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Hermite-Hadamard type integral inequalities for geometric-arithmetically (s, m) convex functions

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Abstract

In this paper, we introduce a definition of geometric-arithmetically (s, m) convex function and give some new inequalities of Hermite-Hadamard type for the geometric-arithmetically (s, m) convex function. Finally, we discuss applications of these inequalities to special means.

Keywords: Integral inequality, Hermite-Hadamard type integral inequality, geometric-arithmetically (s, m) convex function, Hölder inequality.

2020 MSC: 26D15, 26A51.

1. Introduction

The following definition is well known in the literature.

Definition 1.1 ([13]). Let $f(x) : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. The function $f(x)$ is said to be convex on I if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

For such a kind of convex function on I with $a, b \in I$ and $a < b$, we have the double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

The convex function can be generalized and the corresponding the Hermite-Hadamard's integral inequality has been refined and generalized by many mathematicians.

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Definition 1.2 ([9])**.** A function $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0$ is said to be geometric-arithmetically-convex if the inequality

$$f(xy) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$, and $\lambda \in [0, 1]$.

Theorem 1.3 ([20])**.** *Let $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0$ be a differentiable function on I, $a, b \in I$ with $a < b$, and $f' \in L([a, b])$. If $|f'(x)|^q$ is geometric-arithmetically-convex on $[a, b]$ for $q \geqslant 1$, then*

$$\left|[bf(b) - af(a)] - \int_a^b f(x)dx\right| \leqslant \frac{[(b-a)A(a,b)]^{1-\frac{1}{q}}}{2^{\frac{1}{q}}} \times \left\{\left[L\left(a^2, b^2\right) - a^2\right]\left|f'(a)\right|^q + \left[L\left(a^2, b^2\right) b^2 - L\left(a^2, b^2\right)\right]\left|f'(b)\right|^q\right\}^{\frac{1}{q}},$$

where $A(x, y)$ and $L(x, y)$ denote arithmetic and logarithmic mean, respectively, which may be defined in (4.1).

Definition 1.4 ([15])**.** Let $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0$ and $s \in (0, 1]$. If

$$f(x^t y^{1-t}) \leqslant t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$, then $f(x)$ is said to be geometric-arithmetically s-convex function or simply speaking, an s-GA-convex function.

Theorem 1.5 ([12])**.** *Let $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|^p$ is an s-GA-convex function on $[0, b]$, $s \in (0, 1]$ and $p \geqslant 1$, then*

$$\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)dx\right| \leqslant \frac{\ln b - \ln a}{n}\left[L(a^{n+1}, b^{n+1})\right]^{1-\frac{1}{p}}\left[G(s, n+1)\left|f'(b)\right|^p + H(s, n+1)\left|f'(a)\right|^p\right]^{\frac{1}{p}},$$

where $G(s, l), L(x, y), H(s, l)$ are given in (3.1).

Definition 1.6 ([8])**.** For some $(s, m) \in [-1, 1] \times (0, 1]$, a function $f : (0, b] \to \mathbb{R}$ is called to be extended (s, m)-GA-convex on $(0, b]$ if

$$f\left(x^\lambda y^{m(1-\lambda)}\right) \leqslant \lambda^s f(x) + m(1-\lambda)^s f(y)$$

holds for all $x, y \in (0, b]$ and $\lambda \in (0, 1)$.

Definition 1.7 ([18])**.** For some $(s, m) \in [-1, 1] \times (0, 1]$ and $\epsilon \geqslant 0$, a function $f : (0, b] \to \mathbb{R}$ is called to be extended (s, m)-ϵ-GA-convex on $(0, b]$ if

$$f\left(x^\lambda y^{m(1-\lambda)}\right) \leqslant \lambda^s f(x) + m(1-\lambda)^s f(y) + \epsilon.$$

Theorem 1.8 ([18])**.** *Let $(s, m) \in [-1, 1] \times (0, 1]$ and $\lambda \in (0, 1)$ and let $f : (0, b^*] \to \mathbb{R}$ be a differentiable mapping on $(0, b^*]$, where $a, b \in (0, b^*]$, $a < b$, $b^{\frac{1}{m}} < b^*$, and $f' \in L_1([a, b])$. If $|f'|^q$ is extended (s,m)-ϵ-GA-convex on $\left(0, \max\left\{b^{\frac{1}{m}}, b\right\}\right]$ for $q \geqslant 1$, then*

$$\begin{aligned}\left|f(a^\lambda b^{1-\lambda}) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \leqslant \frac{\ln b - \ln a}{6^{\frac{1}{q}}}&\{F^{1-\frac{1}{q}}(a, b; \lambda)(6G(a, b; \lambda, s))\left|f'(a)\right|^q \\ &+ 6mH(a, b; \lambda, s)\left|f'\left(b^{\frac{1}{m}}\right)\right|^q + \epsilon\lambda^2[2\lambda a + (3-2\lambda)b]^{\frac{1}{q}} \\ &+ F^{1-\frac{1}{q}}(b, a; 1-\lambda, s)(6H(b, a, 1-\lambda, s)\left|f'(a)\right|^q) \\ &+ 6mG(b, a; 1-\lambda, s)\left|f'\left(b^{\frac{1}{m}}\right)\right|^q + \epsilon(1-\lambda)^2[(1+2\lambda)a + 2(1-\lambda)b]^{\frac{1}{q}}\},\end{aligned}$$

where

$$F(a,b;\lambda,s)=\int_0^\lambda (1-t)a^t b^{1-t}dt, \qquad F(a,b;1-\lambda,s)=\int_\lambda^1 (1-t)a^t b^{1-t}dt,$$
$$G(x,y;\lambda,s)=\int_0^\lambda t[tx+(1-t)y]t^s dt, \qquad H(x,y;\lambda,s)=\int_0^\lambda t[tx+(1-t)y](1-t)^s dt.$$

Now we introduce the definition of geometric-arithmetically (s,m) convex function.

Definition 1.9. Let $f: I\subseteq \mathbb{R}_0\to\mathbb{R}_0=[0,+\infty)$ and $\lambda\in[0,1]$. A function $f(x)$ is said to be geometric-arithmetically (s,m) convex on I if

$$f\left(x^\lambda y^{m(1-\lambda)}\right)\leqslant \lambda^s f(x)+m(1-\lambda)^s f(y) \tag{1.1}$$

holds for $x,y\in I$ and $s,m\in(0,1]$.

In recent years, a number of mathematicians researched Hermite-Hadamard type inequalities for some kinds of convex functions, for example, [2–7, 10, 11, 14–17, 19, 21, 22]. In this paper, we will establish some integral inequalities of Hermite-Hadamard type related to (s, m)-GA-convex functions and then apply these inequalities to special means.

2. Two lemmas

To establish the inequalities for geometric-arithmetically (s,m) convex functions, we recite the following lemmas.

Lemma 2.1 ([8]). *Let* $f: I\subseteq\mathbb{R}_0=(0,\infty)\to\mathbb{R}$ *be differentiable on* I *and* $a,b\in I$ *with* $a<b$. *If* $f'\in L([a,b])$, *then*

$$\frac{b^n f(b)-a^n f(a)}{n}-\int_a^b x^{n-1}f(x)dx=\frac{\ln b-\ln a}{n}\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}f'(a^{1-t}b^t)dt.$$

Lemma 2.2 ([1]). *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be differentiable on* I *and* $a,b\in I$ *with* $a<b$. *If* $f'\in L([a,b])$, *then*

$$\begin{aligned}&\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\\ &=\frac{(x-a)^2}{b-a}\int_0^1(t-1)f'(tx+(1-t)a)dt+\frac{(b-x)^2}{b-a}\int_0^1(1-t)f'(tx+(1-t)b)dt\end{aligned}$$

for $x\in[a,b]$.

3. Main results

We now set off to establish some integral inequalities of Hermite-Hadamard type for geometric-arithmetically (s,m) convex functions.

Theorem 3.1. *Suppose* $f: I\subseteq\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ *be differentiable on* I, $a,b\in I$ *with* $1<a<b$, $f'\in L([a,b])$ *and* $|f'|$ *be decreasing on* $[a,b]$. *If* $|f'|^p$ *is geometric-arithmetically* (s,m) *convex function on* $[a,b]$ *for* $(s,m)\in(0,1]^2$ *and* $p\geqslant 1$, *then*

$$\begin{aligned}\left|\frac{b^n f(b)-a^n f(a)}{n}-\int_a^b x^{n-1}f(x)dx\right|\leqslant&\frac{\ln b-\ln a}{n}\{L(a^{n+1},b^{n+1})\}^{1-\frac{1}{p}}\\ &\times\{G(s,n+1)\left|f'(b)\right|^p+mH(s,n+1)\left|f'(a)\right|^p\}^{\frac{1}{p}},\end{aligned}$$

where

$$G(s,l)=\int_0^1 t^s a^{l(1-t)}b^{lt}dt,\qquad L(x,y)=\frac{y-x}{\ln y-\ln x},\qquad H(s,l)=\int_0^1 (1-t)^s a^{l(1-t)}b^{lt}dt, \tag{3.1}$$

for all $x>0$, $y>0$, $l\geqslant 0$, *with* $x\neq y$.

Proof. Since $|f'|^p$ is an (s,m)-GA-convex function on $[a,b]$ and $|f'|$ is decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we get

$$\begin{aligned}
&\left|\frac{b^nf(b)-a^nf(a)}{n}-\int_a^b x^{n-1}f(x)dx\right|\\
&\leqslant \frac{\ln b-\ln a}{n}\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{m(1-t)}b^t)\right|dt\\
&\leqslant \frac{\ln b-\ln a}{n}\left[\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}dt\right]^{1-\frac{1}{p}}\left[\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{m(1-t)}b^t)\right|^p\right]^{\frac{1}{p}}\\
&\leqslant \frac{\ln b-\ln a}{n}\left[L\left(a^{n+1},b^{n+1}\right)\right]^{1-\frac{1}{p}}\left[\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}(t^s\left|f'(b)\right|^p+m(1-t)^s\left|f'(a)\right|^p)dt\right]^{\frac{1}{p}}\\
&=\frac{\ln b-\ln a}{n}\left[L\left(a^{n+1},b^{n+1}\right)\right]^{1-\frac{1}{p}}\left[G(s,n+1)\left|f'(b)\right|^p+mH(s,n+1)\left|f'(a)\right|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

□

Theorem 3.2. *Suppose* $f: I\subseteq\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ *be differentiable on* I, $a,b\in I$ *with* $1<a<b$, $f'\in L([a,b])$ *and* $|f'|$ *be decreasing on* $[a,b]$. *If* $|f'|^p$ *is geometric-arithmetically* (s,m) *convex function on* $[a,b]$ *for* $(s,m)\in(0,1]^2$ *and* $p>1$, *then*

$$\begin{aligned}
\left|\frac{b^nf(b)-a^nf(a)}{n}-\int_a^b x^{n-1}f(x)dx\right|\leqslant \frac{\ln b-\ln a}{n}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}\left[L\left(a^{\frac{(n+1)p}{p-1}},b^{\frac{(n+1)p}{p-1}}\right)\right]^{1-\frac{1}{p}}&\\
\times\left[\left|f'(b)\right|^p+m\left|f'(a)\right|^p\right]^{\frac{1}{p}}&.
\end{aligned}$$

Proof. Since $|f'|^p$ is an (s,m)-GA-convex function on $[a,b]$ and $|f'|$ is decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, it follows that

$$\begin{aligned}
&\left|\frac{b^nf(b)-a^nf(a)}{n}-\int_a^b x^{n-1}f(x)dx\right|\\
&\leqslant \frac{\ln b-\ln a}{n}\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{m(1-t)}b^t)\right|dt\\
&\leqslant \frac{\ln b-\ln a}{n}\left[\int_0^1 a^{\frac{(n+1)p(1-t)}{p-1}}b^{\frac{(n+1)pt}{p-1}}dt\right]^{1-\frac{1}{p}}\left[\int_0^1(t^s\left|f'(b)\right|^p+m(1-t)^s\left|f'(a)\right|^p)dt\right]^{\frac{1}{p}}\\
&=\frac{\ln b-\ln a}{n}\left(\frac{1}{s+1}\right)^{\frac{1}{p}}\left[L\left(a^{\frac{(n+1)p}{p-1}},b^{\frac{(n+1)p}{p-1}}\right)\right]^{1-\frac{1}{p}}\left[\left|f'(b)\right|^p+m\left|f'(a)\right|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

□

Theorem 3.3. *Suppose* $f: I\subseteq\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ *be differentiable on* I, $a,b\in I$ *with* $1<a<b$, $f'\in L([a,b])$ *and* $|f'|$ *be decreasing on* $[a,b]$. *If* $|f'|^p$ *is a geometric-arithmetically* (s,m) *convex function on* $[a,b]$ *for* $(s,m)\in(0,1]^2$ *and* $p>1$, *then*

$$\left|\frac{b^nf(b)-a^nf(a)}{n}-\int_a^b x^{n-1}f(x)dx\right|\leqslant\frac{\ln b-\ln a}{n}[G(s,(n+1)p)\left|f'(b)\right|^p+mH(s,(n+1)p)\left|f'(a)\right|^p]^{\frac{1}{p}}.$$

Proof. Since $|f'|^p$ is an (s,m)-GA-convex function on $[a,b]$ and $|f'|$ is decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{b^n f(b)-a^n f(a)}{n}-\int_a^b x^{n-1}f(x)dx\right| \\
&\leqslant \frac{\ln b-\ln a}{n}\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{m(1-t)}b^t)\right|dt \\
&\leqslant \frac{\ln b-\ln a}{n}\left(\int_0^1 1^{\frac{p}{p-1}}dt\right)^{1-\frac{1}{p}}\left[\int_0^1\left[a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{m(1-t)}b^t)\right|\right]^p dt\right]^{\frac{1}{p}} \\
&= \frac{\ln b-\ln a}{n}\left[G(s,(n+1)p)\left|f'(b)\right|^p+mH(s,(n+1)p)\left|f'(a)\right|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

□

Theorem 3.4. *Suppose* $f: I\subseteq \mathbb{R}_0=[0,\infty)\to\mathbb{R}$ *be differentiable on* I, $a,b\in I$ *with* $1<a<b$, $f'\in L([a,b])$ *and* $|f'|$ *be decreasing on* $[a,b]$. *If* $|f'|^p$ *is a geometric-arithmetically* (s,m) *convex function on* $[a,b]$ *for* $(s,m)\in(0,1]^2$ *and* $p>1$, $p>q>0$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}
\left|\frac{b^n f(b)-a^n f(a)}{n}-\int_a^b x^{n-1}f(x)dx\right| \leqslant \frac{\ln b-\ln a}{n}\left[L\left(a^{\frac{(n+1)(p-q)}{p-1}},b^{\frac{(n+1)(p-q)}{p-1}}\right)\right]^{1-\frac{1}{p}} \\
\times\left[G(s,(n+1)q)\left|f'(b)\right|^p+mH(s,(n+1)q)\left|f'(a)\right|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

Proof. Since $|f'|^p$ is an (s,m)-GA-convex function on $[a,b]$ and $|f'|$ is decreasing on $[a,b]$, by Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{b^n f(b)-a^n f(a)}{n}-\int_a^b x^{n-1}f(x)dx\right| \\
&\leqslant \frac{\ln b-\ln a}{n}\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{m(1-t)}b^t)\right|dt \\
&\leqslant \frac{\ln b-\ln a}{n}\left[\int_0^1 a^{\frac{(n+1)(p-q)(1-t)}{p-1}}b^{\frac{(n+1)(p-q)t}{p-1}}dt\right]^{1-\frac{1}{p}}\left[\int_0^1 a^{(n+1)q(1-t)}b^{(n+1)qt}\left|f'(a^{m(1-t)}b^t)\right|^p dt\right]^{\frac{1}{p}} \\
&= \frac{\ln b-\ln a}{n}\left[L\left(a^{\frac{(n+1)(p-q)}{p-1}},b^{\frac{(n+1)(p-q)}{p-1}}\right)\right]^{1-\frac{1}{p}}\left[G(s,(n+1)q)\left|f'(b)\right|^p+mH(s,(n+1)q)\left|f'(a)\right|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

□

Theorem 3.5. *Suppose* $f: I\subseteq \mathbb{R}_0=[0,\infty)\to\mathbb{R}$ *be differentiable on* I, $a,b\in I$ *with* $a<b$, $f'\in L([a,b])$ *and* $|f'|$ *be decreasing on* $[a,b]$. *If* $|f'|^p$ *is a geometric-arithmetically* (s,m) *convex function on* $[a,b]$ *for* $(s,m)\in(0,1]^2$ *and* $p\geqslant 1$, *then*

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leqslant \frac{(x-a)^2}{b-a}\left[\frac{(p+s+1)B(s+1,p+1)|f'(x)|^p+m|f'(a)|^p}{p+s+1}\right]^{1/p} \\
&\quad+\frac{(b-x)^2}{b-a}\left[\frac{(p+s+1)B(s+1,p+1)|f'(x)|^p+m|f'(b)|^p}{p+s+1}\right]^{1/p},
\end{aligned}$$

where

$$B(r,s)=\int_0^1 t^{r-1}(1-t)^{s-1}dt, \tag{3.2}$$

for $r>0$ *and* $s>0$ *is the noted Beta function.*

Proof.

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|\\
&\leqslant \frac{(x-a)^2}{b-a}\int_0^1(1-t)|f'(tx+m(1-t)a)|dt+\frac{(b-x)^2}{b-a}\int_0^1(1-t)|f'(tx+m(1-t)b)|dt\\
&\leqslant \frac{(x-a)^2}{b-a}\int_0^1(1-t)|f'(x^t a^{m(1-t)})|dt+\frac{(b-x)^2}{b-a}\int_0^1(1-t)|f'\left(x^t b^{m(1-t)}\right)|dt\\
&\leqslant \frac{(x-a)^2}{b-a}\left(\int_0^1 1^{\frac{p}{p-1}}dt\right)^{(p-1)/p}\left[\int_0^1(1-t)^p|f'(x^t a^{m(1-t)})|^p dt\right]^{1/p}\\
&\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 1^{\frac{p}{p-1}}dt\right)^{(p-1)/p}\left[\int_0^1(1-t)^p|f'(x^t b^{m(1-t)})|^p dt\right]^{1/p}\\
&\leqslant \frac{(x-a)^2}{b-a}\left[\int_0^1(1-t)^p\left(t^s|f'(x)|^p+m(1-t)^s|f'(a)|^p\right)dt\right]^{1/p}\\
&\quad+\frac{(b-x)^2}{b-a}\left[\int_0^1(1-t)^p\left(t^s|f'(x)|^p+m(1-t)^s|f'(b)|^p\right)dt\right]^{1/p}\\
&=\frac{(x-a)^2}{b-a}\left[\frac{(p+s+1)B(s+1,p+1)|f'(x)|^p+m|f'(a)|^p}{p+s+1}\right]^{1/p}\\
&\quad+\frac{(b-x)^2}{b-a}\left[\frac{(p+s+1)B(s+1,p+1)|f'(x)|^p+m|f'(b)|^p}{p+s+1}\right]^{1/p}.
\end{aligned}$$

Thus, the theorem is proved. □

Corollary 3.6.

1. *If* $p=1$, *we have*

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|\\
&\qquad\leqslant \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|+m(s+1)|f'(a)|}{(s+1)(s+2)}\right]+\frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+m(s+1)|f'(b)|}{(s+1)(s+2)}\right].
\end{aligned}$$

2. *If* $x=\frac{a+b}{2}$, *we have*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\leqslant \frac{b-a}{4}&\left\{\left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p+m|f'(a)|^p}{p+s+1}\right]^{1/p}\right.\\
&\left.+\left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p+m|f'(b)|^p}{p+s+1}\right]^{1/p}\right\}.
\end{aligned}$$

3. *If* $p = 1$ *and* $x = \frac{a+b}{2}$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leqslant \frac{b-a}{4(s+1)(s+2)}\left[2\left|f'\left(\frac{a+b}{2}\right)\right| + m(s+1)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)\right].$$

Theorem 3.7. *Let* $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ *be differentiable on* I, $a, b \in I$ *with* $a < b$, $f' \in L([a, b])$ *and let* $|f'|$ *be decreasing on* $[a, b]$. *If* $|f'|^p$ *is geometric-arithmetically* (s, m) *convex on* $[a, b]$ *for* $(s, m) \in (0, 1]^2$ *and* $p > 1$, *then*

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leqslant \left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\left\{\frac{(x-a)^2}{b-a}\left[\frac{m|f'(a)|^p+|f'(x)|^p}{s+1}\right]^{1/p} + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|^p+m|f'(b)|^p}{s+1}\right]^{1/p}\right\}\end{aligned}$$

for $x \in [a, b]$, $t \in (0, 1]$.

Proof. Since $|f'|$ is decreasing on $[a, b]$, by Lemma 2.2 and Hölder inequality, we have

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leqslant \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+m(1-t)a)|dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+m(1-t)b)|dt \\ &\leqslant \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(x^t a^{m(1-t)})|dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(x^t b^{m(1-t)})|dt \\ &\leqslant \frac{(x-a)^2}{b-a}\left[\int_0^1 (1-t)^{p/(p-1)}dt\right]^{(p-1)/p}\left[\int_0^1 |f'(x^t a^{m(1-t)})|^p dt\right]^{1/p} \\ &\quad + \frac{(b-x)^2}{b-a}\left[\int_0^1 (1-t)^{p/(p-1)}dt\right]^{(p-1)/p}\left[\int_0^1 |f'(x^t b^{m(1-t)})|^p dt\right]^{1/p},\end{aligned}$$

where

$$\int_0^1 (1-t)^{p/(p-1)}dt = \frac{p-1}{2p-1}.$$

Making use of the (s, m)-geometric-arithmetic convexity of $|f'(x)|^p$ on $[a, b]$ again, we get

$$\int_0^1 |f'(x^t a^{m(1-t)})|^p dt \leqslant \int_0^1 (t^s|f'(x)|^p + m(1-t)^s|f'(a)|^p)dt = \frac{|f'(x)|^p + m|f'(a)|^p}{s+1}$$

and

$$\int_0^1 |f'(x^t b^{m(1-t)})|^p dt \leqslant \int_0^1 (t^s|f'(x)|^p + m(1-t)^s|f'(b)|^p)dt = \frac{|f'(x)|^p + m|f'(b)|^p}{s+1}.$$

Therefore, the inequality

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leqslant \left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\left\{\frac{(x-a)^2}{b-a}\left[\frac{m|f'(a)|^p+|f'(x)|^p}{s+1}\right]^{1/p} + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|^p+m|f'(b)|^p}{s+1}\right]^{1/p}\right\}\end{aligned}$$

is derived. □

Corollary 3.8. *Under the conditions of Theorem 3.7, if* $x = \frac{a+b}{2}$*, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leqslant \frac{b-a}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\left\{\left[\frac{m|f'(a)|^p + |f'(\frac{a+b}{2})|^p}{s+1}\right]^{1/p} + \left[\frac{|f'(\frac{a+b}{2})|^p + m|f'(b)|^p}{s+1}\right]^{1/p}\right\}.$$

Theorem 3.9. *Let* $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ *be differentiable on* I*,* $a, b \in I$ *with* $a < b$*,* $f' \in L([a,b])$ *and let* $|f'|$ *be decreasing on* $[a,b]$*. If* $|f'|^p$ *is a geometric-arithmetically* (s,m) *convex on* $[a,b]$ *for* $(s,m) \in (0,1]^2$ *and* $p \geqslant 1$*, then*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \leqslant \left(\frac{1}{2}\right)^{(p-1)/p}\left\{\frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|^p + m(s+1)|f'(a)|^p}{(s+1)(s+2)}\right]^{1/p} + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|^p + m(s+1)|f'(b)|^p}{(s+1)(s+2)}\right]^{1/p}\right\}.$$

Proof. Since $|f'|$ is decreasing on $[a,b]$ and $|f'|^p$ is geometric-arithmetically (s,m) convex on $[a,b]$, by Lemma 2.2 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leqslant \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(tx+m(1-t)a)|dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(tx+m(1-t)b)|dt \\
&\leqslant \frac{(x-a)^2}{b-a}\int_0^1 (1-t)|f'(x^t a^{m(1-t)})|dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)|f'(x^t b^{m(1-t)})|dt \\
&\leqslant \frac{(x-a)^2}{b-a}\left[\int_0^1 (1-t)dt\right]^{(p-1)/p}\left[\int_0^1 (1-t)|f'(x^t a^{m(1-t)})|^p dt\right]^{1/p} \\
&\quad + \frac{(b-x)^2}{b-a}\left[\int_0^1 (1-t)dt\right]^{(p-1)/p}\left[\int_0^1 (1-t)|f'(x^t b^{m(1-t)})|^p dt\right]^{1/p} \\
&\leqslant \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{(p-1)/p}\left[\int_0^1 (1-t)(t^s|f'(x)|^p + m(1-t)^s|f'(a)|^p)dt\right]^{1/p} \\
&\quad + \frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{(p-1)/p}\left[\int_0^1 (1-t)(t^s|f'(x)|^p + m(1-t)^s|f'(b)|^p)dt\right]^{1/p} \\
&= \left(\frac{1}{2}\right)^{(p-1)/p}\left\{\frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|^p + m(s+1)|f'(a)|^p}{(s+1)(s+2)}\right]^{1/p} + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|^p + m(s+1)|f'(b)|^p}{(s+1)(s+2)}\right]^{1/p}\right\}.
\end{aligned}$$

□

Corollary 3.10. *Under the conditions of Theorem 3.9, if* $x = \frac{a+b}{2}$*, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leqslant \left(\frac{1}{2}\right)^{(p-1)/p}\frac{b-a}{4}\left\{\left[\frac{|f'(\frac{a+b}{2})|^p + m(s+1)|f'(a)|^p}{(s+1)(s+2)}\right]^{1/p} + \left[\frac{|f'(\frac{a+b}{2})|^p + m(s+1)|f'(b)|^p}{(s+1)(s+2)}\right]^{1/p}\right\}.$$

4. Application to special means

For positive numbers $b > a > 0$, define

$$A(a,b) = \frac{a+b}{2}, \qquad H(a,b) = \frac{2ab}{a+b}, \qquad L(a,b) = \frac{b-a}{\ln b - \ln a},$$

and

$$L_r(a,b) = \begin{cases} \left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, & r \neq 0, -1, \\ L(a,b), & r = -1, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & r = 0. \end{cases}$$

These quantities are respectively called the arithmetic, harmonic, logarithmic, generalized logarithmic means of two positive numbers a and b.

Now let $f(x) = x^r$ for $x > 0, r \in \mathbb{R}$ with $r \neq 0$, and $(s,m) \in (0,1]^2$. Then

$$\left|f'\left(x^\lambda y^{m(1-\lambda)}\right)\right|^p = |r|^p \left|x^\lambda y^{m(1-\lambda)}\right|^{p(r-1)} \leqslant |r|^p \left[\lambda^s x^{p(r-1)} + m(1-\lambda)^s y^{p(r-1)}\right]$$

for $\lambda \in [0,1]$, $x, y > 0$ and $p \geqslant 1$. We can see a function $|f'|^p$ is said to be geometric-arithmetically (s,m) convex on I. Applying this function to Corollaries 3.6, 3.8, and 3.10 derives the following inequalities for means .

Theorem 4.1. *Let* $B(r,s)$ *be defined by* (3.2) *and let* $b > a > 0, r \in (-\infty,0) \cup (0,1)$, $p \geqslant 1$ *and* $0 < s \leqslant 1, 0 < m \leqslant 1$.

1. *If* $r \neq -1$ *and* $x = \frac{a+b}{2}$, *we have*

$$|A(a^r,b^r) - L_r^r(a,b)| \leqslant \frac{|r|(b-a)}{4(p+s+1)^{1/p}}\left\{\left[(p+s+1)B(s+1,p+1)A^{p(r-1)}(a,b) + ma^{p(r-1)}\right]^{1/p} + \left[(p+s+1)B(s+1,p+1)A^{p(r-1)}(a,b) + mb^{p(r-1)}\right]^{1/p}\right\}.$$

2. *If* $r = -1$ *and* $x = \frac{a+b}{2}$, *we have*

$$\left|\frac{1}{H(a,b)} - \frac{1}{L(a,b)}\right| \leqslant \frac{(b-a)}{4(p+s+1)^{1/p}}\left\{\left[\frac{(p+s+1)B(s+1,p+1)}{A^{2p}(a,b)} + \frac{m}{a^{2p}}\right]^{1/p} + \left[\frac{(p+s+1)B(s+1,p+1)}{A^{2p}(a,b)} + \frac{m}{b^{2p}}\right]^{1/p}\right\}.$$

Proof. According to Corollary 3.6, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right| \leqslant \frac{b-a}{4}\left\{\left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p + m|f'(a)|^p}{p+s+1}\right]^{1/p} + \left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p + m|f'(b)|^p}{p+s+1}\right]^{1/p}\right\}.$$

1. If $r \neq -1$, then $f(x) = x^r$,

$$
\begin{aligned}
|A(a^r, b^r) - L_r^r(a,b)| &= \left| \frac{a^r + b^r}{2} - \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right| \\
&= \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leqslant \frac{b-a}{4}\Bigg\{ \left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p + m|f'(a)|^p}{p+s+1} \right]^{1/p} \\
&\quad + \left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p + m|f'(b)|^p}{p+s+1} \right]^{1/p} \Bigg\} \\
&\leqslant \frac{b-a}{4(p+s+1)^{1/p}}\Bigg\{ \left[(p+s+1)B(s+1,p+1)\left| r\left(\frac{a+b}{2}\right)^{r-1} \right|^p + m|ra^{r-1}|^p \right]^{1/p} \\
&\quad + \left[(p+s+1)B(s+1,p+1)\left| r\left(\frac{a+b}{2}\right)^{r-1} \right|^p + m|ra^{r-1}|^p \right]^{1/p} \Bigg\} \\
&\leqslant \frac{|r|(b-a)}{4(p+s+1)^{1/p}}\Bigg\{ \left[(p+s+1)B(s+1,p+1)A^{p(r-1)}(a,b) + ma^{p(r-1)} \right]^{1/p} \\
&\quad + \left[(p+s+1)B(s+1,p+1)A^{p(r-1)}(a,b) + mb^{p(r-1)} \right]^{1/p} \Bigg\}.
\end{aligned}
$$

2. If $r = -1$, then $f(x) = \frac{1}{x}$,

$$
\begin{aligned}
\left| \frac{1}{H(a,b)} - \frac{1}{L(a,b)} \right| &= \left| \frac{a+b}{2ab} - \frac{\ln b - \ln a}{b-a} \right| \\
&= \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leqslant \frac{b-a}{4}\Bigg\{ \left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p + m|f'(a)|^p}{p+s+1} \right]^{1/p} \\
&\quad + \left[\frac{(p+s+1)B(s+1,p+1)|f'(\frac{a+b}{2})|^p + m|f'(b)|^p}{p+s+1} \right]^{1/p} \Bigg\} \\
&\leqslant \frac{b-a}{4(p+s+1)^{1/p}}\Bigg\{ \left[(p+s+1)B(s+1,p+1)\left| \frac{-1}{(\frac{a+b}{2})^2} \right|^p + m\left| \frac{-1}{a^2} \right|^p \right]^{1/p} \\
&\quad + \left[(p+s+1)B(s+1,p+1)\left| \frac{-1}{(\frac{a+b}{2})^2} \right|^p + m\left| \frac{-1}{b^2} \right|^p \right]^{1/p} \Bigg\} \\
&\leqslant \frac{b-a}{4(p+s+1)^{1/p}}\Bigg\{ \left[(p+s+1)B(s+1,p+1)\frac{2^{2p}}{(a+b)^{2p}} + \frac{m}{a^{2p}} \right]^{1/p} \\
&\quad + \left[(p+s+1)B(s+1,p+1)\frac{2^{2p}}{(a+b)^{2p}} + \frac{m}{b^{2p}} \right]^{1/p} \Bigg\} \\
&\leqslant \frac{(b-a)}{4(p+s+1)^{1/p}}\Bigg\{ \left[\frac{(p+s+1)B(s+1,p+1)}{A^{2p}(a,b)} + \frac{m}{a^{2p}} \right]^{1/p}
\end{aligned}
$$

$$+\left[\frac{(p+s+1)B(s+1,p+1)}{A^{2p}(a,b)}+\frac{m}{b^{2p}}\right]^{1/p}\Bigg\}.$$

□

Theorem 4.2. *Let* $b>a>0,\ r\in(-\infty,0)\cup(0,1),\ p>1$ *and* $0<s\leqslant 1,\ 0<m\leqslant 1.$

1. *If* $r\neq -1$ *and* $x=\frac{a+b}{2}$, *we have*

$$|A(a^r,b^r)-L_r^r(a,b)|\leqslant\frac{|r|(b-a)}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\frac{1}{(s+1)^{1/p}}\times\left\{\left[ma^{p(r-1)}+A^{p(r-1)}(a,b)\right]^{1/p}+\left[A^{p(r-1)}(a,b)+mb^{p(r-1)}\right]^{1/p}\right\}.$$

2. *If* $r=-1$ *and* $x=\frac{a+b}{2}$, *we have*

$$\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right|\leqslant\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\times\frac{b-a}{4(s+1)^{1/p}}\left\{\left[\frac{m}{a^{2p}}+\frac{1}{A^{2p}(a,b)}\right]^{1/p}+\left[\frac{1}{A^{2p}(a,b)}+\frac{m}{b^{2p}}\right]^{1/p}\right\}.$$

Proof. According to Corollary 3.8, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\leqslant\frac{b-a}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\left\{\left[\frac{m|f'(a)|^p+|f'(\frac{a+b}{2})|^p}{s+1}\right]^{1/p}+\left[\frac{|f'(\frac{a+b}{2})|^p+m|f'(b)|^p}{s+1}\right]^{1/p}\right\}.$$

1. If $r\neq -1$, then $f(x)=x^r$,

$$\begin{aligned}|A(a^r,b^r)-L_r^r(a,b)|&=\left|\frac{a^r+b^r}{2}-\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right|\\&=\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\leqslant\frac{b-a}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\\&\quad\times\left\{\left[\frac{m|f'(a)|^p+|f'(\frac{a+b}{2})|^p}{s+1}\right]^{1/p}+\left[\frac{|f'(\frac{a+b}{2})|^p+m|f'(b)|^p}{s+1}\right]^{1/p}\right\}\\&\leqslant\frac{(b-a)}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\frac{1}{(s+1)^{1/p}}\\&\quad\times\left\{\left[m|f'(a)|^p+\left|f'\left(\frac{a+b}{2}\right)\right|^p\right]^{1/p}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^p+m|f'(b)|^p\right]^{1/p}\right\}\\&\leqslant\frac{(b-a)}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\frac{1}{(s+1)^{1/p}}\\&\quad\times\left\{\left[\left|r\left(\frac{a+b}{2}\right)^{r-1}\right|^p+m|ra^{r-1}|^p\right]^{1/p}+\left[\left|r\left(\frac{a+b}{2}\right)^{r-1}\right|^p+m|ra^{r-1}|^p\right]^{1/p}\right\}\\&\leqslant\frac{|r|(b-a)}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\frac{1}{(s+1)^{1/p}}\\&\quad\times\left\{\left[ma^{p(r-1)}+A^{p(r-1)}(a,b)\right]^{1/p}+\left[A^{p(r-1)}(a,b)+mb^{p(r-1)}\right]^{1/p}\right\}.\end{aligned}$$

2. If $r=-1$, then $f(x)=\frac{1}{x}$,

$$\begin{aligned}
&\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right| \\
&=\left|\frac{a+b}{2ab}-\frac{\ln b-\ln a}{b-a}\right| \\
&=\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leqslant \frac{b-a}{4}\left(\frac{p-1}{2p-1}\right)^{(p-1)/p} \\
&\times\left\{\left[\frac{m|f'(a)|^p+|f'(\frac{a+b}{2})|^p}{s+1}\right]^{1/p}+\left[\frac{|f'(\frac{a+b}{2})|^p+m|f'(b)|^p}{s+1}\right]^{1/p}\right\} \\
&\leqslant\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\frac{b-a}{4(s+1)^{1/p}}\times\left\{\left[\frac{2^{2p}}{(a+b)^{2p}}+\frac{m}{a^{2p}}\right]^{1/p}+\left[\frac{2^{2p}}{(a+b)^{2p}}+\frac{m}{b^{2p}}\right]^{1/p}\right\} \\
&\leqslant\left(\frac{p-1}{2p-1}\right)^{(p-1)/p}\frac{b-a}{4(s+1)^{1/p}}\times\left\{\left[\frac{m}{a^{2p}}+\frac{1}{A^{2p}(a,b)}\right]^{1/p}+\left[\frac{1}{A^{2p}(a,b)}+\frac{m}{b^{2p}}\right]^{1/p}\right\}.
\end{aligned}$$

□

Theorem 4.3. *Let* $b>a>0$, $r\in(-\infty,0)\cup(0,1)$, $p\geqslant 1$ *and* $0<s\leqslant 1$, $0<m\leqslant 1$. *If* $r\neq -1$ *and* $x=\frac{a+b}{2}$, *we have*

$$\begin{aligned}
|A(a^r,b^r)-L_r^r(a,b)| \leqslant \left(\frac{1}{2}\right)^{(p-1)/p}\frac{(b-a)|r|}{4[(s+1)(s+2)]^{1/p}}\Big\{&[A^{p(r-1)}(a,b) \\
&+m(s+1)a^{p(r-1)}]^{1/p}+[A^{p(r-1)}(a,b)+m(s+1)b^{p(r-1)}]^{1/p}\Big\}.
\end{aligned}$$

If $r=-1$ *and* $x=\frac{a+b}{2}$, *we have*

$$\begin{aligned}
\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right| \leqslant \left(\frac{1}{2}\right)^{(p-1)/p}&\frac{(b-a)}{4[(s+1)(s+2)]^{1/p}} \\
&\times\left\{\left[\frac{1}{A^{2p}(a,b)}+\frac{m(s+1)}{a^{2p}}\right]^{1/p}+\left[\frac{1}{A^{2p}(a,b)}+\frac{m(s+1)}{b^{2p}}\right]^{1/p}\right\}.
\end{aligned}$$

Proof. According to Corollary 3.10, we have

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leqslant \left(\frac{1}{2}\right)^{(p-1)/p}\frac{b-a}{4}\Bigg\{&\left[\frac{|f'(\frac{a+b}{2})|^p+m(s+1)|f'(a)|^p}{(s+1)(s+2)}\right]^{1/p} \\
&+\left[\frac{|f'(\frac{a+b}{2})|^p+m(s+1)|f'(b)|^p}{(s+1)(s+2)}\right]^{1/p}\Bigg\}.
\end{aligned}$$

1. If $r\neq -1$, then $f(x)=x^r$,

$$\begin{aligned}
|A(a^r,b^r)-L_r^r(a,b)| &= \left|\frac{a^r+b^r}{2}-\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right| \\
&=\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\leqslant\left(\frac{1}{2}\right)^{(p-1)/p}\frac{b-a}{4}\left\{\left[\frac{\left|f'(\frac{a+b}{2})\right|^p+m(s+1)|f'(a)|^p}{(s+1)(s+2)}\right]^{1/p}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{|f'(\frac{a+b}{2})|^p+m(s+1)|f'(b)|^p}{(s+1)(s+2)}\right]^{1/p}\Bigg\}\\
&\leqslant\left(\frac{1}{2}\right)^{(p-1)/p}\frac{(b-a)|r|}{4[(s+1)(s+2)]^{1/p}}\left\{\left[\left(\frac{a+b}{2}\right)^{p(r-1)}+m(s+1)a^{p(r-1)}\right]^{1/p}\right.\\
&\left.+\left[\left(\frac{a+b}{2}\right)^{p(r-1)}+m(s+1)b^{p(r-1)}\right]^{1/p}\right\}\\
&\leqslant\left(\frac{1}{2}\right)^{(p-1)/p}\frac{(b-a)|r|}{4[(s+1)(s+2)]^{1/p}}\left\{\left[A^{p(r-1)}(a,b)+m(s+1)a^{p(r-1)}\right]^{1/p}\right.\\
&\left.+\left[A^{p(r-1)}(a,b)+m(s+1)b^{p(r-1)}\right]^{1/p}\right\}.
\end{aligned}
$$

2. If $r=-1$, then $f(x)=\frac{1}{x}$,

$$
\begin{aligned}
\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right|&=\left|\frac{a+b}{2ab}-\frac{\ln b-\ln a}{b-a}\right|\\
&=\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\\
&\leqslant\left(\frac{1}{2}\right)^{(p-1)/p}\frac{b-a}{4}\left\{\left[\frac{|f'(\frac{a+b}{2})|^p+m(s+1)|f'(a)|^p}{(s+1)(s+2)}\right]^{1/p}\right.\\
&\left.\quad+\left[\frac{|f'(\frac{a+b}{2})|^p+m(s+1)|f'(b)|^p}{(s+1)(s+2)}\right]^{1/p}\right\}\\
&\leqslant\left(\frac{1}{2}\right)^{(p-1)/p}\frac{(b-a)}{4[(s+1)(s+2)]^{1/p}}\\
&\quad\times\left\{\left[\frac{2^{2p}}{(a+b)^{2p}}+\frac{m(s+1)}{a^{2p}}\right]^{1/p}+\left[\frac{2^{2p}}{(a+b)^{2p}}+\frac{m(s+1)}{b^{2p}}\right]^{1/p}\right\}\\
&\leqslant\left(\frac{1}{2}\right)^{(p-1)/p}\frac{(b-a)}{4[(s+1)(s+2)]^{1/p}}\\
&\quad\times\left\{\left[\frac{1}{A^{2p}(a,b)}+\frac{m(s+1)}{a^{2p}}\right]^{1/p}+\left[\frac{1}{A^{2p}(a,b)}+\frac{m(s+1)}{b^{2p}}\right]^{1/p}\right\}.
\end{aligned}
$$

□

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