



The Dirichlet-type Laplace transforms



Diego Caratelli^a, Sandra Pinelas^b, Paolo Emilio Ricci^{c,*}

^a*Electromagnetics Group, Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.*

^b*Academia Militar-Departamento de Ciências Exactas e Engenharia, Av. Conde Castro Guimarães, 2720-113, Amadora, Portugal.*

^c*Sezione di Matematica, International Telematic University UniNettuno, 39 Corso Vittorio Emanuele II, I-00186 Rome, Italy.*

Abstract

We show that it is possible to define extensions of the Laplace transform that use a general Dirichlet series as a kernel. These transforms, denoted by DLTs, further generalize those, considered in previous papers, in which the kernels were related to Laguerre-type exponentials or Bell polynomials. Computational techniques, exploiting expansions in Laguerre polynomials, and using Tricomi's method, have been considered. Since it turns out that the transforms considered are obtained as linear combinations of ordinary Laplace transforms, it is also possible to define an approximation of the relevant inverse transforms. Numerical experiments, performed with the algebra program Mathematica, show that the introduced technique is fast and efficient.

Keywords: General Dirichlet series, Laguerre-type exponentials, Bell polynomials, Laplace transform.

2020 MSC: 30B50, 44A10, 05A15, 05A40, 11B83.

©2022 All rights reserved.

1. Introduction

The classical Laplace Transform (shortly LT) [8, 19] writes:

$$\mathcal{L}(f) := \int_0^{\infty} \exp^{-st} f(t) dt = F(s).$$

Similarly to the Fourier transform, it is one of the most useful tools in applied mathematics, with numerous applications in ordinary and partial differential equations problems, as well as in signal analysis and image processing [1].

The Laplace operator converts a function of a real variable t (often representing the time) into a function of a complex variable s (the complex frequency) and transforms differential into algebraic equations and convolution into multiplication. It can be applied to functions belonging to $L^1_{loc}[0, +\infty)$ and converges in each half plane $\operatorname{Re}(s) > a$, with a denoting the *convergence abscissa*, which depends on the growth behavior of $f(t)$.

*Corresponding author

Email addresses: d.caratelli@tue.nl (Diego Caratelli), sandra.pinelas@gmail.com (Sandra Pinelas), paoloemilioricci@gmail.com (Paolo Emilio Ricci)

doi: [10.22436/jnsa.015.03.05](https://doi.org/10.22436/jnsa.015.03.05)

Received: 2022-01-08 Revised: 2022-01-26 Accepted: 2022-03-10

Tables of LTs, together with the relative anti-transforms, are widely used in the solution of differential problems. For the numerical approximation of the LT of functions not included in the aforementioned tables, a method introduced by F.G. Tricomi [17, 18] can be used. Such method relies on a link between the LT and series expansions in Laguerre polynomials. Tricomi's technique has been generalized by taking advantage of more general expansions in orthogonal polynomials [13]; however the original formulation turned out to be computationally more efficient.

The extensions of the LT considered in [12] have been examined in [4, 5] from a numerical point of view, using the following procedure:

1. The kernel of the transform was approximated by a partial sum of a general Dirichlet series, obtained through the matrix pencil method.
2. The function to be transformed was expanded in a series of Laguerre polynomials (a sufficiently high order of the partial sum of this expansion was chosen).
3. The Tricomi's method was applied in order to derive the LT of each term.
4. The approximate generalized LT was obtained by a simple linear combination of terms.

Before exposing our new operator, which will be referred to as Dirichlet-type Laplace transform (shortly DLT), we first recall the kernels considered in [4, 5, 12].

A first extension (called Laguerre-Laplace transform) is obtained by substituting the original kernel $\exp^{-1}(st)$ with the function $e_k^{-1}(st)$, where $e_k(x)$ is the k -th Laguerre-Type exponential introduced in [7] and extensively studied in [9–11].

In a further generalization, the kernel of the LT was replaced by an expansion whose coefficients are related to Bell polynomials [2, 6, 14], exploiting a transformation introduced in [12] and based on the Blissard formula [3], which is a typical tool of umbral calculus [15, 16].

In this article, upon noting that the reciprocal exponential term $e^{-1}(x)$ is just the first term of a general Dirichlet series with unitary coefficients, a generalization of the LT is introduced by assuming, as a kernel of the transformation, a general Dirichlet series with complex coefficients. In this way, the transformed function is found to be the combination of LTs, which can be evaluated by means of the Tricomi's method. Such property allows defining the inverse of a DLT as a linear combination of ordinary LTs. This holds true even in the case of Laguerre-type LTs [5] and Bell-type LTs [4], this addressing an open point in the previous articles published on this topic.

In Section 2, we recall the definition of a general Dirichlet series, as well as the method due to F.G. Tricomi [17, 18] which allows evaluating the pair of LT and anti-LT as soon as the involved function is expanded in a series of Laguerre polynomials.

In Section 3, we introduce the extension of the LT using a general Dirichlet series as the kernel of the transformation. An approximation of the DLT operator is obtained by considering a partial sum of the corresponding Dirichlet series, i.e., a linear combination of exponential terms each of which can be Laplace transformed by a simple change of variable.

In Section 4, several test cases are investigated using the computer algebra program Mathematica[®] in order to prove the effectiveness of the computational procedure for DLTs described in this article.

2. General Dirichlet series and Tricomi's method

A general Dirichlet series is an infinite series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad (2.1)$$

where $\{\lambda_n\}$ is a strictly increasing sequence of nonnegative real numbers tending to infinity, the coefficients a_n and the variable s are complex numbers.

It is worth to note that in the particular case $\lambda_n = \log n$ we find the classical Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

while, assuming $\lambda_n = n$, the power series in the variable e^{-s} ,

$$\sum_{n=1}^{\infty} a_n (e^{-s})^n,$$

is recovered.

The abscissa of convergence σ_c of the series (2.1) is defined as

$$\sigma_c := \inf\{\sigma \in \mathbf{R} : \text{the series (2.1) converges } \forall s \text{ for which } \Re(s) > \sigma\}.$$

In order to avoid convergence problems, we will suppose in what follows to consider only general Dirichlet series converging in the whole complex plane, that is for which $\sigma_c = -\infty$.

Actually, dealing with approximations, this condition is always satisfied even for the partial sums of such a general Dirichlet series.

In what follows, we use a method due to Tricomi that gives a pair of Laplace transforms and anti-transforms of a function expanded in series of Laguerre polynomials. This method applies naturally when the kernel of the transformation is a general Dirichlet series, which is nothing but a linear combination of exponential functions. In the particular case of the Laplace transform, the general Dirichlet series trivially reduces to a single term. The Tricomi method is described below by Proposition 1.

We denote the LT of $f(t)$ by the symbol $\mathcal{L}(f(t))$, and by $L_m(t)$ Laguerre polynomial of degree m .

Tricomi, starting from the equation

$$\mathcal{L}(L_m(t)) := \int_0^{\infty} \exp^{-1(st)} L_m(t) dt = F(s) = \frac{1}{s} \left(\frac{s-1}{s} \right)^m,$$

proved in [17, 18] the following result.

Proposition 1. *If the analytic function $F(s)$ is regular at infinity and we can find a real number h such that it can be represented with a series of the form*

$$F(s) = \frac{1}{s+h} \sum_{m=0}^{\infty} \gamma_m \left(\frac{s+h-1}{s+h} \right)^m, \quad (2.2)$$

then it is the Laplace transform of the sum of the series of Laguerre polynomials

$$f(t) = e^{-ht} \sum_{m=0}^{\infty} \gamma_m L_m(t),$$

which is absolutely and uniformly convergent for $t > 0$.

Then Tricomi, under the above conditions, found in particular the Laplace (transform and anti-transform) pairs:

$$f(t) = \sum_{m=0}^{\infty} \gamma_m L_m(t) \quad \leftrightarrow \quad F(s) = \frac{1}{s} \sum_{m=0}^{\infty} \gamma_m \left(\frac{s-1}{s} \right)^m,$$

which have been widely used in applications.

3. Dirichlet-type Laplace transforms

We introduce now an extension of the LT whose kernel is related to a general Dirichlet series, satisfying suitable conditions. We call this a Dirichlet-type Laplace Transform (shortly DLT). Such a transform, depends on the umbral symbol $\mathfrak{a} := \{a_1, a_2, a_3, \dots\}$ for the coefficients, and on the constants $\lambda := \{\lambda_1, \lambda_2, \lambda_3, \dots\}$ for the exponents:

$$\mathcal{DL}(\mathfrak{a}; \lambda)(f) := \int_0^\infty \sum_{n=1}^\infty a_n e^{-\lambda_n s t} f(t) dt = F(s). \quad (3.1)$$

We suppose that the functions f is absolutely integrable and that the sequence $\{a_1, a_2, a_3, \dots\}$ is bounded, so that there exists a number K such that $\forall n, |a_n| < K$.

We prove the following result.

Theorem 1. *A numerical approximation of the transform in equation (3.1) is obtained considering a partial sum of the type*

$$\mathcal{DL}(\mathfrak{a}; \lambda)_N(f) := \int_0^\infty \sum_{n=1}^N a_n e^{-\lambda_n s t} f(t) dt = F_N(s), \quad (3.2)$$

because for any fixed $\varepsilon > 0$ it is possible to find an integer N such that

$$|F(s) - F_N(s)| < \varepsilon, \quad \forall s : \Re(s) > \sigma. \quad (3.3)$$

Proof. Putting

$$\mathcal{DL}(\mathfrak{a}; \lambda)(f) = \mathcal{DL}(\mathfrak{a}; \lambda)_N(f) + \int_0^\infty \sum_{n=N+1}^\infty a_n e^{-\lambda_n s t} f(t) dt,$$

since the sequence of exponents $\{\lambda_n\}$ is positive, increasing and divergent, it results

$$\forall n \geq N+1, e^{-\lambda_n s t} \leq e^{-\lambda_{N+1} s t},$$

and consequently we can choose N is such that the absolute value of the remainder term is bounded by the inequality:

$$K \left| \int_0^\infty e^{-\lambda_{N+1} s t} |f(t)| dt \right| = \frac{K}{\lambda_{N+1}} \left| \mathcal{L} \left(\left| f \left(\frac{t}{\lambda_{N+1}} \right) \right| \right) \right| < \varepsilon, \quad (3.4)$$

where ε is less than the machine precision.

In fact, we can split the integration interval in the form $[0, A] \cup [A, +\infty)$ where A is such that

$$K \left| \int_A^\infty e^{-\lambda_{N+1} s t} |f(t)| dt \right| < \varepsilon/2.$$

This is possible for every choice of N , since the LT of an absolutely integrable function is $O(s^{-1})$, for $s \rightarrow +\infty$. Furthermore, it is possible to choose N in such a way that

$$K \left| \int_0^A e^{-\lambda_{N+1} s t} |f(t)| dt \right| < \varepsilon/2,$$

since the integral is bounded and the sequence $\lambda_N \rightarrow +\infty$. Therefore, the inequality in (3.4) is proved, and consequently the result in equation (3.3) follows. \square

For the numerical computation of the approximate DLT, we use the Tricomi’s method, expanding the function $f(t)$ in a series of Laguerre polynomials, of course considering a partial sum of this series in such a way that the remainder term is again smaller than ε .

Putting:

$$f(t) = \sum_{m=0}^M \gamma_m L_m(t) + R_M(t), \quad R_M(t) := \sum_{m=M+1}^{\infty} \gamma_m L_m(t), \quad \text{and} \quad |R_M(t)| < \varepsilon, \quad \forall t \in [0, +\infty),$$

we find the approximate DLT:

$$\mathcal{DL}(a; \lambda)(f) \simeq \frac{1}{s} \sum_{m=0}^M \gamma_m \sum_{n=1}^N \frac{a_n}{\lambda_n} \left(1 - \frac{1}{\lambda_n s}\right)^m.$$

As a consequence of Proposition 1, for every fixed N satisfying equation (3.3), Tricomi’s method allows to define an approximation of the inverse of the DLT, by noting following.

Theorem 2. *Considering the DLT defined in (3.1), whose truncated kernel is given by $\sum_{n=1}^N a_n e^{-\lambda_n s t}$, with coefficients a_n and exponents γ_n subject to the above recalled restrictions, the following approximate DLT and inverse DLT pairs hold:*

$$f(t) = \sum_{m=0}^M \gamma_m L_m(t) \quad \leftrightarrow \quad F(s) = \frac{1}{s} \sum_{m=0}^M \gamma_m \sum_{n=1}^N \frac{a_n}{\lambda_n} \left(1 - \frac{1}{\lambda_n s}\right)^m. \tag{3.5}$$

Proof. As the kernel of the approximate operator in equation (3.2) is a linear combination of exponentials which can be reduced by a simple change of variable to the Laplace’s kernel, it is sufficient to use equation (2.2) in order to prove equation (3.5). □

Of course, the inverse of the considered DLT follows by letting $N \rightarrow +\infty$.

4. Numerical Results

In what follows we will show some numerical examples concerning the computation of DLTs.

Example 4.1 (Dirichlet-Laplace transform of $e^t \Gamma(t)$). Let us consider the umbral symbols $\lambda = \{\lambda_n = \sqrt{n}\}$, $a = \{a_n = \frac{\sin n}{n}\}$, and the function:

$$f(t) = e^t \Gamma(t),$$

whose Laguerre polynomial series approximant of order K is defined as:

$$\tilde{f}(t) = \sum_{m=0}^M \gamma_m L_m(t), \tag{4.1}$$

with:

$$\gamma_m = \int_0^{\infty} e^{-t} f(t) L_m(t) dt = \frac{1}{m+1}.$$

Hence, using Tricomi’s method, the Dirichlet-Laplace transform of $f(t)$ with coefficients a and exponents λ can be evaluated as:

$$\tilde{F}(s) \simeq \frac{1}{s} \sum_{m=0}^M \gamma_m \Psi_m(a; \lambda; s), \tag{4.2}$$

where,

$$\Psi_m(a; \lambda; s) = \sum_{n=1}^{+\infty} \frac{a_n}{\lambda_n} \left(1 - \frac{1}{\lambda_n s}\right)^m. \tag{4.3}$$

The approximated expression (4.2) can be compared against the following rigorous expansion:

$$F(s) = \sum_{n=1}^{\infty} a_n \mathcal{L}(f)(\lambda_n s), \tag{4.4}$$

with

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st} f(t) dt = \frac{\log(s)}{s-1}. \tag{4.5}$$

Upon selecting the expansion order $M = 200$, one can readily verify that the approximant $\tilde{f}(t)$ in (4.1) is characterized by the distribution shown in Figure 1. As it can be noticed from the cut sections for $\omega = \text{Im}\{s\} = 1$ and $\sigma = \text{Re}\{s\} = 5$ reported in Figures 2 and 3, respectively, the agreement between the Dirichlet-Laplace transform $F(s)$ defined by (4.4) and the relevant Tricomi’s approximation $\tilde{F}(s)$ as calculated by (4.2) is very good. To this end, please note that the number of terms retained in the expansions (4.3) and (4.5) has been limited to $N = 100$, that is sufficient to yield a pretty accurate representation of the kernel of the transform:

$$\Lambda(a; \lambda; x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x},$$

as it readily follows from the visual inspection of Figure 4.

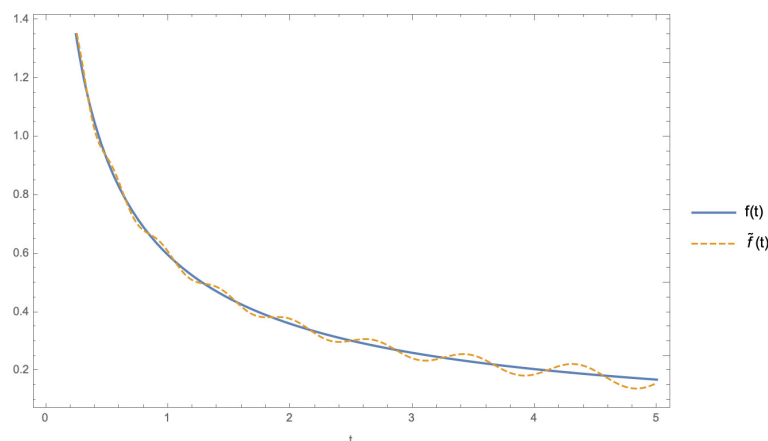
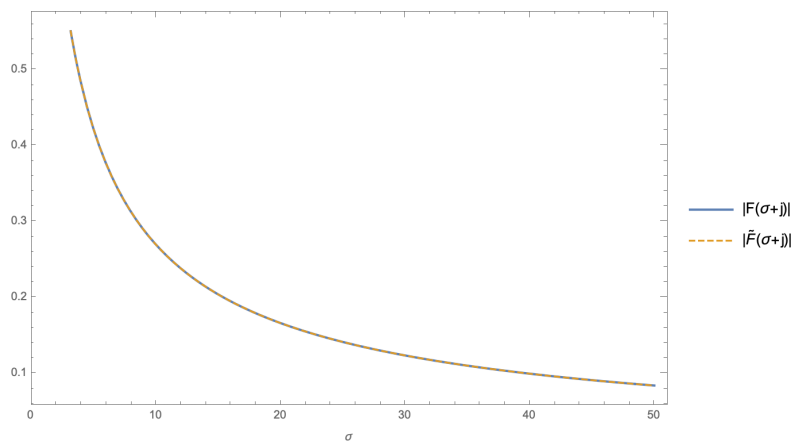
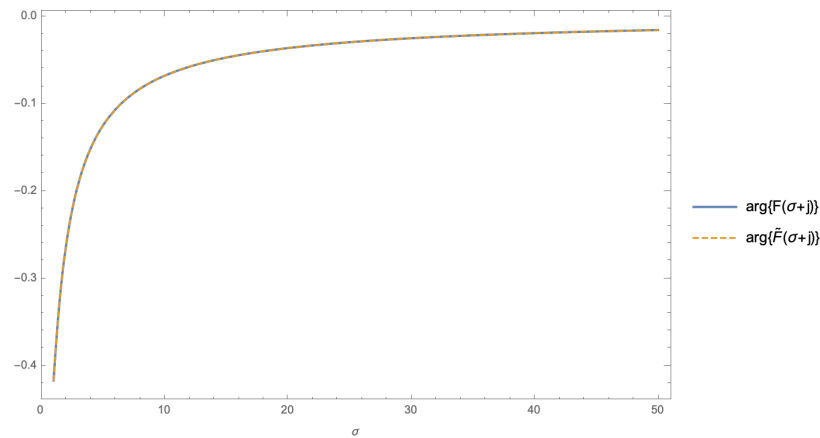


Figure 1: Distribution of the function $f(t) = e^t \Gamma(t)$ as compared to the relevant Laguerre polynomial series approximant $\tilde{f}(t)$ of order $M = 200$.

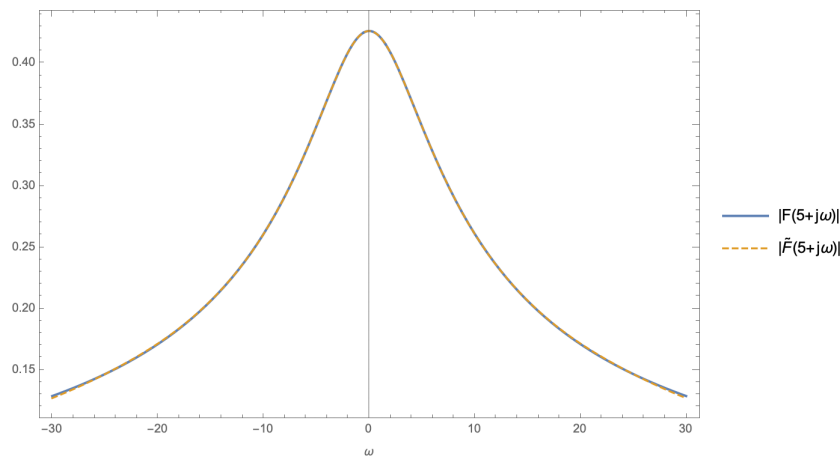


(a)

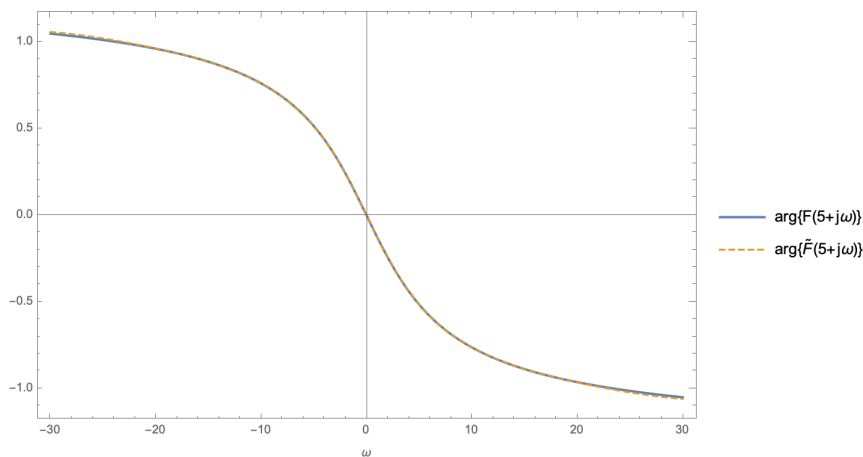


(b)

Figure 2: Magnitude (a) and argument (b) of the Dirichlet-Laplace transform with coefficients $a = \{a_n = \frac{\sin n}{n}\}$ and exponents $\lambda = \{\lambda_n = \sqrt{n}\}$ relevant to $f(t) = e^t \Gamma(t)$ as a function of the complex variable $s = \sigma + i\omega$ for $\omega = 1$ when computed using the rigorous integral expression $F(s)$ and the corresponding Tricomi's series approximant $\tilde{F}(s)$ of order $M = 200$.



(a)



(b)

Figure 3: Magnitude (a) and argument (b) of the Dirichlet-Laplace transform with coefficients $a = \{a_n = \frac{\sin n}{n}\}$ and exponents $\lambda = \{\lambda_n = \sqrt{n}\}$ relevant to $f(t) = e^t \Gamma(t)$ as a function of the complex variable $s = \sigma + i\omega$ for $\sigma = 5$ when computed using the rigorous integral expression $F(s)$ and the corresponding Tricomi's series approximant $\tilde{F}(s)$ of order $M = 200$.

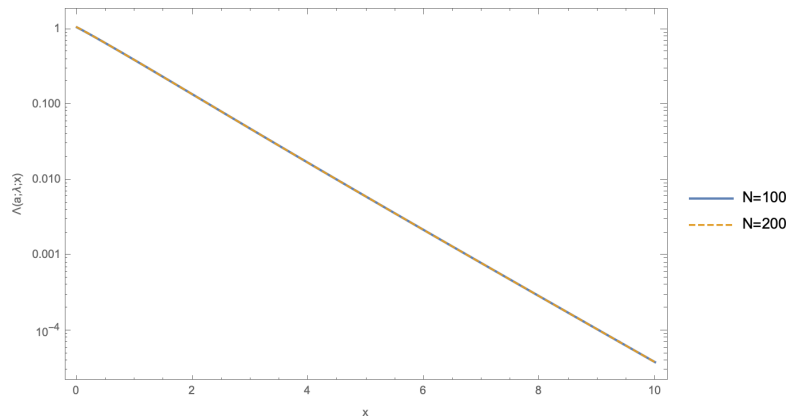


Figure 4: Distribution of the kernel $\Lambda_k(a; \lambda; x)$ relevant to the Dirichlet-Laplace transform with coefficients $a = \{a_n = \frac{\sin n}{n}\}$ and exponents $\lambda = \{\lambda_n = \sqrt{n}\}$ for different truncation orders N of the corresponding series expansion representation.

Example 4.2 (Dirichlet-Laplace Transform of $J_0(2\sqrt{t})$). Let us consider the umbral symbols $\lambda = \{\lambda_n = \log(n + 1)\}$, $a = \{a_n = e^{-n} \arctan n\}$, and the function:

$$f(t) = J_0(2\sqrt{t}),$$

whose Laguerre polynomial series approximant of order K is defined as:

$$\tilde{f}(t) = \sum_{m=0}^M \gamma_m L_m(t), \tag{4.6}$$

with

$$\gamma_m = \int_0^\infty e^{-t} f(t) L_m(t) dt = \frac{1}{em!}.$$

Hence, using Tricomi’s method, the Dirichlet-Laplace transform of $f(t)$ with coefficients a and exponents λ can be evaluated as:

$$\tilde{F}(s) \simeq \frac{1}{s} \sum_{m=0}^M \gamma_m \Psi_m(a; \lambda; s), \tag{4.7}$$

where,

$$\Psi_m(a; \lambda; s) = \sum_{n=1}^{+\infty} \frac{a_n}{\lambda_n} \left(1 - \frac{1}{\lambda_n s}\right)^m. \tag{4.8}$$

The approximated expression (4.7) can be compared against the following rigorous expansion:

$$F(s) = \sum_{n=1}^\infty a_n \mathcal{L}(f)(\lambda_n s), \tag{4.9}$$

with

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt = \frac{e^{-1/s}}{s}. \tag{4.10}$$

Upon selecting the expansion order $M = 100$, one can readily verify that the approximant $\tilde{f}(t)$ in (4.6) is characterized by the distribution shown in Figure 5. As it can be noticed from the cut sections for $\omega = \text{Im}\{s\} = 1$ and $\sigma = \text{Re}\{s\} = 1$ reported in Figures 6 and 7, respectively, the agreement between the Dirichlet-Laplace transform $F(s)$ defined by (4.9) and the relevant Tricomi’s approximation $\tilde{F}(s)$ as calculated by (4.7) is excellent. To this end, please note that the number of terms retained in the expansions (4.8) and (4.10) has been limited to $N = 100$, that is sufficient to yield a pretty accurate representation of the kernel of the transform

$$\Lambda(a; \lambda; x) = \sum_{n=1}^\infty a_n e^{-\lambda_n x},$$

as it readily follows from the visual inspection of Figure 8.

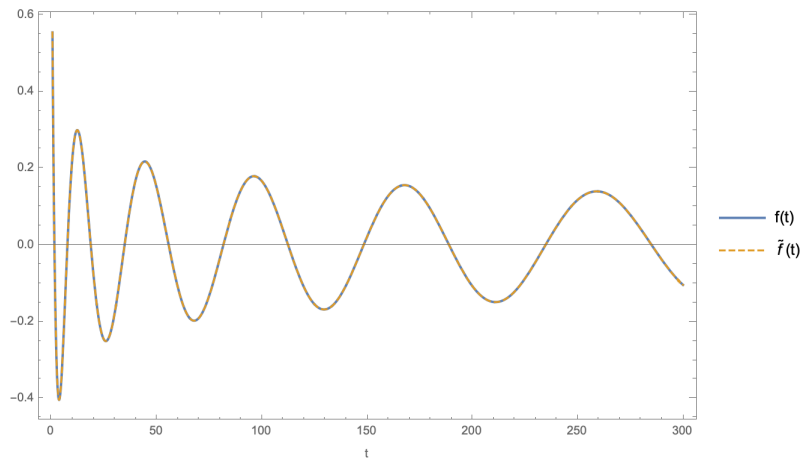
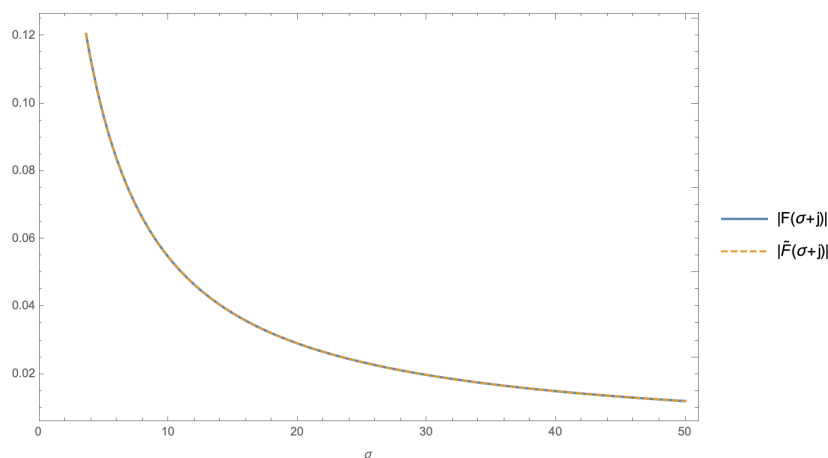


Figure 5: Distribution of the function $f(t) = J_0(2\sqrt{t})$ as compared to the relevant Laguerre polynomial series approximant $\tilde{f}(t)$ of order $M = 100$.



(a)

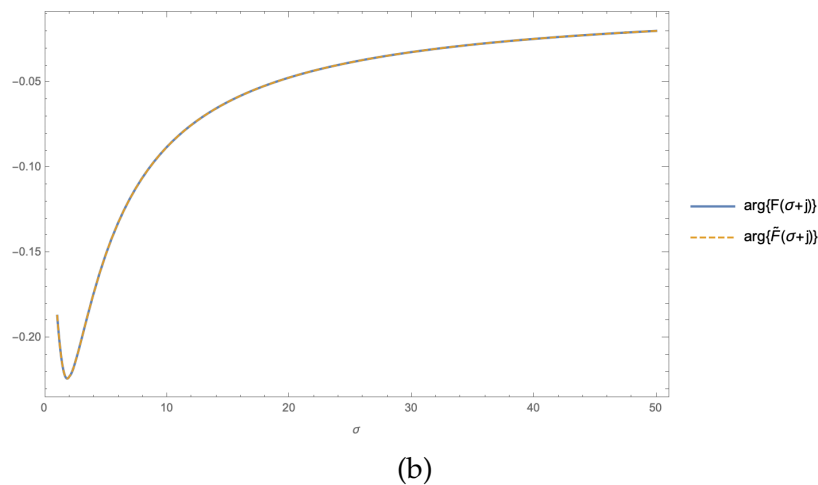


Figure 6: Magnitude (a) and argument (b) of the Dirichlet-Laplace transform with coefficients $a = \{a_n = e^{-n} \arctan n\}$ and exponents $\lambda = \{\lambda_n = \log(n + 1)\}$ relevant to $f(t) = J_0(2\sqrt{t})$ as a function of the complex variable $s = \sigma + i\omega$ for $\omega = 1$ when computed using the rigorous integral expression $F(s)$ and the corresponding Tricomi's series approximant $\tilde{F}(s)$ of order $M = 100$.

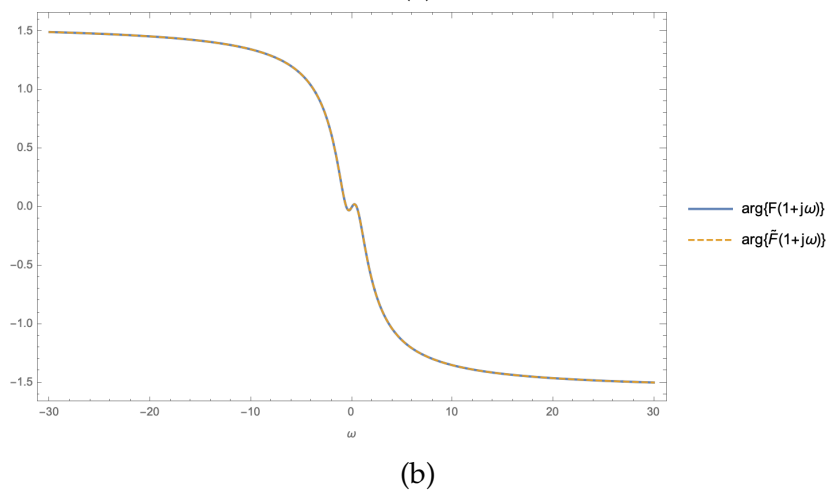
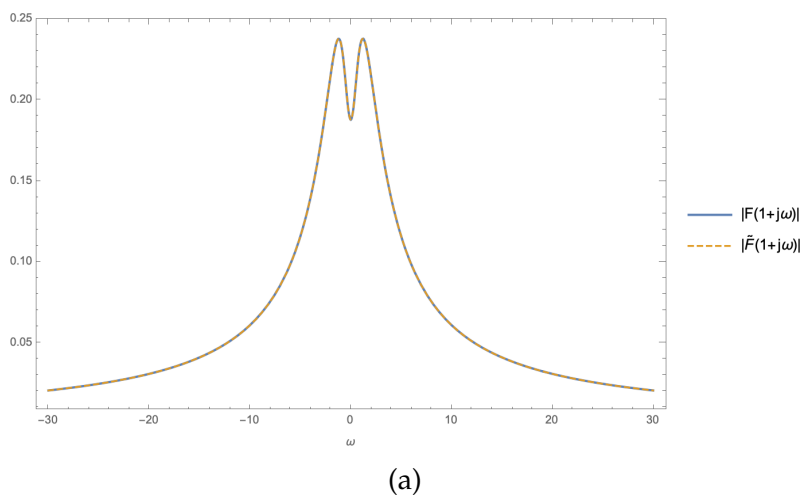


Figure 7: Magnitude (a) and argument (b) of the Dirichlet-Laplace transform with coefficients $a = \{a_n = e^{-n} \arctan n\}$ and exponents $\lambda = \{\lambda_n = \log(n + 1)\}$ relevant to $f(t) = J_0(2\sqrt{t})$ as a function of the complex variable $s = \sigma + i\omega$ for $\sigma = 1$ when computed using the rigorous integral expression $F(s)$ and the corresponding Tricomi's series approximant $\tilde{F}(s)$ of order $M = 100$.

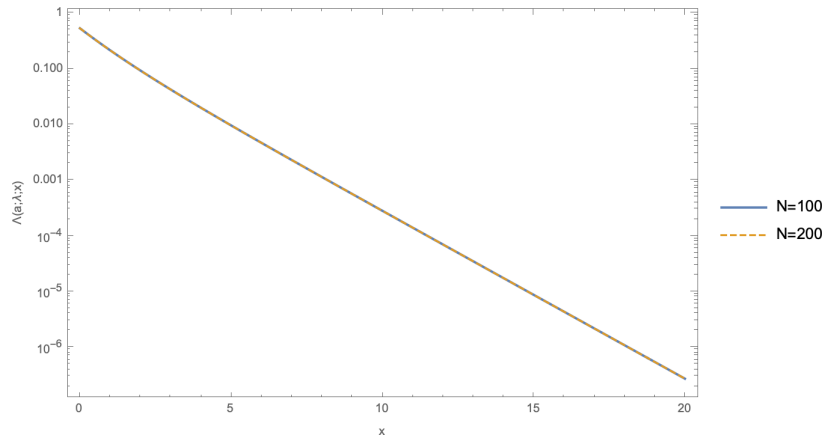


Figure 8: Distribution of the kernel $\Lambda_k(a; \lambda; x)$ relevant to the Dirichlet-Laplace transform with coefficients $a = \{a_n = e^{-n} \arctan n\}$ and exponents $\lambda = \{\lambda_n = \log(n + 1)\}$ for different truncation orders N of the corresponding series expansion representation.

Example 4.3 (Dirichlet-Laplace Transform of $e^{i\pi t}$). Let us consider the umbral symbols $\lambda = \{\lambda_n = n^2\}$, $a = \{a_n = \frac{1}{n^2} \exp \frac{2n\pi i}{3}\}$, and the function:

$$f(t) = e^{i\pi t},$$

whose Laguerre polynomial series approximant of order K is defined as:

$$\tilde{f}(t) = \sum_{m=0}^M \gamma_m L_m(t), \tag{4.11}$$

with

$$\gamma_m = \int_0^\infty e^{-t} f(t) L_m(t) dt = \frac{i}{\pi} \left(1 + \frac{i}{\pi}\right)^{-m-1}.$$

Hence, using Tricomi’s method, the Dirichlet-Laplace transform of $f(t)$ with coefficients a and exponents λ can be evaluated as

$$\tilde{F}(s) \simeq \frac{1}{s} \sum_{m=0}^M \gamma_m \Psi_m(a; \lambda; s), \tag{4.12}$$

where,

$$\Psi_m(a; \lambda; s) = \sum_{n=1}^{+\infty} \frac{a_n}{\lambda_n} \left(1 - \frac{1}{\lambda_n s}\right)^m. \tag{4.13}$$

The approximated expression (4.12) can be compared against the following rigorous expansion

$$F(s) = \sum_{n=1}^\infty a_n \mathcal{L}(f)(\lambda_n s), \tag{4.14}$$

with

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt = \frac{1}{s - i\pi}. \tag{4.15}$$

Upon selecting the expansion order $M = 200$, one can readily verify that the approximant $\tilde{f}(t)$ in (4.11) is characterized by the distribution shown in Figure 9. As it can be noticed from the cut sections for $\omega = \text{Im}\{s\} = 1$ and $\sigma = \text{Re}\{s\} = 5$ reported in Figures 10 and 11, respectively, the agreement between the Dirichlet-Laplace transform $F(s)$ defined by (4.14) and the relevant Tricomi's approximation $\tilde{F}(s)$ as calculated by (4.12) is excellent. To this end, please note that the number of terms retained in the expansions (4.13) and (4.15) has been limited to $N = 100$, that is sufficient to yield a pretty accurate representation of the kernel of the transform:

$$\Lambda(a; \lambda; x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x},$$

as it readily follows from the visual inspection of Figure 12.

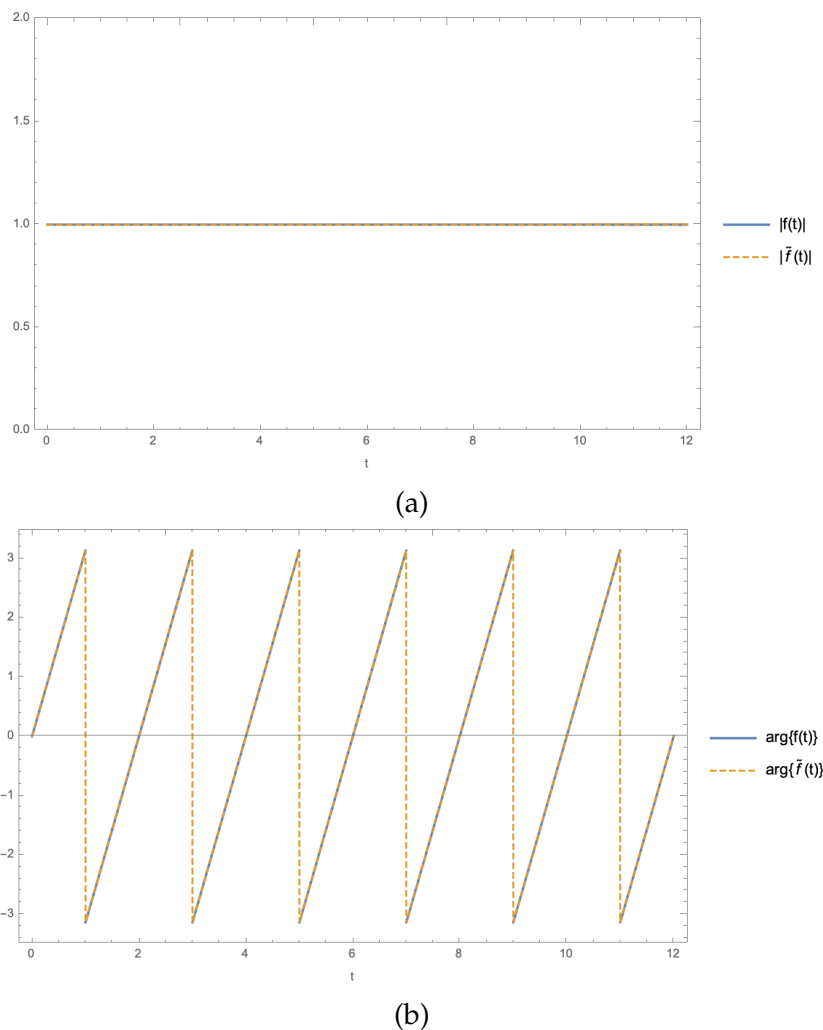
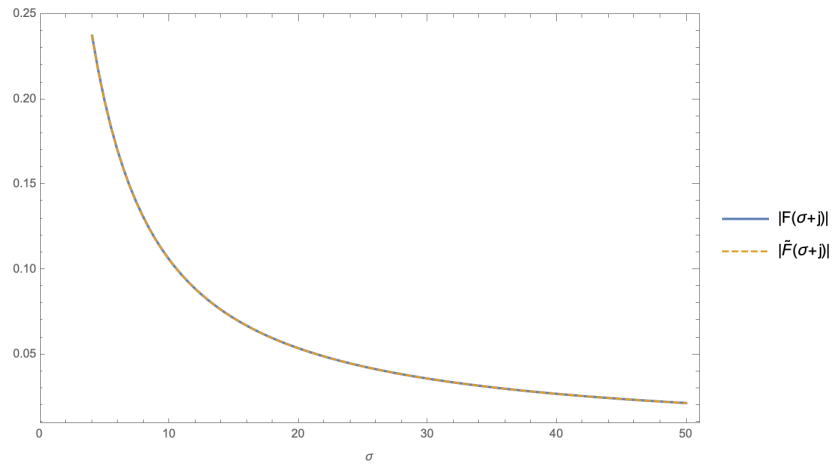
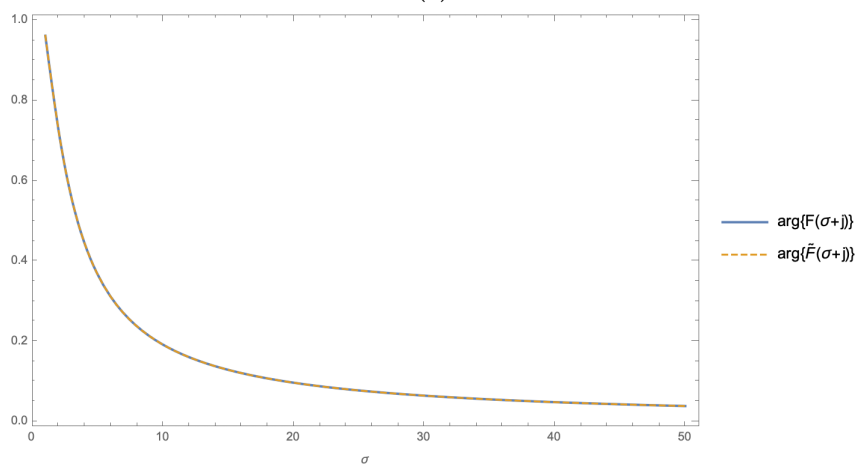


Figure 9: Magnitude (a) and argument (b) of the function $f(t) = e^{i\pi t}$ as compared to the relevant Laguerre polynomial series approximant $\tilde{f}(t)$ of order $M = 200$.

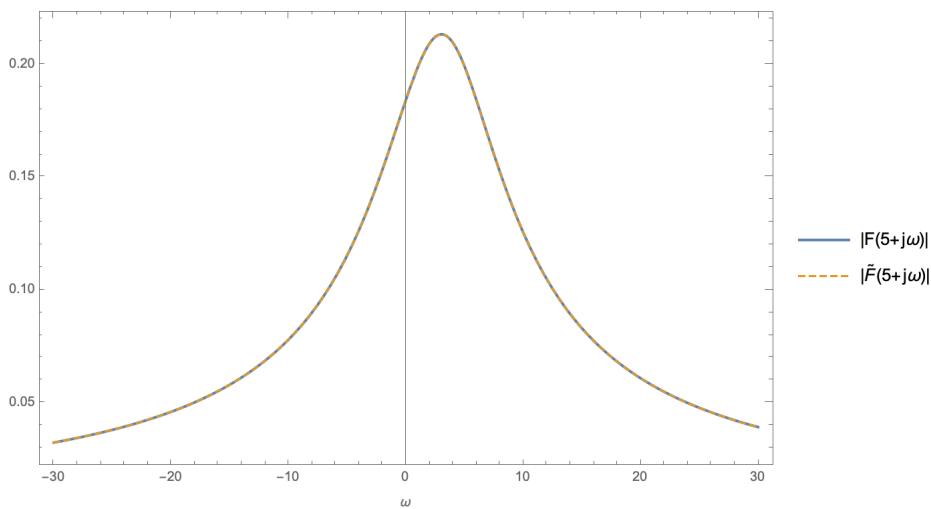


(a)



(b)

Figure 10: Magnitude (a) and argument (b) of the Dirichlet-Laplace transform with coefficients $a = \{a_n = \frac{1}{n^2} \exp \frac{2n\pi i}{3}\}$ and exponents $\lambda = \{\lambda_n = n^2\}$ relevant to $f(t) = e^{i\pi t}$ as a function of the complex variable $s = \sigma + i\omega$ for $\omega = 1$ when computed using the rigorous integral expression $F(s)$ and the corresponding Tricomi's series approximant $\tilde{F}(s)$ of order $M = 200$.



(a)

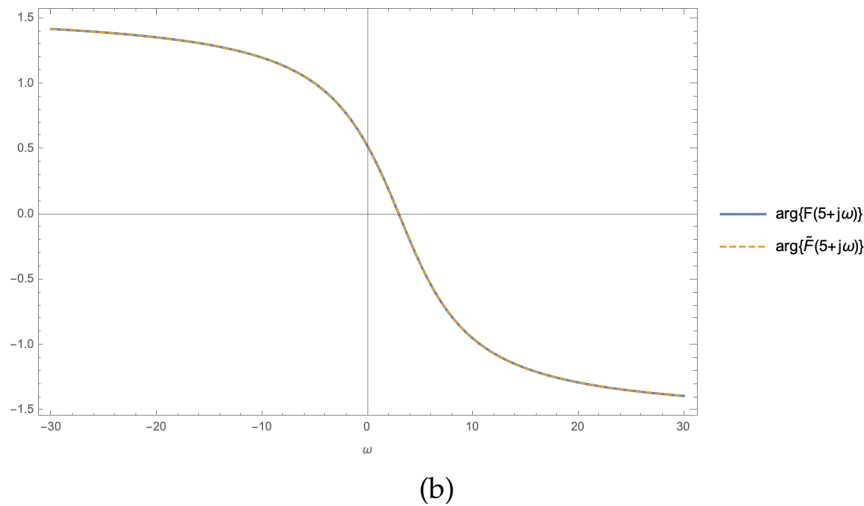


Figure 11: Magnitude (a) and argument (b) of the Dirichlet-Laplace transform with coefficients $a = \{a_n = \frac{1}{n^2} \exp \frac{2n\pi i}{3}\}$ and exponents $\lambda = \{\lambda_n = n^2\}$ relevant to $f(t) = e^{i\pi t}$ as a function of the complex variable $s = \sigma + i \omega$ for $\sigma = 5$ when computed using the rigorous integral expression $F(s)$ and the corresponding Tricomi’s series approximant $\tilde{F}(s)$ of order $M = 200$.

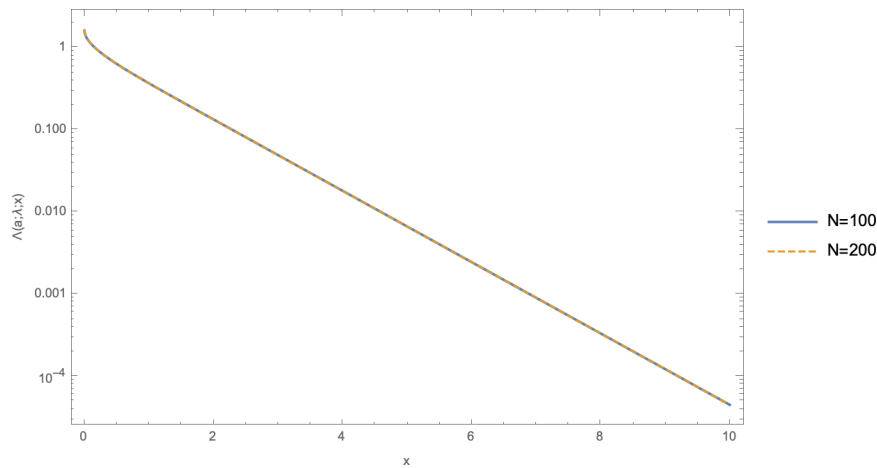


Figure 12: Distribution of the kernel $\Lambda_k(a; \lambda; x)$ relevant to the Dirichlet-Laplace transform with coefficients $a = \{a_n = \frac{1}{n^2} \exp \frac{2n\pi i}{3}\}$ and exponents $\lambda = \{\lambda_n = n^2\}$ for different truncation orders N of the corresponding series expansion representation.

5. Conclusion

In preceding articles [5, 12] we have shown that generalized forms of the Laplace transform can be defined through the Laguerre-type exponentials on, in more general form, by exploiting the Blissard problem, which is connected with Bell’s polynomials.

In this article a wider generalization is obtained by considering as a kernel a general Dirichlet series subject to suitable conditions. The use of Tricomi’s method reduces the transform to a linear combination of classical LTs. This property allows to define the inverse of the DLTs by combining the inverse of classical LTs.

The effectiveness of the proposed methodology has been shown by several test cases involving complex functions typically encountered in Mathematical Physics.

References

- [1] R. J. Beerends, H. G. ter Morsche, J. C. van den Berg, E. M. van de Vrie, *Fourier and Laplace Transforms*, Cambridge University Press, Cambridge, (2003). 1
- [2] E. T. Bell, *Exponential polynomials*, Ann. of Math. (2), **35** (1934), 258–277. 1
- [3] J. Blissard, *Theory of generic functions*, Quarterly J. Pure Appl. Math., **4** (1861), 279–305. 1
- [4] D. Caratelli, C. Cesarano, P. E. Ricci, *Computation of the Bell-Laplace transforms*, Dolomites Res. Notes Approx., **14** (2021), 74–91. 1, 1
- [5] D. Caratelli, P. E. Ricci, *Approximating generalized Laplace transforms*, J. Comput. Math., **6** (2021), 15–40. 1, 1, 5
- [6] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, (1974). 1
- [7] G. Dattoli, P. E. Ricci, *Laguerre-type exponentials, and the relevant L-circular and L-hyperbolic functions*, Georgian Math. J., **10** (2003), 481–494. 1
- [8] A. Ghizzetti, A. Ossicini, *Trasformate di Laplace e calcolo simbolico*, (Italian), Unione Tipografico-Editrice Torinese (UTET), Turin, (1971). 1
- [9] V. Kiryakova, *From the hyper-Bessel operators of Dimovski to the generalized fractional calculus*, Fract. Calc. Appl. Anal., **17** (2014), 977–1000. 1
- [10] V. S. Kiryakova, A. C. McBride, *Explicit solution of the nonhomogeneous hyper-Bessel differential equation*, C. R. Acad. Bulgare Sci., **46** (1993), 23–26.
- [11] É. Le Roy, *Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle*, (French), Darboux Bull. (2), **4** (1899), 245–268. 1
- [12] P. E. Ricci, *Bell polynomials and generalized Laplace transforms*, arXiv, **2021** (2021), 17 pages. 1, 1, 5
- [13] P. E. Ricci, D. Caratelli, F. Mainardi, *Tricomi's Method for the Laplace Transform and Orthogonal Polynomials*, Symmetry, **13** (2021), 13 pages. 1
- [14] J. Riordan, *Introduction to Combinatorial Analysis*, John Wiley & Sons, New York, (1958). 1
- [15] S. Roman, *The umbral calculus*, Academic Press, New York, (1984). 1
- [16] S. M. Roman, G.-C. Rota, *The umbral calculus*, Advances in Math., **27** (1978), 95–188. 1
- [17] F. G. Tricomi, *Ancora sull'inversione della trasformazione di Laplace*, (Italian), Rend. Lincei. (6) **21**, (1935), 420–426. 1, 1, 2
- [18] F. G. Tricomi, *Trasformazione di Laplace e polinomi di Laguerre, I. Inversione della trasformazione; II. Alcune nuove formule sui polinomi di Laguerre*, (Italian), Rend. Accad. Lincei (vi), **21** (1935), 235–242. 1, 1, 2
- [19] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, (1946). 1