.21), (3.22), and the previous equality, we have

$$\lim_{i\to\infty} \|u_{n_i} - x_{n_i}\| = 0.$$

The characterization of the projection P_C compels

$$\langle w_{n_i} - \mu\gamma^{m(w_{n_i})} f(w_{n_i}) - y_{n_i}, y_{n_i} - p\rangle \geqslant 0, \text{ for all } p \in C.$$

It follows that

$$\langle f(w_{n_i}), p - w_{n_i}\rangle \geqslant \langle f(w_{n_i}), y_{n_i} - w_{n_i}\rangle + \frac{1}{\mu\gamma^{m(w_{n_i})}}\langle y_{n_i} - p, y_{n_i} - w_{n_i}\rangle, \text{ for all } p \in C. \tag{3.24}$$

The boundedness of the sequences $\{f(w_{n_i})\}$ and $\{y_{n_i}\}$, and Remark 3.3 imply $\frac{\gamma\theta}{\kappa} < \mu\gamma^{m(w_{n_i})} \leqslant M$. Using (3.21) and (3.24), we have

$$\liminf_{i\to\infty} \langle f(w_{n_i}), p - w_{n_i}\rangle \geqslant 0, \text{ for all } p \in C,$$

which implies that there can be found a positive real number sequence $\{\zeta_j\}$ which satisfies $\lim_{j\to\infty} \zeta_j = 0$. There exists a smallest positive integer k_i such that

$$\langle f(w_{n_ij}), p - w_{n_ij}\rangle + \zeta_j \geqslant 0, \text{ for all } j \geqslant k_i. \tag{3.25}$$

From Remark 3.5, we get that for each $j > 0$, $f(w_{n_ij}) \neq 0$. If we denote by $\omega(w_{n_ij}) = \frac{f(w_{n_ij})}{\|f(w_{n_ij})\|^2}$, then $\langle f(w_{n_ij}), \omega(w_{n_ij})\rangle = 1$. From (3.25), we obtain

$$\langle f(w_{n_{i_j}}), p + \zeta_j \omega(w_{n_{i_j}}) - w_{n_{i_j}}\rangle \geqslant 0. \tag{3.26}$$

Furthermore, (3.26) together with the fact that f is pseudomonotone on H imply

$$\langle f(p + \zeta_j \omega(w_{n_{i_j}})), p + \zeta_j \omega(w_{n_{i_j}}) - w_{n_{i_j}}\rangle \geqslant 0.$$

Therefore,

$$\langle f(p), p - w_{n_{i_j}}\rangle \geqslant \langle f(p) - f(p + \zeta_j \omega(w_{n_{i_j}})), p + \zeta_j \omega(w_{n_{i_j}}) - w_{n_{i_j}}\rangle + \langle f(p), -\zeta_j \omega(w_{n_{i_j}})\rangle. \tag{3.27}$$

Since the sequence $\{w_{n_{i_j}}\}$ is bounded, without loss of generality, we may assume that $w_{n_ij} \rightharpoonup v \in C$ as $j \to \infty$. Having in mind that $\|w_n - x_n\| \to 0$ and since $\{x_n\}$ is bounded, we have that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_{i_j}} \rightharpoonup v$.

Moreover, due to the fact that f is weakly sequentially continuous $f(w_{n_ij}) \rightharpoonup f(v)$.

Presume that $f(v) \neq 0$ (otherwise, $v \in VI(C, f)$) and $\omega(w_n) \subset VI(C, f)$. It follows that

$$\liminf_{j\to\infty} \|f(w_{n_{i_j}})\| \geqslant \|f(v)\| > 0,$$

and

$$\lim_{j\to\infty} \|\zeta_j \omega(w_{n_{i_j}})\| = \lim_{j\to\infty} \frac{\zeta_j}{\|f(w_{n_{i_j}})\|} = 0.$$

Using relation (3.27) together with the fact that f is Lipschitz continuous, we have

$$\langle f(p), p - v\rangle \geqslant 0,$$

and we get that $v \in VI(C, f)$, and hence $\omega(w_n) \subset VI(C, f)$.

Since T is Lipschitzian, the next relations hold true

$$\begin{aligned}\|u_n - Tu_n\| &\leqslant \|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\| + \|T[(1-\delta_n)u_n + \delta_n Tu_n] - Tu_n\| \\ &\leqslant \|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\| + L\delta_n\|u_n - Tu_n\| \\ &\leqslant \frac{1}{1-\delta_n L}\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n\| \to 0.\end{aligned} \tag{3.28}$$

From (3.20), (3.23), and (3.28), we get

$$\lim_{j\to\infty}\|w_{n_ij} - Tw_{n_ij}\| = 0.$$

Taking advantage of (3.2), assumption (C2) and the previous equality, we obtain

$$\begin{aligned}\|x_{n_{i_j}} - Tx_{n_{i_j}}\| &\leqslant \|x_{n_{i_j}} - w_{n_{i_j}}\| + \|w_{n_{i_j}} - Tw_{n_{i_j}}\| + \|Tw_{n_{i_j}} - Tx_{n_{i_j}}\| \\ &\leqslant (1+L)\|w_{n_{i_j}} - x_{n_{i_j}}\| + \|w_{n_{i_j}} - Tw_{n_{i_j}}\| \to 0.\end{aligned} \tag{3.29}$$

Applying Lemma 2.5 to (3.29), we conclude that $v \in F(T)$, hence $v \in VI(C, f) \cap F(T) = \Gamma$.

Step 3: We show that $x_n \to P_\Gamma h(x^*)$. Notice that in Case I or Case II,

$$\limsup_{n\to\infty}\langle h(x^*) - x^*, x_{n+1} - x^*\rangle = \langle h(x^*) - x^*, v - x^*\rangle \leqslant 0.$$

Let $x^* = P_\Gamma h(x^*)$. Then, from (3.2) and inequality (3.14), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leqslant (1 - 2\alpha_n(1-\beta))\|x_n - x^*\|^2 \\ &\quad + \alpha_n\Bigg[-(1-\alpha_n)^2\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n\|^2}{\alpha_n} \\ &\quad - \frac{(1-\alpha_n)^2(2-\tau)\tau(1-\theta)^2\|w_n - y_n\|^4}{\alpha_n\|w_n - y_n + \mu\gamma^{m(w_n)}f(y_n)\|^2} \\ &\quad + \beta_n + 2\langle h(x^*) - x^*, x_{n+1} - x^*\rangle] \\ &\leqslant (1 - 2\alpha_n(1-\beta))\|x_n - x^*\|^2 + \alpha_n\beta_n + 2\alpha_n\langle h(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leqslant (1-\mu_n)\|x_n - x^*\|^2 + \gamma_n,\end{aligned} \tag{3.30}$$

where $\mu_n = 2\alpha_n(1-\beta) \in (0,1)$, for all n, $\gamma_n = \alpha_n\beta_n + 2\alpha_n\langle h(x^*) - x^*, x_{n+1} - x^*\rangle$.

In order to prove that $\{x_n\}$ converges strongly to x^*, we shall apply Lemma 2.4 (ii). Observe that either in Case 1 or Case 2, it is not difficult to see that $\limsup_{n\to\infty}\frac{\gamma_n}{\mu_n} \leqslant 0$, as $\beta_n \to 0$. Hence, from Lemma 2.4 (i) and (3.30), we conclude that $x_n \to x^* := P_\Gamma h(x^*)$. This completes the proof. □

Remark 3.7. It is obvious that the monotonicity implies pseudomonotonicity. Hence, Theorem 3.6 holds when the involved operator f is monotone.

Algorithm 3.8 (Inertia-based iterative procedures for VI)**.** Let x_0, $x_1 \in C$ be fixed, and let $\gamma \in (0,1)$, $\mu \in (0,1)$, $\theta \in (0,1)$, and $\tau \in (0,2)$, respectively.

Step 1. Let $w_0 \in C$ be an initial value. Set $n = 0$.

Step 2. Assume that the sequence $\{w_n\}$ in C has been constructed and then calculate $P_C[w_n - f(w_n)]$. If $P_C[w_n - f(w_n)] = w_n$, then stop; else go to the steps below.

Step 3. If $P_C[w_n - f(w_n)] \neq w_n$, then reckon the sequence $\{y_n\}$ as follows:

$$y_n = P_C[w_n - \mu\gamma^{m(w_n)}f(w_n)],$$

where $m(w_n) = \min\{0, 1, 2, 3, \dots\}$ and satisfies

$$\mu\gamma^{m(w_n)}\|f(w_n) - f(y_n)\| \leqslant \theta\|w_n - y_n\|,$$

and compute the sequences $\{w_n\}$, $\{u_n\}$, $\{\xi_n\}$, and $\{x_{n+1}\}$ by the following rule

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \ n \geqslant 1, \\ u_n = P_C[w_n - \tau(1-\theta)\|w_n - y_n\|^2 \frac{w_n - y_n + \mu\gamma^{m(w_n)} f(y_n)}{\|w_n - y_n + \mu\gamma^{m(w_n)} f(y_n)\|^2}], \\ x_{n+1} = \alpha_n h(\xi_n) + (1-\alpha_n)u_n, \ \text{for all } n \geqslant 1. \end{cases} \tag{3.31}$$

Step 4. Set $n := n + 1$, and return to Step 2.

Corollary 3.9. *Suppose that* $VI(C, f) \neq \emptyset$. *Assume that the iterative parameter* $\{\alpha_n\}$ *satisfies condition* (C1) *in Theorem* 3.6. *Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.8 *converges strongly to* $x^* = P_{VI(C,f)}h(x^*)$.

Algorithm 3.10 (Inertia-based iterative procedures for FP)**.** Let x_0, $x_1 \in C$ be fixed, $\gamma \in (0, 1)$, $\mu \in (0, 1)$, $\theta \in (0, 1)$, and $\tau \in (0, 2)$.

Step 1. Let $w_0 \in C$ be an initial value. Set $n = 0$.

Step 2. Assume that the sequence $\{w_n\}$ in C has been constructed. Let $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$. Compute the sequences $\{w_n\}$, $\{\xi_n\}$, and $\{x_{n+1}\}$ via the following iterations

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \ n \geqslant 1, \\ \xi_n = (1-\sigma_n)w_n + \sigma_n T[(1-\delta_n)w_n + \delta_n T w_n], \\ x_{n+1} = \alpha_n h(\xi_n) + (1-\alpha_n)\xi_n, \text{ for all } n \geqslant 1. \end{cases}$$

Step 3. Set $n := n + 1$ and return to Step 2.

Corollary 3.11. *Suppose that* $F(T) \neq \emptyset$. *Assume that the iterative parameter sequences* $\{\alpha_n\}$, $\{\sigma_n\}$, *and* $\{\delta_n\}$ *satisfy Conditions* (C1) *and* (C2) *in Theorem* 3.6. *Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.10 *converges strongly to* $x^* = P_{F(T)}h(x^*)$.

Remark 3.12. The application given in [24] to strict pseudocontractive maps carries over to the inertia based results presented in here.

Remark 3.13. Yao et al. [24] introduced a Halpern-type iterative algorithm in order to solve pseudomonotone variational inequalities and fixed point problems for pseudocontractive operators. Theorem 3.6 and the corollaries therein of our work improve the results of their paper by involving the inertia term, which is known to accelerate the convergence of such algorithms.

4. Numerical examples

Example 4.1. Define $f\colon \mathbb{R}^m \to \mathbb{R}^m$, $fx = e^{-x^TQx} + \beta$. The variational inequality problem for thie operator, f is equivalent to the operator $A(x) = Px + q$, where Q is a positive definite matrix (i.e., $x^TQx \geqslant \alpha\|x\|^2$, for all $x \in \mathbb{R}^m$), P is a positive semi-definite matrix, $q \in \mathbb{R}^m$, and $\beta > 0$. Observe that A is differentiable and there exists $M > 0$ such that $\|\nabla Ax\| \leqslant M$, for all $x \in \mathbb{R}^m$. Therefore, by the mean value theorem, A is Lipschitz continuous. Also, A is pseudomonotone, but not monotone (see, e.g., [4]). This is a popular numerical example for variational inequalities with pseudomonotone cost function, which shows that the class of pseudomonotone variational inequalities properly contains the class of monotone variational inequalities and has been considered by many authors (see, e.g., [4]).

Our interest in this example is to compare Algorithm 2 (3.31) of our work with Algorithm 2 of [24].

Algorithm 4.2 ([24])**.** Let x_0, $u \in C$ be fixed. Let $\gamma \in (0, 1)$, $\mu \in (0, 1)$, $\theta \in (0, 1)$, and $\tau \in (0, 2)$, respectively.

Step 1. Set $n = 0$.

Step 2. Assume that the sequence $\{x_n\}$ has been constructed and then calculate $P_C[x_n - f(x_n)]$. If $P_C[x_n - f(x_n)] = x_n$, then stop, else go to the steps below.

Step 3. If $P_C[x_n - f(x_n)] \neq x_n$, then compute the sequence $\{y_n\}$ as follows:

$$y_n = P_C[x_n - \mu\gamma^{m(x_n)} f(x_n)],$$

where $m(x_n) = \min\{0, 1, 2, 3, \ldots\}$ satisfies the inequality

$$\mu\gamma^{m(x_n)} \|f(x_n) - f(y_n)\| \leqslant \theta \|x_n - y_n\|.$$

Calculate the sequences $\{u_n\}$ and $\{x_{n+1}\}$ by the following rule:

$$\begin{cases} u_n = P_C[x_n - \tau(1-\theta)\|x_n - y_n\|^2 \frac{x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2}], \\ x_{n+1} = \alpha_n u + (1-\alpha_n) u_n, \text{ for all, } n \geqslant 1. \end{cases}$$

Step 4. Set $n := n + 1$ and return to Step 2.

Table 1: Numerical results comparing our Algorithm 3.8 with Algorithm 2 of [24].

		Our Algorithm 3.8		Algorithm 2 of [24]	
		CPU time (sec)	No. of iteration	CPU time (sec)	No. of iteration
Case 1	$m = 50$	0.8647	179	10.1159	1133
Case 2	$m = 100$	0.0571	153	10.1676	1453
Case 3	$m = 150$	0.0431	240	10.2162	2183
Case 4	$m = 200$	0.1015	540	10.4026	2697

Note: Here m stands for the dimension of the space.

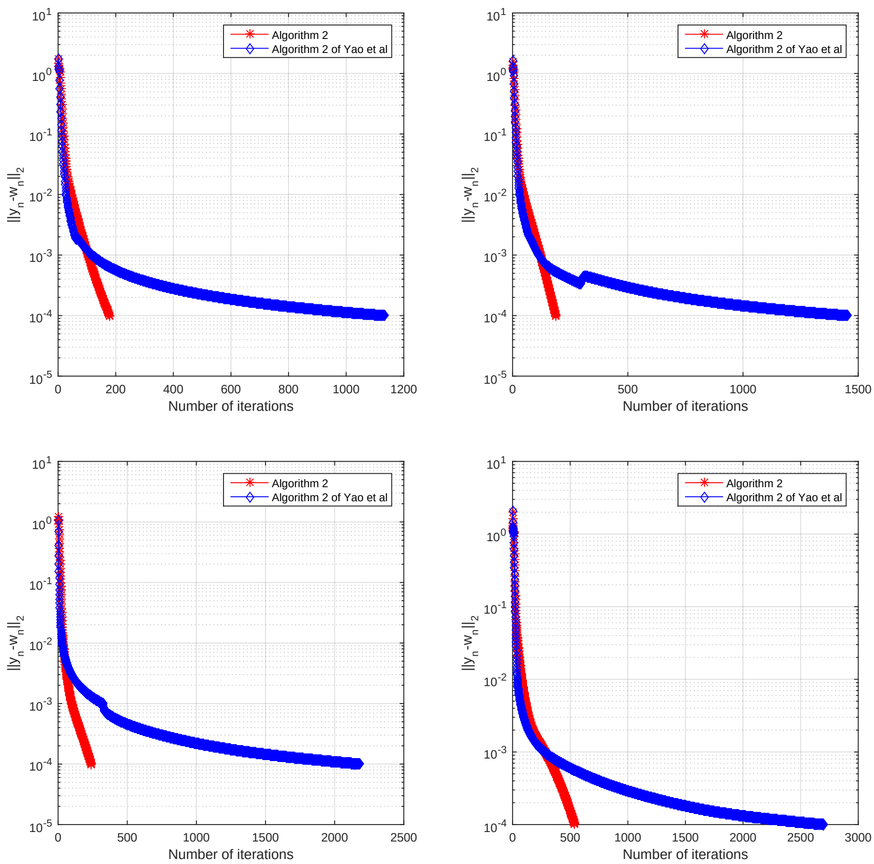

Figure 1: $\|y_n - w_n\|$ vs number of iterations (n): top left: **Case 1**; top right: **Case 2**; bottom left: **Case 3**; bottom right: **Case 4**.

Example 4.3. Define $fx = Mx + q$, $M = B^T B + S + D$, where $S, D \in \mathbb{R}^{m \times m}$ are randomly generated matrices such that S is skew-symmetric (hence it does not arise from an optimization problem), D is a positive definite diagonal matrix (hence the variational inequalities has a unique solution) and $q = 0$. Suppose the feasible set $C := \{x \in \mathbb{R}^m | Bx \leqslant b\}$, for some random matrix $B \in \mathbb{R}^{m \times k}$, and random vector $b \in \mathbb{R}^k$ with non-negative entries. The unique solution of $VI(f, C)$ here is $x^* = \{0\}$, and the Lipschitz constant $\kappa = \|M\|$ (see [17]).

Next, consider the operator $T \colon K \to K$ defined by

$$Tx = \begin{cases} x + x', & \text{if } x \in K_1, \\ \frac{x}{\|x\|} - x + x', & \text{if } x \in K_2, \end{cases}$$

where $K := \{x \in \mathbb{R}^2 | \|x\| \leqslant 1\}$, $K_1 := \{x \in \mathbb{R}^2 | \|x\| \leqslant \frac{1}{2}\}$, $K_2 := \{x \in \mathbb{R}^2 | \frac{1}{2} \leqslant \|x\| \leqslant 1\}$. Then T is a Lipschitz pseudocontractive map (see [8]), with the unique fixed point zero.

Table 2: Numerical results comparing our Algorithm 3.1 with Algorithm 1 of [24].

		Our Algorithm 3.8		Algorithm 2 of [24]	
		CPU time (sec)	No. of iteration	CPU time (sec)	No. of iteration
Case 1	$m = 50$	0.0310	224	2.8768	8910
Case 2	$m = 100$	0.0475	311	2.8646	8579
Case 3	$m = 150$	0.1176	1087	9.2769	26160
Case 4	$m = 200$	0.0514	351	12.9830	38133

Note: m stands for the dimension of the space.

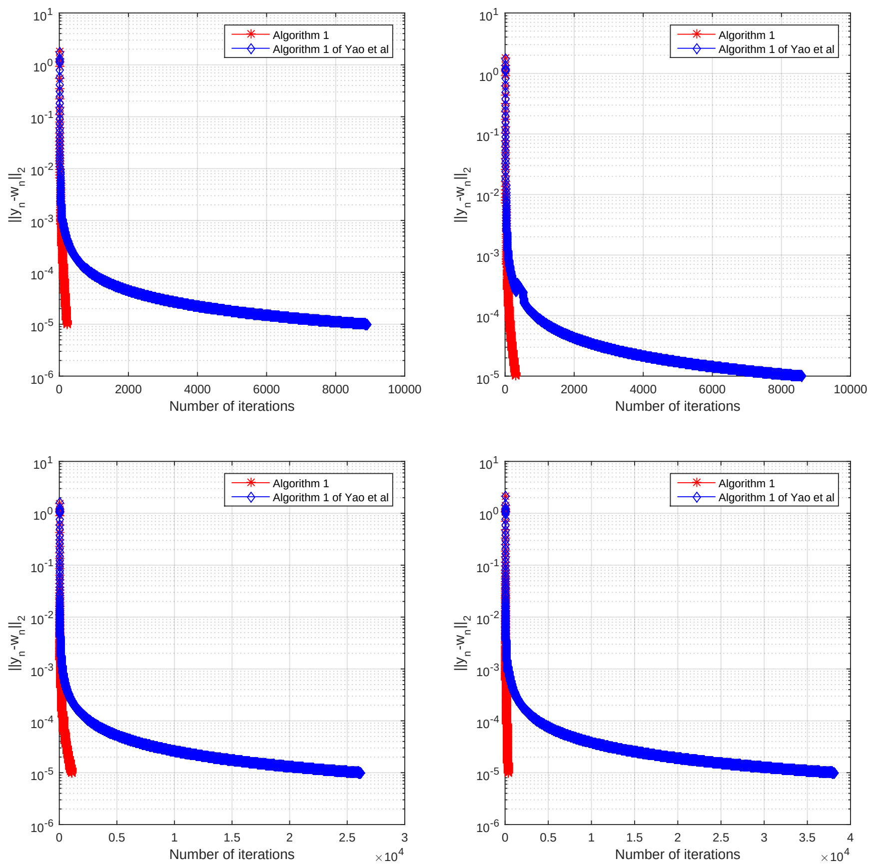

Figure 2: $\|y_n - w_n\|$ vs number of iterations (n): top left: **Case 1**; top right: **Case 2**; bottom left: **Case 3**; bottom right: **Case 4**.

Remark 4.4.

(1). Using Example 4.1 to compare the efficiency of Algorithm 3.8 of our work and Algorithm 2 in the result of Yao et al. [24], it can be noticed from Table 1 and Figure 1, that our algorithm performs better than that of Yao et al. [24].

(2). In Example 4.3, we have compared our Algorithm 3.1 with Algorithm 1 of Yao et al. [24]. It can be observed from Table 2 and Figure 2, that our algorithm performs better than the algorithm of Yao et al. [24].

(3). From the foregoing, we infer that the addition of an inertial term to Algorithms 1 and 2 of Yao et al. [24] improves the efficiency of the algorithms.

(4). The algorithms considered by Yao et al. [24], and the algorithms considered in our work, respectively, are the so-called Armijo-like line search method, which is known to have some deficiencies. To be more precise, the computer takes more time to achieve convergence, and the step size depends on the Lipschitz constant, which in many cases is difficult to approximate. Recently, self-adaptive line search methods have been developed to solve VI problems in literature (see, e.g., [17] and the references therein), and it is known to be superior to the former. However, this observation has no negative effect on our results because one of the principal purposes of our work is to check if the addition of an inertia-term will improve the performance of the algorithms of Yao et al. [24]. We recall the fact that in some cases, the addition of an inertial term to some iterative algorithms used for solving VI problems does not lead to any improvement on such schemes (see, e.g., [9, 11]).

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