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Qualitative analysis, chaos and coexisting attractors in an asymmetric four-well ϕ^8 -generalized Liénard oscillator driven by parametric and external excitations



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Abstract

In this paper, we study the qualitative dynamical analysis, routes to chaos and the coexistence of attractors in a four-well ϕ^8 -generalized Liénard oscillator under external and parametric excitations. The local analysis of the autonomous system reveals saddles, nodes, spirals or centers for appropriate choice of stiffness and damping coefficients. The existence of a Hopf bifurcation is proved during the stability analysis of the equilibrium points. The routes to chaos and the prediction of coexisting attractors have been investigated numerically by using the fourth order Runge-Kutta algorithm. The bifurcation structures obtained show that the system displays a rich variety of bifurcation phenomena, such as symmetry breaking, symmetry restoring, period-doubling, period windows, period-m bubbles, reverse period windows, antimonotonicity, intermittency, quasiperiodic, and chaos. In addition, remerging chaotic band attractors and remarkable routes to chaos occur in the system. Further, it is found that the system presents various coexistence of two attractors as well as the monostability and bistability phenomena. On the other hand, for large amplitude of the parametric excitation and with $\omega = 1$, the coexistence of asymmetric periodic bursting oscillations of different topologies takes place in the system. It has also been shown numerically that for appropriate values of system parameters and initial conditions, the presented system can exhibit up to five types of coexisting multiple attractors.

Keywords: Generalized Liénard oscillator, four well potential, parametric excitation, local stability, coexisting attractors. **2020 MSC:** 34C60, 37D45, 37G35, 65P20.

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1. Introduction

Physical processes are obviously nonlinear in nature. To this end, a better understanding of these processes requires to take into account during the mathematical modeling, the fundamental nonlinearities of the physical systems. Thus, many physical systems are represented by nonlinear ordinary differential equations. Due to inherent nonlinearities, the nonlinear dynamical systems display a rich variety of different long-term behaviors, such as fixed points, limit cycles, quasiperiodic, and chaotic responses. One of the classical equations often used to describe these behaviors besides chaotic behavior is the

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Liénard-type equation of the form:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$
 (1.1)

where a dot represents the derivative with respect to time t. f and g are arbitrary functions of x.

The Liénard equation (1.1) plays an important role in many areas of physics, biology, mechanics, chemistry, seismology, cosmology and engineering [3, 4, 8, 23]. In view of importance of Liénard-type equation (1.1), several researchers investigated the mathematical properties of this equation from both mathematical and physical points of view, and its study remains up to now an active field of research. For example, Han and Romanovski [5] considered a polynomial Liénard system of arbitrary degree on the plane, and developed a new method to obtain a lower bound of the maximal number of limit cycles. Sun [26] obtained more limit cycles by considering a Liénard system of type (7, n) with $n \gg 6$. Recently, Yang and Ding [31] investigated limit cycles of a class of Liénard systems with restoring forces of seventh degree. Recently again, Wu et al. [30] obtained some new and better lower bounds of the maximal number of small amplitude limit cycles of Z₂-equivariant generalized Liénard systems using the method of normal form theory. In most these studies, ϕ^8 potential is often used. It is then important to point out that, the Liénard system with ϕ^8 potential presents substantial and additional difficulties in the qualitative analysis of perturbed systems. Recently, the dynamics study of Liénard-type system (1.1) driven by parametric or external periodic excitation has received considerable attention. For instance, Maccari [18] used an asymptotic perturbation method based on Fourier expansion and time rescaling to calculate an approximate solution of a generalized Van der Pol-Duffing oscillator in resonance with a periodic excitation. The same method has been applied in [19] to a parametrically excited Liénard system. On the other hand, Rayleigh-Liénard oscillator is also considered as generalization of the Liénard system (1.1) [6]. To that end, Maccari [20] investigated a bifurcation analysis of parametrically excited Rayleigh-Liénard oscillator. An asymptotic perturbation method based on Fourier expansion and time rescaling has been used to calculate an approximate analytic solution. Moreover, Floquet's theory has also been used to determine the stability of the periodic solutions. Miwadinou et al. [22] investigated chaotic motions in Rayleigh-Liénard oscillator with external and parametric periodic excitations by using the Melnikov method. Recently, Kpomahou et al. [12] studied regular and chaotic oscillations in a modified Rayleigh-Liénard system under parametric excitation. But it is important to underline that, the chaotic dynamics of a forced Liénard system (1.1) with ϕ^8 potential is very few studied in the open literature. One can recently notice that, Koudahoun et al. [10] investigated the chaotic dynamics of an extended forced Duffing oscillator with double-well ϕ^8 potential. More recently, Miwadinou et al. [21] studied the stability and chaotic dynamics of a forced ϕ^8 -generalized Liénard oscillator. The obtained results by these authors have shown in the nonlinear dynamics study of perturbed systems, the existence of a symmetric four-well potential following an appropriate choice of stiffness parameters. Speaking of the four-well potential, Warminski [29] showed that the nonlinear systems with self, parametric and external excitations can vibrate chaotically and hyperchaotically when the amplitude of parametric excitation becomes large enough. Thus, the dynamic response of a nonlinear oscillator with ϕ^8 potential can provide in certain conditions more information than ϕ^6 and ϕ^4 potentials since it is well known that the dynamics of ϕ^6 is more rich than ϕ^4 and ϕ^2 [15]. Therefore in the fields of mathematical physics and structural mechanics, chaotic dynamics for nonlinear oscillators with four-well potential may be investigated numerically for a better understanding of the nonlinear behavior of these oscillators types.

Another interesting characteristic found in nonlinear dynamical systems is the coexistence of multiple attractors in some regions of parameter space [2, 24, 27]. Note that multiple attractors coexisting as a typical bifurcation lead to unpredictable behavior of trajectories and are considered as a source of unpredictability of a nonlinear system [2]. Thus, it affects the performance of the system to some extent [1, 9]. However, the coexistence of attractors offers another importance to nonlinear dynamical systems. For example, it provides multiple optional steady states for a system to respond to different needs. In other word, it gives rise to the possibility of hysteresis. This later phenomenon is encountered in mechanical systems [16], electromagnetism [28], chemical kinetics [25], nonlinear optics [17], and so on. Contrary to chaos that has been widely studied through a variety of nonlinear oscillators, the study of coexistence of

attractors is still in its infancy for nonlinear dissipative systems. The prediction of this phenomenon in some nonlinear oscillators has been recently made in open literature [7, 11, 13, 14]. However, the transition of chaos and coexisting attractors in a parametrically and self excited generalized Liénard oscillator with asymmetric four well potential have not been yet performed.

Our objective here is to explore dynamical analysis and coexisting attractors in a generalized Liénard oscillator with asymmetric four-well potential driven by parametric and external excitations. In order to attain our objective, we first present the description of the model and we perform the qualitative analysis of the autonomous system (Sect. 2). Second, we investigate numerically the transitions from regular to chaotic motions as well as the coexistence of attractors by using the fourth-order Runge-Kutta algorithm (Sect. 3). Finally we end with a conclusion (Sect. 4).



Figure 1: Potential function of the unforced and undamped system given by (2.5) exhibiting the effect of: (a) α_1 and (b) α_2 . The used basic values are: $\alpha_0 = 2$, $\alpha_1 = 0.68$, and $\alpha_2 = -0.2$.



Figure 2: Phase portraits of the generalized Liénard system (2.2) with the parameters of Table 1.

2. Description of the model and qualitative analysis of the autonomous system

2.1. Description of the model

In this paper, we investigate the qualitative analysis and dynamical transitions to chaos in an asymmetric four-well ϕ^8 -generalized Liénard oscillator driven by external and parametric excitations. The model in question is defined as follows:

$$\ddot{\mathbf{x}} + (\mathbf{x}^3 - \mathbf{x})(\mathbf{x}^2 - \alpha_0)(\alpha_1 \mathbf{x}^2 + \alpha_2 \mathbf{x} - \alpha_0) + (\beta_0 + \beta_1 \mathbf{x}^2 + \beta_2 \mathbf{x}^4 + \beta_3 \mathbf{x}^6 + \beta_4 \mathbf{x}^8)\dot{\mathbf{x}} = \mathsf{F}(1 + \eta \mathbf{x})\cos\omega t, \quad (2.1)$$

where $\beta_i(i = \overline{0,4})$, $\alpha_j(j = \overline{0,2})$, F, η and ω are constant parameters. Physically, β_i and α_j represent the damping and stiffness coefficients, respectively. F and η denote the amplitude of external and parametric periodic forcing respectively and, ω is the corresponding frequency. When F = $\alpha_2 = 0$ and $\alpha_1 = 1$, the corresponding model has been studied by Sun [26] and the obtained results have shown that this system admits 13 limit cycles with $0 < \alpha_0 < 4$. So, the study of (2.1) is then important since it contains some particular cases already studied in the open literature.

At present, we perform in the following subsection, the qualitative analysis of the autonomous system.

2.2. Qualitative analysis of the autonomous system

It is well known that the fixed points play a key role on the dynamics of nonlinear systems. In other word, the study of the fixed points bifurcation allows us to know for a dynamical system the existence of certain complex dynamics. Thus, we derive in this paragraph the fixed points corresponding to the system (2.1) when it is unperturbed and we analyze their asymptotic stability by using the first Lyapunov technique.

If we let F = 0, (2.1) is considered as an unperturbed system and can be rewritten as

$$\dot{\mathbf{x}} = \mathbf{y}, \quad \dot{\mathbf{y}} = -(\mathbf{x}^3 - \mathbf{x})(\mathbf{x}^2 - \alpha_0)(\alpha_1 \mathbf{x}^2 + \alpha_2 \mathbf{x} - \alpha_0) - (\beta_0 + \beta_1 \mathbf{x}^2 + \beta_2 \mathbf{x}^4 + \beta_3 \mathbf{x}^6 + \beta_4 \mathbf{x}^8)\mathbf{y}.$$
(2.2)

From (2.2), by posing $\dot{x} = \dot{y} = 0$, we find the following results.

Lemma 2.1. The following statements are true for $\alpha_0 \in \mathbb{R}^*_+ \setminus \{1\}$.

- If $\Delta = \alpha_2^2 + 4\alpha_0\alpha_1 > 0$, then system (2.2) has seven equilibrium points such as: $P_1(0,0), P_{2,3}(\pm 1,0), P_{4,5}(\pm\sqrt{\alpha_0},0)$ and $P_{6,7}\left(\frac{-\alpha_2\pm\sqrt{\Delta}}{2\alpha_1},0\right)$.
- If $\Delta = 0$, $\alpha_1 < 0$ and $\alpha_0 + \alpha_1 \neq 0$, then the system (2.2) admits six equilibrium points: $P_1, P_{2,3}, P_{4,5}$ and $P_6(x_6^*, 0)$, with $x_6^* = -\sqrt{\frac{-\alpha_0}{\alpha_1}}$ for $\alpha_2 > 0$ or $x_6^* = \sqrt{\frac{-\alpha_0}{\alpha_1}}$ for $\alpha_2 < 0$.
- If $\Delta < 0$, then the system of (2.2) possess five equilibrium points such as: P₁, P_{2,3} and P_{4,5}.

Lemma 2.2. The following statements are true for $\alpha_0 < 0$.

- If $\Delta > 0$, then system (2.2) possess five equilibrium points: P₁, P_{2,3} and P_{6,7}.
- If $\Delta = 0, \alpha_1 > 0$ and $\alpha_0 + \alpha_1 \neq 0$, then the system (2.2) has four equilibrium points: $P_1, P_{2,3}$, and $P_6(x_6^*, 0)$, with $x_6^* = -\sqrt{\frac{-\alpha_0}{\alpha_1}}$ for $\alpha_2 > 0$ or $x_6^* = \sqrt{\frac{-\alpha_0}{\alpha_1}}$ for $\alpha_2 < 0$.
- If $\Delta < 0$, then the system (2.2) admits three equilibrium points such as: P₁ and P_{2,3}.

In order to analyze the stability of these equilibrium points, it is necessary to determine the Jacobian matrix associated to the system (2.2). Thus, this matrix is given by:

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -f'(x)y - g'(x) & -f(x) \end{pmatrix},$$

where the prime (') denotes the differentiation with respect to x. Evaluating of this matrix at equilibrium point $P_i(x_i^*, 0)$ gives

$$J(\mathbf{x}_{i}^{*}, \mathbf{0}) = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -g'(\mathbf{x}_{i}^{*}) & -f(\mathbf{x}_{i}^{*}) \end{pmatrix},$$

with

$$g'(x_i^*) = 7\alpha_1 x_i^{*6} + 6\alpha_2 x_i^{*5} - 5(\alpha_0 + \alpha_1 + \alpha_1 \alpha_0) x_i^{*4} - 4\alpha_2(1 + \alpha_0) x_i^{*3} + 3\alpha_0(\alpha_1 + \alpha_0 + 1) x_i^{*2} + 2\alpha_0 \alpha_2 x_i^* - \alpha_0^2.$$

The stability process depends on the sign of eigenvalues λ of (2.2) which are given through the following

characteristic equation:

$$\lambda^{2} + f(x_{i}^{*})\lambda + g'(x_{i}^{*}) = 0, \qquad (2.3)$$

Then, from (2.3), the eigenvalues can be written as follows:

$$\lambda_{1,2} = \frac{1}{2} \left[-f(x_i^*) \pm \sqrt{|Q(x_i^*)|} \right],$$
(2.4)

where $Q(x^\ast_i) = f^2(x^\ast_i) - 4g^{\,\prime}(x^\ast_i).$

From (2.4), the following propositions are hold.

Proposition 2.3. For each equilibrium point $P_i(x_i^*, 0), i = \overline{2, 7}$, we have the following statements for $g'(x_i^*) > 0$.

- If $Q(x_i^*) > 0$, then the equilibrium points are nodes for $f(x_i^*) > \sqrt{Q(x_i^*)}$ or saddles for $f(x_i^*) < \sqrt{Q(x_i^*)}$.
- If $Q(x_i^*) = 0$, then the equilibrium points are saddles.
- If $Q(x_i^*) < 0$, then the equilibrium points are spirals.

Proof. For the case $Q(x_i^*) > 0$, we obtain from (2.4) the real eigenvalues having the form:

$$\lambda_{1,2} = \frac{1}{2} \left[-f(x_i^*) \pm \sqrt{Q(x_i^*)} \right].$$

This case corresponds to nodes if $f(x_i^*) > \sqrt{Q(x_i^*)}$ and to saddles if $f(x_i^*) < \sqrt{Q(x_i^*)}$.

The case $Q(x_i^*) = 0$ gives also the real eigenvalues of the form: $\lambda_{1,2} = \pm \sqrt{g'(x_i^*)}$. Hence we have saddles for this case. The case $Q(x_i^*) < 0$ gives from (2.4) two complex conjugate eigenvalues of the form: $\lambda_{1,2} = \frac{1}{2} \left[-f(x_i^*) \pm j\sqrt{-Q(x_i^*)} \right]$ with $j^2 = -1$. This case corresponds to spirals.

Proposition 2.4. For each equilibrium point $P_i(x_i^*, 0)$, $i = \overline{1, 7}$, we have the following statement for $g'(x_i^*) < 0$.

• If $Q(x_i^*) > 0$, then the equilibrium points are only saddles.

Proof. For this case, $Q(x_i^*) > 0$. Thus, we get from (2.4) $\lambda_{1,2} = \frac{1}{2} \left[-f(x_i^*) \pm \sqrt{Q(x_i^*)} \right]$. This leads us only to the existence of saddles because $f(x_i^*) < \sqrt{Q(x_i^*)}$.

At present, we study the case of the unforced and undamped system. For this system, we replace in the previous relations the damping coefficients β_i by zero. Thus, the system (2.2) becomes

$$\dot{\mathbf{x}} = \mathbf{y}, \quad \dot{\mathbf{y}} = -(\mathbf{x}^3 - \mathbf{x})(\mathbf{x}^2 - \alpha_0)(\alpha_1 \mathbf{x}^2 + \alpha_2 \mathbf{x} - \alpha_0).$$
 (2.5)

The system (2.5) corresponds to an integrable Hamiltonian system with a potential function

$$V(x) = \frac{\alpha_1}{8}x^8 + \frac{\alpha_2}{7}x^7 - \frac{\alpha_0 + \alpha_1(1+\alpha_0)}{6}x^6 - \frac{\alpha_2(1+\alpha_0)}{5}x^5 + \frac{\alpha_0(1+\alpha_0+\alpha_1)}{4}x^4 + \frac{\alpha_0\alpha_2}{3}x^3 - \frac{\alpha_0^2}{2}x^2$$

and the associated Hamiltonian function is given by

$$H(x,y) = \frac{1}{2}y^2 + \frac{\alpha_1}{8}x^8 + \frac{\alpha_2}{7}x^7 - \frac{\alpha_0 + \alpha_1(1+\alpha_0)}{6}x^6 - \frac{\alpha_2(1+\alpha_0)}{5}x^5 + \frac{\alpha_0(1+\alpha_0+\alpha_1)}{4}x^4 + \frac{\alpha_0\alpha_2}{3}x^3 - \frac{\alpha_0^2}{2}x^2.$$

For this system, we have the following results.

Proposition 2.5. For each equilibrium point $P_i(x_i^*, 0), i = \overline{1, 7}$, we have the following statements.

- If $g'(x_i^*) < 0$, then, the equilibrium points are saddles.
- If $g'(x_i^*) > 0$, then the equilibrium points are centers and system (2.5) has a Hopf bifurcation when β_0 passes through the critical value $\beta_{0H} = -\Sigma_{n=0}^4 \beta_n x_i^{*2n}$.

Proof. From (2.3) by posing $f(x_i^*) = 0$, we have for each equilibrium $P_i(x_i^*, 0), i = \overline{1, 7}$, the following eigenvalues:

$$\lambda_{1,2} = \pm \sqrt{-g'(x_i^*)}.$$
 (2.6)

Therefore, from (2.6), the equilibrium points are saddles if and only if $g'(x_i^*) < 0$. In this case, the

eigenvalues of the Jacobian matrix are real numbers of different sign. However, when $g'(x_i^*) > 0$, we obtain the following relationship

$$\lambda_{1,2} = \pm j \sqrt{g'(x_i^*)}.$$

Then, the equilibrium points verifying this condition are centers. Now, replacing $\lambda = \pm j \sqrt{g'(x_i^*)}$ into (2.3) we obtain $\beta_{0H} = -\Sigma_{n=0}^4 \beta_n x_i^{*2n}$. Second, differentiating both sides of (2.3) with respect to β_0 gives

$$\frac{d\lambda}{d\beta_0} = -\frac{\lambda}{f(x_i^*) + 2\lambda}.$$

Then,

$$\operatorname{Re}\left(\frac{d\lambda}{d\beta_0}|_{\beta 0=\beta_{0H},\lambda=\pm j\sqrt{g'(x_i^*)}}\right)=-\frac{1}{2}\neq 0.$$

Therefore, the condition for a Hopf bifurcation to occur is satisfied.

Since we are interested to the case of a four-well potential, then we fix the parameter values to be $\alpha_0 = 2$, $\alpha_1 = 0.68$ or 0.6 and $\alpha_2 = -0.2$ throughout this paper. The potential function of the system given by (2.5) illustrating the influence of α_1 and α_2 is shown in Figure 1. From this figure, we notice that when α_1 decreases, the depth of right well increases. The same observation is made for negative values of α_2 . However, for positive values of α_2 , the depth of left well increases as α_2 increases.

We now summarize in Table 1, the stability of fixed points of the unforced and damped system (2.2) for a few choices of damping coefficients. The stiffness parameters leading to an asymmetric four-well potential are kept constant. The corresponding phase portraits plotted in Figure 2 (a)-(d) confirm the obtained analytical results.

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$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_2(-1,0)$	-0.4225 ± 1.4358 j	focus	stable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_3(1,0)$	0.2775 ± 1.7213 j	focus	unstable
$ (b) \ \beta_0 = 1.5, \ \beta_1 = -0.7, \ \beta_3 = 0.02 $ $ \begin{array}{c} P_5(\sqrt{2}, 0) & -1.6236, 2.2736 & saddle & unstable \\ P_6(-1.5742, 0) & -0.8110 \pm 1.3950j & focus & stable \\ P_7(1.8683, 0) & 0.0262 \pm 4.0289j & focus & unstable \\ P_1(0, 0) & -2.8860, 1.3860 & saddle & unstable \\ P_2(-1, 0) & -1.1425 \pm 0.9668j & focus & stable \\ P_3(1, 0) & -0.4425 \pm 1.6865j & focus & stable \\ P_3(1, 0) & -0.4425 \pm 1.6865j & focus & stable \\ P_5(\sqrt{2}, 0) & -2.3565, 1.5665 & saddle & unstable \\ P_6(-1.5742, 0) & -1.5104 \pm 0.5106j & focus & stable \\ P_7(1.8683, 0) & -0.6938 \pm 3.9695 & focus & stable \\ P_2(-1, 0) & 0.4275 \pm 1.4343j & focus & unstable \\ P_2(-1, 0) & 0.4275 \pm 1.4343j & focus & unstable \\ P_3(1, 0) & -0.5725 \pm 1.6469j & focus & stable \\ P_3(1, 0) & -0.5725 \pm 1.6469j & focus & unstable \\ P_5(\sqrt{2}, 0) & -2.9891, 1.2349 & saddle & unstable \\ P_5(\sqrt{2}, 0) & -2.9891, 1.2349 & saddle & unstable \\ P_5(\sqrt{2}, 0) & -2.9891, 1.2349 & saddle & unstable \\ P_5(\sqrt{2}, 0) & -2.9891, 1.2349 & saddle & unstable \\ P_6(-1.5742, 0) & 0.5271 \pm 1.5254j & focus & unstable \\ P_6(-1.5742, 0) & 0.5271 \pm 1.7334j & focus & unstable \\ P_6(-1.5742, 0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_6(-\sqrt{2}, 0) & -2.995, 0.5496 & saddle & unstable \\ P_5(\sqrt{2}, 0) & -1.9566, 1.8866 & saddle & unstable \\ P_5(-1.5742, 0) & -1.3637 \pm 0.8631j & focus & unstable \\ P_5(-1.5742, 0) & -1.3637 \pm 0.8631j & focus & unstable \\ P_5(-1.5742, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & $		$P_4(-\sqrt{2},0)$	-2.0328, 0.7028	saddle	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_5(\sqrt{2},0)$	-1.6236, 2.2736	saddle	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_6(-1.5742,0)$	$-0.8110 \pm 1.3950 \mathrm{j}$	focus	stable
$ (b) \ \beta_0 = 1.5, \ \beta_1 = -0.7, \ \beta_3 = 0.02 \qquad \begin{array}{c} P_1(0,0) & -2.8860, 1.3860 & saddle & unstable \\ P_2(-1,0) & -1.1425 \pm 0.9668j & focus & stable \\ P_3(1,0) & -0.4425 \pm 1.6865j & focus & stable \\ P_3(1,0) & -2.3565, 1.5665 & saddle & unstable \\ P_5(\sqrt{2},0) & -2.3565, 1.5665 & saddle & unstable \\ P_6(-1.5742,0) & -1.5310 \pm 0.5106j & focus & stable \\ P_7(1.8683,0) & -0.6938 \pm 3.9695 & focus & stable \\ P_2(-1,0) & 0.4275 \pm 1.4343j & focus & unstable \\ P_2(-1,0) & 0.4275 \pm 1.6469j & focus & stable \\ P_3(1,0) & -0.5725 \pm 1.6469j & focus & stable \\ P_3(1,0) & -0.5725 \pm 1.6469j & focus & stable \\ P_5(\sqrt{2},0) & -2.9891, 1.2349 & saddle & unstable \\ P_5(\sqrt{2},0) & -2.9891, 1.2349 & saddle & unstable \\ P_7(1.8683,0) & -1.549 \pm 3.7147j & focus & unstable \\ P_2(-1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_3(1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_3(1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_5(\sqrt{2},0) & -2.9950, 0.5496 & saddle & unstable \\ P_5(-1.5742,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_5(-1.5742,0) & -1.5955, 0.5496 & saddle & unstable \\ P_5(-1.5742,0) & -1.9566, 1.8866 & saddle & unstable \\ P_5(-1.5742,0) & -1.637 \pm 0.8631j & focus & stable \\ P_5(-1.5742,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0$		$P_7(1.8683,0)$	$0.0262 \pm 4.0289 \mathrm{j}$	focus	unstable
$ (b) \ \beta_0 = 1.5, \ \beta_1 = -0.7, \ \beta_3 = 0.02 \qquad \begin{array}{lll} P_2(-1,0) & -1.1425 \pm 0.9668j & focus & stable \\ P_3(1,0) & -0.4425 \pm 1.6865j & focus & stable \\ P_3(1,0) & -2.3565, 1.5665 & saddle & unstable \\ P_5(\sqrt{2},0) & -2.3565, 1.5665 & saddle & unstable \\ P_6(-1.5742,0) & -1.5310 \pm 0.5106j & focus & stable \\ P_7(1.8683,0) & -0.6938 \pm 3.9695 & focus & stable \\ P_2(-1,0) & 0.4275 \pm 1.4343j & focus & unstable \\ P_2(-1,0) & 0.4275 \pm 1.4343j & focus & unstable \\ P_3(1,0) & -0.5725 \pm 1.6469j & focus & stable \\ P_4 = (-\sqrt{2},0) & -0.5725 \pm 1.6469j & focus & unstable \\ P_5(\sqrt{2},0) & -2.9891, 1.2349 & saddle & unstable \\ P_5(\sqrt{2},0) & -2.9891, 1.2349 & saddle & unstable \\ P_6(-1.5742,0) & 0.5271 \pm 1.5254j & focus & unstable \\ P_7(1.8683,0) & -1.549 \pm 3.7147j & focus & stable \\ P_2(-1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_2(-1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_3(1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_6(-1.5742,0) & 0.5271 \pm 1.5254j & focus & unstable \\ P_6(-1.5742,0) & 0.5271 \pm 1.5254j & focus & unstable \\ P_7(1.8683,0) & -1.549 \pm 3.7147j & focus & unstable \\ P_2(-1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_5(\sqrt{2},0) & -2.9995, 0.5496 & saddle & unstable \\ P_6(-1.5742,0) & -1.3637 \pm 0.8631j & focus & stable \\ P_6(-1.5742,0) & -1.3637 \pm 0.8631j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1$	(b) $\beta_0 = 1.5, \beta_1 = -0.7, \beta_3 = 0.02$	$P_1(0,0)$	-2.8860, 1.3860	saddle	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_2(-1,0)$	-1.1425 ± 0.9668 j	focus	stable
$ \begin{array}{ c c c c c c c c } (b) \ \beta_0 = 1.5, \ \beta_1 = -0.7, \ \beta_3 = 0.02 & P_4(-\sqrt{2}, 0) & 3.2144, 0.4444 & saddle & unstable \\ P_5(\sqrt{2}, 0) & -2.3565, 1.5665 & saddle & unstable \\ P_6(-1.5742, 0) & -1.5310 \pm 0.5106j & focus & stable \\ P_7(1.8683, 0) & -0.6938 \pm 3.9695 & focus & stable \\ P_7(1.8683, 0) & -0.6938 \pm 3.9695 & focus & stable \\ P_2(-1, 0) & 0.4275 \pm 1.4343j & focus & unstable \\ P_3(1, 0) & -0.5725 \pm 1.6469j & focus & stable \\ P_3(1, 0) & -0.5725 \pm 1.6469j & focus & stable \\ P_5(\sqrt{2}, 0) & -2.9891, 1.2349 & saddle & unstable \\ P_5(\sqrt{2}, 0) & -2.9891, 1.2349 & saddle & unstable \\ P_6(-1.5742, 0) & 0.5271 \pm 1.5254j & focus & unstable \\ P_7(1.8683, 0) & -1.549 \pm 3.7147j & focus & stable \\ P_1(0, 0) & -2.0302, 1.9702 & saddle & unstable \\ P_6(-1.5742, 0) & 0.5271 \pm 1.5254j & focus & unstable \\ P_7(1.8683, 0) & -0.5125 \pm 1.4062j & focus & stable \\ P_3(1, 0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_5(\sqrt{2}, 0) & -2.5995, 0.5496 & saddle & unstable \\ P_5(-1.5742, 0) & -1.3637 \pm 0.8631j & focus & stable \\ P_6(-1.5742, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ $		P ₃ (1,0)	$-0.4425 \pm 1.6865 \mathrm{j}$	focus	stable
$ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ \left(d \right) \ \beta_0 = 0.06, \ \beta_1 = 0.2 \\ \left(d \right)$		$P_4(-\sqrt{2},0)$	3.2144, 0.4444	saddle	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_5(\sqrt{2},0)$	-2.3565, 1.5665	saddle	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_6(-1.5742,0)$	-1.5310 ± 0.5106 j	focus	stable
$ (c) \ \beta_0 = 0.06, \ \beta_1 = 1, \ \beta_3 = 0.02 \qquad \begin{array}{c} P_1(0,0) & -2.0302, 1.9702 & \text{saddle} & \text{unstable} \\ P_2(-1,0) & 0.4275 \pm 1.4343j & \text{focus} & \text{unstable} \\ P_3(1,0) & -0.5725 \pm 1.6469j & \text{focus} & \text{stable} \\ P_4 = (-\sqrt{2},0) & -0.7733, 1.8475 & \text{saddle} & \text{unstable} \\ P_5(\sqrt{2},0) & -2.9891, 1.2349 & \text{saddle} & \text{unstable} \\ P_6(-1.5742,0) & 0.5271 \pm 1.5254j & \text{focus} & \text{unstable} \\ P_7(1.8683,0) & -1.549 \pm 3.7147j & \text{focus} & \text{stable} \\ P_2(-1,0) & -0.5125 \pm 1.4062j & \text{focus} & \text{stable} \\ P_2(-1,0) & -0.5125 \pm 1.4062j & \text{focus} & \text{stable} \\ P_3(1,0) & 0.1875 \pm 1.7334j & \text{focus} & \text{unstable} \\ P_4(-\sqrt{2},0) & -2.5995, 0.5496 & \text{saddle} & \text{unstable} \\ P_5 = (\sqrt{2},0) & -1.9566, 1.8866 & \text{saddle} & \text{unstable} \\ P_5(-1.5742,0) & -1.3637 \pm 0.8631j & \text{focus} & \text{stable} \\ P_5(-1.5742,0) & -1.0705 \pm 3.8849j & \text{focus} & \text{stable} \\ \end{array} $		$P_7(1.8683,0)$	-0.6938 ± 3.9695	focus	stable
$ (c) \ \beta_0 = 0.06, \ \beta_1 = 1, \ \beta_3 = 0.02 $ $ \begin{array}{c} P_2(-1,0) & 0.4275 \pm 1.4343j & \text{focus} & \text{unstable} \\ P_3(1,0) & -0.5725 \pm 1.6469j & \text{focus} & \text{stable} \\ P_4 = (-\sqrt{2},0) & -0.7733, 1.8475 & \text{saddle} & \text{unstable} \\ P_5(\sqrt{2},0) & -2.9891, 1.2349 & \text{saddle} & \text{unstable} \\ P_6(-1.5742,0) & 0.5271 \pm 1.5254j & \text{focus} & \text{unstable} \\ P_7(1.8683,0) & -1.549 \pm 3.7147j & \text{focus} & \text{stable} \\ \end{array} $ $ \begin{array}{c} P_1(0,0) & -2.0302, 1.9702 & \text{saddle} & \text{unstable} \\ P_2(-1,0) & -0.5125 \pm 1.4062j & \text{focus} & \text{stable} \\ P_3(1,0) & 0.1875 \pm 1.7334j & \text{focus} & \text{unstable} \\ P_3(1,0) & 0.1875 \pm 1.7334j & \text{focus} & \text{unstable} \\ P_5 = (\sqrt{2},0) & -2.5995, 0.5496 & \text{saddle} & \text{unstable} \\ P_5 = (\sqrt{2},0) & -1.9566, 1.8866 & \text{saddle} & \text{unstable} \\ P_6(-1.5742,0) & -1.3637 \pm 0.8631j & \text{focus} & \text{stable} \\ \end{array} $	(c) $\beta_0 = 0.06, \beta_1 = 1, \beta_3 = 0.02$	$P_1(0,0)$	-2.0302, 1.9702	saddle	unstable
$ \begin{array}{ c c c c c c c } (c) \ \beta_0 = 0.06, \ \beta_1 = 1, \ \beta_3 = 0.02 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = 1, \ \beta_3 = 0.02 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = 1, \ \beta_3 = 0.02 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.06, \ \beta_1 = 0.2 \\ (c) \ \beta_0 = 0.2 \\ (c) \ \beta_0 = 0.2 $		$P_2(-1,0)$	$0.4275 \pm 1.4343 \mathrm{j}$	focus	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_3(1,0)$	-0.5725 ± 1.6469 j	focus	stable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_4 = (-\sqrt{2}, 0)$	-0.7733, 1.8475	saddle	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_5(\sqrt{2},0)$	-2.9891, 1.2349	saddle	unstable
$\begin{array}{ c c c c c c c c }\hline & P_7(1.8683,0) & -1.549 \pm 3.7147j & focus & stable \\ \hline & P_1(0,0) & -2.0302, 1.9702 & saddle & unstable \\ P_2(-1,0) & -0.5125 \pm 1.4062j & focus & stable \\ P_3(1,0) & 0.1875 \pm 1.7334j & focus & unstable \\ P_4(-\sqrt{2},0) & -2.5995, 0.5496 & saddle & unstable \\ P_5 = (\sqrt{2},0) & -1.9566, 1.8866 & saddle & unstable \\ P_6(-1.5742,0) & -1.3637 \pm 0.8631j & focus & stable \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & focus & stable \\ \hline \end{array}$		$P_6(-1.5742,0)$	0.5271 ± 1.5254 j	focus	unstable
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_7(1.8683,0)$	-1.549 ± 3.7147 j	focus	stable
$ \left(\begin{array}{cccc} (d) \ \beta_0 = 0.06, \beta_1 = -0.7, \beta_3 = 0.2 \end{array} \right) \left(\begin{array}{cccc} P_2(-1,0) & -0.5125 \pm 1.4062j & \text{focus} & \text{stable} \\ P_3(1,0) & 0.1875 \pm 1.7334j & \text{focus} & \text{unstable} \\ P_4(-\sqrt{2},0) & -2.5995, 0.5496 & \text{saddle} & \text{unstable} \\ P_5 = (\sqrt{2},0) & -1.9566, 1.8866 & \text{saddle} & \text{unstable} \\ P_6(-1.5742,0) & -1.3637 \pm 0.8631j & \text{focus} & \text{stable} \\ P_7(1.8683,0) & -1.0705 \pm 3.8849j & \text{focus} & \text{stable} \\ \end{array} \right) $	(d) $\beta_0 = 0.06, \beta_1 = -0.7, \beta_3 = 0.2$	P ₁ (0,0)	-2.0302, 1.9702	saddle	unstable
$ \begin{array}{ c c c c c c c c } (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_1 = 0.2 \\ (d) \ \beta_0 = 0.2 \\ (d$		$P_2(-1,0)$	$-0.5125 \pm 1.4062 \mathrm{j}$	focus	stable
$ \begin{array}{ c c c c c c c c } (d) \ \beta_0 = 0.06, \ \beta_1 = -0.7, \ \beta_3 = 0.2 \\ & P_4(-\sqrt{2}, 0) & -2.5995, \ 0.5496 & saddle & unstable \\ P_5 = (\sqrt{2}, 0) & -1.9566, \ 1.8866 & saddle & unstable \\ P_6(-1.5742, 0) & -1.3637 \pm 0.8631j & focus & stable \\ P_7(1.8683, 0) & -1.0705 \pm 3.8849j & focus & stable \\ \end{array} $		$P_3(1,0)$	$0.1875 \pm 1.7334 j$	focus	unstable
$ \begin{array}{ c c c c c c c c } P_5 = (\sqrt{2}, 0) & -1.9566, 1.8866 & saddle & unstable \\ P_6 (-1.5742, 0) & -1.3637 \pm 0.8631 j & focus & stable \\ P_7 (1.8683, 0) & -1.0705 \pm 3.8849 j & focus & stable \\ \end{array} $		$P_4(-\sqrt{2},0)$	-2.5995, 0.5496	saddle	unstable
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$		$P_5 = (\sqrt{2}, 0)$	-1.9566, 1.8866	saddle	unstable
$P_7(1.8683,0)$ -1.0705 ± 3.8849 focus stable		$P_6(-1.5742,0)$	$-1.3637 \pm 0.8631 \mathrm{j}$	focus	stable
		$P_7(1.8683,0)$	-1.0705 ± 3.8849 j	focus	stable

Table 1: Stability of fixed points of the unforced and damped system (2.2). The following parameters are kept constant: $\alpha_0 = 2$, $\alpha_1 = 0.68$, $\alpha_2 = -0.2$, $\beta_2 = 0.06$, and $\beta_4 = 0.005$.

3. Bifurcation and chaos

In this section, we investigate numerically by using the fourth-order Runge-Kutta algorithm, the eventual transitions to chaos and the coexistence of attractors when the control parameter varies. For this, the tools as bifurcation diagram, Lyapunov exponent, time series, Poincaré map and phase portrait are used to quantify the presence of regular and irregular motions in our system. To this end, for a better understanding of the effects of the system parameters on the dynamical behavior of our system, we first investigate the dynamics of the presented system in absence of dissipation and parametric excitation forces. Thus, we set $\eta = \beta_i = 0$ and we consider a set of sample parameter values $\alpha_0 = 2$, $\alpha_1 = 0.68$, $\alpha_2 = -0.2$ and $\omega = 1$ to plot the bifurcation diagram and the corresponding Lyapunov exponent versus the amplitude of the external excitation with the initial conditions $x_0 = 0.5$ and $y_0 = 0.5$ (see Figure 3). From this figure, we note that the chaotic oscillations are more abundant than periodic oscillations. Moreover, the geometric shape of chaotic attractors differs as shown in Figure 4. However, the geometrical complexity of chaotic attractors is identical since the Kaplan-Yorke dimension calculated for F = 3 and F = 15.69 is equal to 2.

Second, taking into account the dissipation and parametric excitation forces, that is $\beta_0 = \beta_2 = 0.06$, $\beta_1 = -0.7$, $\beta_3 = 0.02$, $\beta_4 = 0.005$ and $\eta = 0.00025$, we observe that the periodic oscillations are more important than chaotic oscillations (see Figure 4). One can clearly see through this figure that for F = 3and F = 15.69, the presented system displays chaotic oscillations. Thus, the Kaplan-Yorke dimension that characterizes the geometric complexity of chaotic attractors is equal to 1.405 and 1.1031, respectively. Therefore, we can say that the dissipation and parametric excitation forces reduce the geometrical complexity of chaotic attractors. On the other hand, we note that the system (2.1) displays a rich variety of bifurcation phenomena, such as symmetry breaking, symmetry restoring, period doubling, period windows, reverse period bifurcations, antimonotonicity, intermittency, quasiperiodic, chaos and so on (see Figures 5 and 6). In addition, period-3 motion, period-5 motion, period-m windows and period bubbling routes to chaos occur in the system. Further, remerging chaotic band attractors, monostable oscillations, bistable oscillations as well as the coexistence of different attractors occur in the system. Figure 7 illustrates some attractors exhibited by the bifurcation diagrams of Figure 5 for several different values of F. The effect of parametric excitation amplitude, η on the bifurcation diagrams of Figure 5 is investigated and the obtained results are presented in Figure 8. Through this figure, we notice that the system presents bistable chaotic oscillations for F = 0.42. However for F > 0.42, the coexistence of asymmetric periodic bursting oscillations appears in the system. These observations are confirmed by the phase portraits and its corresponding time series shown in Figure 9 for three different values of F. We can finally conclude that with these system parameters, chaotic oscillations disappear and the periodic bursting oscillations take place in the system for large n. In other word, the amplitude of the parametric excitation can be used to control the chaotic oscillations in the system.

When β_1 is used as control parameter with $\alpha_1 = 0.6$, $\omega = 4$, F = 4, and $\eta = 0.25$, we also notice through Figure 10 that the domain where chaotic oscillations appear decreases. Moreover, the system exhibits various bifurcations, including symmetry restoring, symmetry breaking, period-doubling, and reverse periodic windows. On the other hand, antimonitonicity and period-1 orbit to chaos occur in the system. Further, the coexistence of attractors, bistable and monostable oscillations appear in the system. These phenomena are illustrated in Figure 11 for several different values of β_1 .

When $\omega = 4$, $\eta = 0.00025$ and F = 7.5, the system presents remarkable routes to chaos such a period-1 orbit, period-3 orbit and period bubble to chaos (see Figure 12). Moreover, period windows and reverse period windows occur in the system when β_1 evolves. Various coexisting behaviors of asymmetric attractors also appears in the system. For example, when $\beta_1 = 0.05$, left-period-2 orbit and right-period-3 orbit coexist. As $\beta_1 = 0.1$, left-period-1 orbit and right-period-3 orbit coexist. On the other hand, for $\beta_1 = 0.25$, left-period-1 orbit coexists with right-chaotic attractor. In addition, we observe that left-period-1 orbit coexists with right-period-3.

For $\beta_3 = 1.2, 1.25$, and several sets of the initial conditions, we found that the presented system

displays multiple coexisting attractors' behaviors for certain values of η (see Figures 13 and 14). In Figure 13 (a), left-period-1 limit cycle of small size coexists with small right-period-1 limit cycle and large right-period-2 limit cycle. In Figure 13 (b), two left-chaotic attractors of different topologies coexist with left-period-1 limit cycle and right-period-1 limit cycle all of them of small size. Figure 13 (c) shows the coexistence of left-period-1 limit cycle of small size, right-period-1 limit cycle of small size with period-4 limit cycle. In Figure 14 (a), the presented system exhibits the coexistence of left-point, left-period-1 limit cycle of small size, larger right-period-1 limit cycle and small right-period-2 limit cycle. However, Figure 14 (b) presents the coexistence of small left-period-1 limit cycle. In Figure 14 (c), small left-period-1 limit cycle, four larger right-chaotic attractor of different topologies coexist with small left period-1 limit cycle.

Therefore, we can conclude that the presented system can exhibit up to five coexisting multiple attractors' behaviors for appropriate values of system parameters and initial conditions.



Figure 3: Bifurcation diagram and its corresponding Lyapunov exponent versus the amplitude of the external excitation F with $\alpha_0 = 2$, $\alpha_1 = 0.68$, $\alpha_2 = -0.2$, $\omega = 1$, $\beta_1 = 0$, and $\eta = 0$.



Figure 4: Phase portraits and its Poincaré maps of the generalized Liénard system (2.1) with the parameters of Fig. 3 for two different values of F.



Figure 5: Bifurcation diagrams and its corresponding Lyapunov exponents versus the amplitude of the external excitation F. The specific initial conditions are: (0.5, 0.5) (blue) and (-0.5, -0.5) (red). The used parameter values are: $\alpha_0 = 2$, $\alpha_1 = 0.68$, $\alpha_2 = -0.2$, $\omega = 1$, $\beta_0 = \beta_2 = 0.06$, $\beta_1 = -0.7$, $\beta_3 = 0.02$, $\beta_4 = 0.005$ and $\eta = 0.00025$.



Figure 6: Enlargement of Fig. 5 in the range: (a) $2.1 \le F \le 2.78$ and (b) $2.79 \le F \le 2.88$.



Figure 7: Various phase portraits showing monostable oscillations, bistable oscillations, and coexisting attractors for different values of F with the parameters of Fig. 5.



Figure 8: Bifurcation diagrams in the (F - x) plane with the parameters of Fig. 5 for $\eta = 12.5$.



Figure 9: Phase portraits and its corresponding time series illustrating bistability phenomenon and coexisting asymmetric periodic attractors for three different values of F with the parameters of Fig. 8.



Figure 10: Bifurcation diagrams and its corresponding Lyapunov exponents in the $(\beta_1 - x)$ plane with $\alpha_1 = 0.6$, $\omega = 4$, F = 4, and $\eta = 0.25$. The other parameter values are kept constant.

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Figure 11: Various phase portraits exhibiting bistable oscillations and coexisting asymmetric attractors for some values of β_1 with the parameters of Fig. 10.



Figure 12: Bifurcation diagrams and its corresponding Lyapunov exponents in the $(\beta_1 - x)$ plane with the parameter values of Fig. 10 for F = 7.5 and $\eta = 0.00025$.



Figure 13: Phase portraits showing coexisting of multiple attractors in generalized Liénard system for different values of η and initial conditions: (a) $\eta = 1.75$; (b) $\eta = 2$ and $\eta = 2.5$. The other parameters values of Fig. 10 are kept constant with $\beta_3 = 1.2$ and F = 2.5.



Figure 14: Phase portraits exhibiting coexisting of multiple attractors in generalized Liénard system for different initial conditions and certain values of η : (a) $\eta = 0.75$; (b) $\eta = 1.5$ and (c) $\eta = 2$. The other parameters values of Fig. 10 are kept constant with $\beta_3 = 1.25$ and F = 2.5.

4. Conclusion

In this paper, we have investigated the problem of stability of fixed points, transitions to chaos and the coexistence of attractors in an asymmetric four-well ϕ^8 -generalized Liénard oscillator driven by external and parametric excitations. The local stability analysis of fixed points of the autonomous system has revealed saddles, nodes, spirals or centers for appropriate choice of stiffness and damping parameters. The existence of a Hopf bifurcation is demonstrated during the stability analysis of the equilibrium points. The chaotic dynamics and coexisting attractors of our system have been examined numerically by using the fourth-order Runge-Kutta algorithm. When the external and parametric excitations are taken into account, the system exhibits for $\omega = 1$, complex dynamical behaviors, such as symmetry breaking, symmetry restoring, standard period-doubling, period windows, period-m bubbles, reverse period bifurcations, antimonotonicity, intermittency, quasiperiodic and chaos. In addition, the system presents remerging chaotic band attractors and various coexistence of attractors. Further, monostable and bistable phenomena occur in the system as well as remarkable routes to chaos. It is also found that when $\omega = 1$ and for large amplitude of the parametric excitation, the coexistence of asymmetric periodic bursting oscillations of different topologies takes place in the system. On the other hand, it has been found that for $\omega = 4$, the chaotic region decreases for appropriate values of F and η . In this case of oscillation, the considered system exhibits coexisting behaviors of asymmetric attractors. Moreover, for specific initial conditions, it has also been numerically shown that the generalized Liénard system displays up to five types of coexisting multiple attractors.

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