J. Nonlinear Sci. Appl., 15 (2022), 267–275

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Lie group classification of the nonlinear transmission line model and exact traveling wave solutions

Tarik Amtout*, Mustapha Er-Riani, Mustapha El Jarroudi

Department of Mathematics, Laboratory of Mathematics and Applications (LMA), Faculty of Sciences and Techniques, University Abdel Malek Essaadi, UAE, BP 416, Km 10, Ziaten, 90000, Tangier, Morocco.

Abstract

A nonlinear transmission line (NLTL) model is very essential tools in understanding of propagation of electrical solitons which can propagate in the form of voltage waves in nonlinear dispersive media. These models are often formulated using nonlinear partial differential equations. One of the basic tools available to study these equations are numerical methods such as finite difference method, finite element method, etc, have been developed for nonlinear partial differential equations. These methods require a great amount of time and memory due to the discretization and usually the effect of round-off error causes loss of accuracy in the results. So in this paper, we use one of the most famous analytical methods the Lie group analysis due to Sophus Lie. One of the advantages of this approach is that requires only algebraic calculations. The main aim of this study is to explore the nonlinear transmission line model with arbitrary capacitor's voltage dependence, through the use of Lie group classification, we show that the specifying form of arbitrary capacitor's voltage are power law nonlinearity, exponential law nonlinearity and constant capacitance. The exact solutions and similarity reductions generated from the symmetries are also provided. Furthermore, translational symmetries were utilized to find a family of traveling wave solutions via the tanh-method of the governing nonlinear problem.

Keywords: Lie group classification, non-linear line transmission, traveling wave solutions.

2020 MSC: 76M60, 35C07, 35C08.

©2022 All rights reserved.

1. Introduction

Starting from the papers of Asfari et al. [1], El-borai et al. [6], and Mostafa [13], concerned a nonlinear transmission line (NLTL) model, we apply the Lie group classification method (see [4, 9, 14, 15]) to investigate the symmetry groups of the governing equations. The NLTLs are very convenient tools to study the propagation of electrical solitons which can propagate in the form of voltage waves in nonlinear dispersive media. These transmissions (see [5, 11, 16]) are powerful tools involved in nonlinear transmission phenomena. A transmission line is a specialized medium or other structure designed to carry alternating current of radio frequency, that is, currents with a frequency high enough that their wave nature must be taken into account. Transmission lines are used for purposes such as connecting radio transmitters and

*Corresponding author

Email addresses: tarik.amtout@gmail.com (Tarik Amtout), merriani@uae.ac.ma (Mustapha Er-Riani), meljarroudi@uae.ac.ma (Mustapha El Jarroudi)

doi: 10.22436/jnsa.015.04.02

Received: 2022-02-18 *Revised:* 2022-04-23 *Accepted:* 2022-05-20

receivers with their antennas, distributing cable television signals, computer network connections and high speed computer data buses.

For the large applications of this method in engineering and physics, many authors (see [2, 3, 7, 8]) explained the applications of Lie symmetries to partial or fractional partial differential equations.

The NLTL model used in this work is the same as given by [6], with inductors l, and voltage dependent capacitors, $c(V)$. By applying Kirchhoff current law at node n, whose voltage with respect to ground is V_n, and applying Kirchhoff voltage law across the two inductors connected to this node, the voltages of adjacent nodes on this NLTL are related via:

$$l\frac{d}{dt}\left(c(V_n)\frac{dV_n}{dt}\right)=(V_{n+1}-2V_n+V_{n-1})\,. \tag{1.1}$$

The right-hand side of (1.1) can be approximated with partial derivatives with respect to distance x, from the beginning of the line, assuming that the spacing between two adjacent sections is δ (i.e., $x_n=n\delta$). An approximate continuous partial differential equation can be obtained by using the Taylor expansions of $V(x-\delta)$, $V(x)$, and $V(x+\delta)$ to evaluate the right-hand side of (1.1). Assuming a small δ, and ignoring the high order terms, we obtain

$$L\frac{\partial}{\partial t}\left(C(V)\frac{\partial V}{\partial t}\right)=\frac{\partial^2 V}{\partial x^2}+\frac{\delta^2}{12}\frac{\partial^4 V}{\partial x^4}, \tag{1.2}$$

where C and L are the capacitance and inductance per unit length, respectively. For clarity of the results, we will change the variables, replacing V by u and $C(V)$ by $f(u)$, the equation (1.2) becomes

$$L\frac{\partial}{\partial t}\left[f(u)\frac{\partial u}{\partial t}\right]=\frac{\partial^2 u}{\partial x^2}+\frac{\delta^2}{12}\frac{\partial^4 u}{\partial x^4}, \tag{1.3}$$

where u is the voltage dependent variable, $f(u)$ the capacitance, L is the inductance, and δ an arbitrary constant.

This paper is organized as follows. Section 2 is devoted to investigate a Lie group classification of (1.3) and to determine the classifying relations (determining equations for the arbitrary element $f(u)$). Section 3 is devoted to computing the symmetry group and the reduced solutions. In Section 4, we deal with similarity transformations of equations using symmetry group and provide all possible reduction equations. In Section 5, we consider two cases of the capacitance function $f(u)$, the exact traveling wave solutions are constructed by tanh-method. Finally, the conclusion is presented in the last section.

2. Lie group classification

In this section, we perform the Lie group classification method of (1.3). Consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the equation (1.3),

$$\begin{aligned} t &\to t+\epsilon\tau(t,x,u)+O(\epsilon^2),\\ x &\to x+\epsilon\xi(t,x,u)+O(\epsilon^2),\\ u &\to u+\epsilon\eta(t,x,u)+O(\epsilon^2), \end{aligned}$$

with a small parameter $\epsilon \ll 1$ and where τ, ξ, and η are the the unknowns infinitesimals functions of the transformations for the independent and dependent variables, respectively. The infinitesimal generator V associated with the above group of transformations can be written as

$$V=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}, \tag{2.1}$$

where τ, ξ, and η are functions of t, x, and u.

The operator (2.1) generates a one-parameter symmetry group of (1.3), if and only if the invariance conditions holds, since the system has at most fourth-order derivatives, we prolong the infinitesimal generator V to the fourth order in the following form

$$\Pr^{(4)}(V)(Eq.(1.3)) = 0.$$

If the vector field (2.1) forms a symmetry of (1.3), the infinitesimal generator must satisfy the following invariance criterion given as

$$\Pr^{(4)}(V) = V + \chi_1 \frac{\partial}{\partial u_t} + \chi_{11} \frac{\partial}{\partial u_{tt}} + \chi_{22} \frac{\partial}{\partial u_{xx}} + \chi_{2222} \frac{\partial}{\partial u_{xxxx}},$$

where the coefficient functions of the extended infinitesimals χ_i, $i = t, x$, are explicitly given by

$$\begin{aligned}
\chi_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\chi_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\chi_{11} &= D_t(\chi_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi),\\
\chi_{22} &= D_x(\chi_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\\
\chi_{222} &= D_x(\chi_{22}) - u_{txx} D_x(\tau) - u_{xxx} D_x(\xi),\\
\chi_{2222} &= D_x(\chi_{222}) - u_{txxx} D_x(\tau) - u_{xxxx} D_x(\xi),
\end{aligned}$$

and where the operators D_t and D_x denote total derivatives with respect to t and x.

Applying the fourth order prolongation $\Pr^{(4)}$ onto (1.3) yields the following determining equations

$$2Lf_u u_t \chi_1 + Lf\chi_{11} - \chi_{22} - \frac{\delta^2}{12}\chi_{2222} = 0. \tag{2.2}$$

The invariance condition (2.2) results in an over-determined linear system of determining equations for the coefficients τ, ξ, and η. Manipulation of these determining equations is very tedious. In order to decrease the number of calculations, we take advantage of a computer algebra system to solve these set of over-determining equations. Thus, we have obtained the following determining equations:

$$\begin{aligned}
\mathrm{Det} = \big\{&\delta^2 f\tau_{uuuu} = 0,\ \delta^2 f\xi_{uuuu} = 0,\quad -24Lf^2\xi_u = 0,\quad -2\delta^2 f\tau_u = 0,\ 4\delta^2 f\tau_u = 0,\\
&6\delta^2 f\tau_u = 0,\quad 4\delta^2 f\tau_x = 0,\quad 3\delta^2 f\tau_{uu} = 0,\quad 4\delta^2 f\tau_{uu} = 0,\quad 6\delta^2 f\tau_{uu} = 0,\\
&12\delta^2 f\tau_{uu} = 0,\quad 4\delta^2 f\tau_{ux} = 0,\quad 12\delta^2 f\tau_{ux} = 0,\quad 6\delta^2 f\tau_{xx} = 0,\quad 4\delta^2 f\tau_{uuu} = 0,\\
&6\delta^2 f\tau_{uuu} = 0,\quad 12\delta^2 f\tau_{uux} = 0,\quad 4\delta^2 f\tau_{uuux} = 0,\quad 4\delta^2 f\xi_u = 0,\quad 10\delta^2 f\xi_u = 0,\\
&10\delta^2 f\xi_{uu} = 0,\quad 15\delta^2 f\xi_{uu} = 0,\quad 10\delta^2 f\xi_{uuu} = 0,\quad -24f\tau_u + 6\delta^2 f\tau_{uxx} = 0,\\
&24f\tau_u + 12\delta^2 f\tau_{uxx} = 0,\quad -12Lf^2\tau_{uu} + 12Lf_u f\tau_u = 0,\quad -12Lf^2\xi_{uu} - 12Lf_u f\xi_u = 0,\\
&-3\delta^2 f\eta_{uu} + 12\delta^2 f\xi_{xu} = 0,\quad -6\delta^2 f\eta_{uuu} + 24\delta^2 f\xi_{xuu} = 0,\quad -\delta^2 f\eta_{uuuu} + 4\delta^2 f\xi_{xuuu} = 0,\\
&6\delta^2 f\tau_{uuxx} + 12f\tau_{uu} = 0,\quad 16\delta^2 f\xi_{ux} - 4\delta^2 f\eta_{uu} = 0,\quad 6\delta^2 f\xi_{xx} - 4\delta^2 f\eta_{ux} = 0,\\
&-12f\eta_{xx} + 12Lf^2\eta_{tt} - \delta^2 f\eta_{xxxx} = 0,\quad 24f\tau_x + 4\delta^2 f\tau_{xxx} - 24Lf^2\xi_t = 0,\\
&24f\xi_u + 18\delta^2 f\xi_{uxx} - 12\delta^2 f\eta_{uux} = 0,\quad 12f\xi_{uu} + 6\delta^2 f\xi_{uuxx} - 4\delta^2 f\eta_{uuux} = 0,\\
&-2\delta^2 f\tau_t + \delta^2 f_u\eta + 4\delta^2 f\xi_x = 0,\quad 24f\xi_{ux} + 4\delta^2 f\xi_{uxxx} - 6\delta^2 f\eta_{uuxx} - 12f\eta_{uu} = 0,\\
&-24Lf^2\xi_{ut} + 4\delta^2 f\tau_{uxxx} + 24f\tau_{ux} - 24Lf_u f\xi_t = 0,\\
&12f\tau_{xx} + 24Lf_u f\eta_t + 24Lf^2\eta_{tu} - 12Lf^2\tau_{tt} + \delta^2 f\tau_{xxxx} = 0,\\
&-24Lf^2\tau_{tu} + 12Lf_u f\eta_u + 12Lf_{uu} f\eta + 12Lf^2\eta_{uu} - 12Lf_u^2\eta = 0,\\
&-6f\delta^2\eta_{uxx} - 24f\tau_t + 12f_u\eta + 4\delta^2 f\xi_{xxx} + 24f\xi_x = 0,\\
&\delta^2 f\xi_{xxxx} - 4\delta^2 f\eta_{uxxx} - 24f\eta_{ux} - 12Lf^2\xi_{tt} + 12f\xi_{xx} = 0\big\}.
\end{aligned}$$

Solving the above determining equations for arbitrary value of f, we obtain

$$\tau = c_1 t, \quad \xi = c_2 x, \quad \eta = 0,$$

where c_1 and c_2 are arbitrary constants. Obviously, for arbitrary form of f one obtains the two dimensional principal Lie algebra $L_{\mathcal{P}}$ of (1.3), which is spanned by the following vector generators

$$V_1 = \partial_t, \quad V_2 = \partial_x.$$

The complete Lie group classification of the nonlinear transmission line equation is showed in the following Table 1,

$$\frac{\partial^2 F}{\partial t^2} - \frac{\delta^2}{12Lf_0}\frac{\partial^4 F}{\partial x^4} - \frac{1}{Lf_0}\frac{\partial^2 F}{\partial x^2} = 0. \tag{2.3}$$

Table 1: Lie group classification: $k \neq 0, -\frac{4}{3}, -4$ is constant, where $F(t,x)$ satisfies the equation (2.3).

Case	Forms of $f(u)$	Extensions of $L_{\mathcal{P}}$
I	$f(u) = e^u$	$V_3 = \frac{1}{2}t\partial_t + \partial_u$
II	$f(u) = u^k$	$V_3 = \frac{1}{2}kt\partial_t + u\partial_u$
III	$f(u) = u^{-\frac{4}{3}}$	$V_3 = -\frac{2}{3}t\partial_t + u\partial_u$ $V_4 = t^2\partial_t - 3ut\partial_u$
IV	$f(u) = u^{-4}$	$V_3 = -2t\partial_t + u\partial_u$
V	$f(u) = f_0$	$V_3 = u\partial_u$ $V_4 = F(t,x)\partial_u$

3. Symmetry groups and symmetry reductions of different cases of equation (1.3)

In this section, we compute the corresponding one-parameter groups G_i generated by the vector V_i, by solving the Lie equations

$$\frac{d\bar{t}}{d\epsilon} = \tau(t,x,u), \quad \frac{d\bar{x}}{d\epsilon} = \xi(t,x,u), \quad \frac{d\bar{u}}{d\epsilon} = \eta(t,x,u),$$

subject to the initial conditions

$$\bar{t}|_{\epsilon=0} = t, \quad \bar{x}|_{\epsilon=0} = x, \quad \bar{u}|_{\epsilon=0} = u.$$

By solving this system of ordinary differential equations for different cases, we obtain the one-parameter groups G_i generated by the vector field V_i which are formulated in the second column of Table 2.

The new corresponding solutions $u^{(i)}$ are given. Since each G_i is a symmetry group, it implies that if $u = f(t,x)$ is a known solution of (1.3), then by using the above groups G_i, the corresponding new solutions $u^{(i)}$ are given in the third column of the same Table 2.

4. Similarity variables and its reduction equations

In this section, we derive symmetry reductions of (1.3) associated with the vector generators V_i, by using similarity variables and we will calculate the reduced equation; see Table 3.

Table 2: Symmetry group and symmetry reductions for specified form of $f(u)$.

Cases	Symmetry group	Symmetry reductions
Arbitrary	$G_1: \quad (t,x,u) \to (t+\epsilon, x, u)$ $G_2: \quad (t,x,u) \to (t, x+\epsilon, u)$	$u^{(1)} = f(t-\epsilon, x)$ $u^{(2)} = f(t, x-\epsilon)$
I	$G_1: \quad (t,x,u) \to (t+\epsilon, x, u),$ $G_2: \quad (t,x,u) \to (t, x+\epsilon, u),$ $G_3: \quad (t,x,u) \to (e^{\epsilon}t, x, u+2\epsilon),$	$u^{(1)} = f(t-\epsilon, x),$ $u^{(2)} = f(t, x-\epsilon),$ $u^{(3)} = f(te^{-\epsilon}, x) + 2\epsilon,$
II	$G_1: \quad (t,x,u) \to (t+\epsilon, x, u),$ $G_2: \quad (t,x,u) \to (t, x+\epsilon, u),$ $G_3: \quad (t,x,u) \to (e^{k\epsilon}t, x, e^{2\epsilon}u),$	$u^{(1)} = f(t-\epsilon, x),$ $u^{(2)} = f(t, x-\epsilon),$ $u^{(3)} = f(te^{-k\epsilon}, x)e^{2\epsilon},$
III	$G_1: \quad (t,x,u) \to (t+\epsilon, x, u),$ $G_2: \quad (t,x,u) \to (t, x+\epsilon, u),$ $G_3: \quad (t,x,u) \to (-\frac{t}{\epsilon t-1}, x, -(\epsilon t-1)^3 u),$ $G_4: \quad (t,x,u) \to (e^{-2\epsilon}t, x, e^{3\epsilon}u),$	$u^{(1)} = f(t-\epsilon, x),$ $u^{(2)} = f(t, x-\epsilon),$ $u^{(3)} = -f(\frac{t}{\epsilon t+1}, x)(\epsilon t-1)^3,$ $u^{(4)} = f(te^{2\epsilon}, x)e^{3\epsilon},$
IV	$G_1: \quad (t,x,u) \to (t+\epsilon, x, u),$ $G_2: \quad (t,x,u) \to (t, x+\epsilon, u),$ $G_3: \quad (t,x,u) \to (t, x, e^{\epsilon}u),$	$u^{(1)} = f(t-\epsilon, x),$ $u^{(2)} = f(t, x-\epsilon),$ $u^{(3)} = f(te^{2\epsilon}, x)e^{\epsilon},$
V	$G_1: \quad (t,x,u) \to (t+\epsilon, x, u),$ $G_2: \quad (t,x,u) \to (t, x+\epsilon, u),$ $G_3: \quad (t,x,u) \to (e^{-2\epsilon}t, x, e^{\epsilon}u),$	$u^{(1)} = f(t-\epsilon, x),$ $u^{(2)} = f(t, x-\epsilon),$ $u^{(3)} = f(t, x)e^{\epsilon},$

Table 3: Essential generators, similarity variable, similarity form, and reduced ODEs for different form of $f(u)$.

Case	Essential generators	$r(t,x)$	$u(t,x)$	ODEs
$f(u) = e^u$	$V_1 = \partial_t$ $V_2 = \partial_x$ $V_3 = \frac{1}{2}t\partial_t + \partial_u$	x t x	$h(r)$ $h(r)$ $h(r) + 2\ln(t)$	$h'' + \frac{\delta^2}{12}h^{(4)} = 0$ $e^h(h')^2 + e^h h'' = 0$ $-2t^{-2}Le^h - h'' - \frac{\delta^2}{12}h^{(4)} = 0$
$f(u) = u^k$	$V_1 = \partial_t$ $V_2 = \partial_x$ $V_3 = \frac{1}{2}kt\partial_t + \partial_u$	x t x	$h(r)$ $h(r)$ $t^{\frac{2}{k}}h(r)$	$h'' + \frac{\delta^2}{12}h^{(4)} = 0$ $h^{k-1}k(h')^2 + h^k h'' = 0$ $L\frac{2k^2+4k}{k^2}h^{k+1} - h'' - \frac{\delta^2}{12}h^{(4)} = 0$
$f(u) = u^{-\frac{4}{3}}$	$V_1 = \partial_t$ $V_2 = \partial_x$ $V_3 = -\frac{2}{3}t\partial_t + u\partial_u$ $V_4 = t^2\partial_t - 3ut\partial_u$	x t x x	$h(r)$ $h(r)$ $t^{-3}h(r)$ $t^{-\frac{3}{2}}h(r)$	$h'' + \frac{\delta^2}{12}h^{(4)} = 0$ $-\frac{4}{3}h^{-\frac{7}{3}}(h')^2 + h^{-\frac{4}{3}}h'' = 0$ $h'' + \frac{\delta^2}{12}h^{(4)} = 0$ $\frac{3}{4}Lh^{-\frac{1}{3}} - h'' - \frac{\delta^2}{12}h^{(4)} = 0$
$f(u) = u^{-4}$	$V_1 = \partial_t$ $V_2 = \partial_x$ $V_3 = -2t\partial_t + u\partial_u$	x t x	$h(r)$ $h(r)$ $t^{-\frac{1}{2}}h(r)$	$h'' + \frac{\delta^2}{12}h^{(4)} = 0$ $4(h')^2 - hh'' = 0$ $-\frac{1}{4}Lh^{-3} - h'' - \frac{\delta^2}{12}h^{(4)} = 0$
$f(u) = f_0$	$V_1 = \partial_t$ $V_2 = \partial_x$	x t	$h(r)$ $h(r)$	$h'' + \frac{\delta^2}{12}h^{(4)} = 0$ $h'' = 0$

5. Traveling wave solutions

In this section, we will focus on two cases: the first one when $f(u) = f_0$, and the second one when $f(u) = f_0(1 - bu)$.

5.1. *First case when* $\mathbf{f}(\mathbf{u}) = \mathbf{f_0}$

In this first case, we use the capacitor's voltage dependence using the following constant relationship $f(u) = f_0$, where f_0 is an arbitrary constant. Equation (1.3) reduces to

$$Lf_0 u_{tt} - u_{xx} - \frac{\delta^2}{12} u_{xxxx} = 0. \tag{5.1}$$

By using the following group invariant solution $u(t, x) = U(X)$ and $X = x - ct$, the traveling wave transformation of (5.1), reduces to the following ordinary differential equation:

$$(Lf_0c^2 - 1)U'' - \frac{\delta^2}{12} U^{(4)} = 0. \tag{5.2}$$

By integrating (5.2) twice and taking integration constant zero, we obtain the following nonlinear ODE:

$$(Lf_0c^2 - 1)U - \frac{\delta^2}{12} U'' = 0.$$

By solving the ODE, we obtain, the exact traveling wave solution:

$$u(t, x) = C_1 \sin\left(\frac{2\sqrt{-3Lf_0c^2 + 3}}{\delta}(x - ct)\right) + C_2 \cos\left(\frac{2\sqrt{-3Lf_0c^2 + 3}}{\delta}(x - ct)\right),$$

where C_1 and C_2 are arbitrary constants.

5.2. *Second case when* $\mathbf{f}(\mathbf{u}) = \mathbf{f_0}(\mathbf{1} - \mathbf{bu})$

In this case, we apply tanh-function method when $f(u) = f_0(1 - bu)$. Herein, we give a brief description of this method. The tanh-method is first developed by Malfliet [12]. It is based on the fundamental concept that the traveling wave solution can be written in terms of tanh function. By using the tanh-method one can find out exact traveling wave solutions or solitary solutions of nonlinear partial differential equations by converting them into nonlinear ordinary differential equations taking traveling wave transformation. The main steps of the tanh-method are as follows, for more details see [17–19].

Step 1. We consider the nonlinear evolutionary partial differential equation with two independent variables (t, x) for which we wish to find traveling wave solutions, given as

$$H(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, \ldots) = 0, \tag{5.3}$$

where $u = u(t, x)$ is an unknown function, H is a polynomial in u and its derivatives.

Step 2. We use the traveling wave transformation $u(t, x) = U(X)$, $X = x - ct$, where c represents wave velocity. By making use of this transformation, the nonlinear partial differential equation (5.3) is transformed to the following ordinary differential equation

$$H(U, U', U'', U^{(3)}, \ldots) = 0. \tag{5.4}$$

Step 3. We start by setting $U = F(Y)$ and introduce the following sum

$$F(Y) = \sum_{i=0}^{M} a_i Y^i, \tag{5.5}$$

where M is a positive integer, which is to be determined by balancing the highest order nonlinear term with the highest order linear term in the resulting equation, and $Y = \tanh(X)$ is the new independent variable. We replace all the derivatives in the resulting Equation (5.4) with the following change of derivatives:

$$\begin{aligned}
\frac{d}{dX} &= (1-Y^2)\frac{d}{dY},\\
\frac{d^2}{dX^2} &= -2Y(1-Y^2)\frac{d}{dY} + (1-Y^2)^2\frac{d^2}{dY^2},\\
\frac{d^3}{dX^3} &= 2(1-Y^2(3Y^2-1))\frac{d}{dY} - 6Y(1-Y^2)^2\frac{d^2}{dY^2} + (1-Y^2)^3\frac{d^3}{dY^3},\\
&\vdots
\end{aligned}$$

In order to find the value of the index M, we use the following scheme:

$$\begin{aligned}
F &\to M,\\
F^n &\to nM,\\
F^{'} &\to M+1,\\
F^{''} &\to M+2,\\
F^{(s)} &\to M+s.
\end{aligned}$$

Step 4. In this step, we substitute the value of U and all its derivatives into Equation (5.4), then we collect the coefficients of Y^i $(i = 1, 2, 3, \dots)$ and equating them to zero, we obtained system of nonlinear algebraic equations with unknown parameters a_i, $i = 1, 2, \dots, M$. This system of algebraic equations can be solved either manually or by any symbolic program such as Maple. Thus, one can get the traveling wave solution of (5.3) by putting the values of unknowns into (5.5).

In the following, we implement the tanh to our NLTL model. To this end, we use the capacitor's voltage dependence using the following first-order linear relationship $f(u) = f_0(1-bu)$ where f_0 and b are arbitrary constants. In this case, equation (1.3) reduces to

$$Lf_0(1-bu)u_{tt} - Lf_0bu_t^2 - u_{xx} - \frac{\delta^2}{12}u_{xxxx} = 0. \tag{5.6}$$

We obtained the following group invariant solution: $u(t, x) = U(X)$ and $X = x - ct$. Now, by using the traveling wave transformation, equation (5.6) reduces to the following nonlinear ordinary differential equation:

$$Lf_0c^2\left((1-bU)U^{''} - b(U^{'})^2\right) - U^{''} - \frac{\delta^2}{12}U^{(4)} = 0. \tag{5.7}$$

Integrating (5.7) twice and taking integration constant zero, we obtain the following nonlinear ODE:

$$-12Lf_0c^2U + 6Lf_0c^2bU^2 + 12U + U^{''} = 0. \tag{5.8}$$

Let us take the solution of equation (5.8) in the following form:

$$F(Y) = \sum_{i=0}^{M} a_iY^i, \tag{5.9}$$

where a_i $(i = 1, 2, \cdots, M)$ and M are the unknown parameters. Now, we determine the parameter M, by balancing the linear term of highest-order with the highest order nonlinear terms. Therefore, we balance F^2 and F_{YY} and get: $2M = M + 2 \Rightarrow M = 2$. Therefore, the finite expression in (5.9) reduces to the following expression:

$$F(Y) = a_0 + a_1Y + a_2Y^2.$$

After putting the values of F, F^2, F_Y, and F_{YY} in (5.8) and then equating the coefficients of Y_i, $i = 0, 1, 2, 3, 4$, we obtained the system of nonlinear algebraic equations as follows:

coefficients of Y^0: $3Lf_0c^2ba_2^2 + 3\delta^2 a_2 = 0$,
coefficients Y^1: $6Lf_0c^2ba_1a_2 + \delta^2 a_1 = 0$,
coefficients Y^2: $6Lf_0c^2ba_0a_1 - 6Lf_0c^2a_1 - \delta^2 a_1 = 0$,
coefficients Y^3: $-6Lf_0c^2a_0 + \delta^2 a_2 + 3Lf_0c^2ba_0^2 + 6a_0 = 0$,
coefficients Y^4: $6Lf_0c^2ba_0a_2 + 6a_2 - 4\delta^2 a_2 + 3Lf_0c^2ba_1^2 - 6Lf_0c^2a_2 = 0$.

By solving above system of nonlinear algebraic equations, we obtain the following sets of unknown parameters:

Set 1: $c = c, \quad a_0 = a_0, \quad a_1 = a_1, \quad a_2 = a_2.$
Set 2: $c = c, \quad a_0 = 0, \quad a_1 = 0, \quad a_2 = 0.$
Set 3: $c = c, \quad a_0 = \frac{2(Lf_0c^2-1)}{Lf_0c^2b}, \quad a_1 = 0, \quad a_2 = 0.$
Set 4: $c = \sqrt{\frac{3-\delta^2}{3Lf_0}}, \quad a_0 = -\frac{\delta^2}{(-3+\delta^2)b}, \quad a_1 = 0, \quad a_2 = \frac{3\delta^2}{(-3+\delta^2)b}.$
Set 5: $c = -\sqrt{\frac{3-\delta^2}{3Lf_0}}, \quad a_0 = -\frac{\delta^2}{(-3+\delta^2)b}, \quad a_1 = 0, \quad a_2 = \frac{3\delta^2}{(-3+\delta^2)b}.$
Set 6: $c = \sqrt{\frac{3+\delta^2}{3Lf_0}}, \quad a_0 = \frac{3\delta^2}{(3+\delta^2)b}, \quad a_1 = 0, \quad a_2 = -\frac{3\delta^2}{(3+\delta^2)b}.$
Set 7: $c = -\sqrt{\frac{3+\delta^2}{3Lf_0}}, \quad a_0 = \frac{3\delta^2}{(3+\delta^2)b}, \quad a_1 = 0, \quad a_2 = -\frac{3\delta^2}{(3+\delta^2)b}.$

Therefore, the traveling wave solutions of (5.6) are as follows:

Set 1: in this case, we found the constant solution $u(t,x) = a_0$;
Set 2: in this case, we found the trivial solution $u(t,x) = 0$;
Set 3: in this case, we found the constant solution

$$u(t,x) = \frac{2Lf_0c^2 - 2}{Lf_0c^2b};$$

Set 4:

$$u(t,x) = -\frac{\delta^2}{(-3+\delta^2)b} + \frac{3\delta^2 \tanh\left(\frac{3xLf_0 - i\sqrt{3}\sqrt{-3+\delta^2}\sqrt{L}\sqrt{f_0}t}{3Lf_0}\right)^2}{(-3+\delta^2)b};$$

Set 5:

$$u(t,x) = -\frac{\delta^2}{(-3+\delta^2)b} + \frac{3\delta^2 \tanh\left(\frac{3xLf_0 + i\sqrt{3}\sqrt{-3+\delta^2}\sqrt{L}\sqrt{f_0}t}{3Lf_0}\right)^2}{(-3+\delta^2)b};$$

Set 6:

$$u(t,x) = \frac{3\delta^2}{(3+\delta^2)b} - \frac{3\delta^2 \tanh\left(\frac{3xLf_0 - \sqrt{3}\sqrt{3+\delta^2}\sqrt{L}\sqrt{f_0}t}{3Lf_0}\right)}{(3+\delta^2)b};$$

Set 7:

$$u(t,x) = \frac{3\delta^2}{(3+\delta^2)b} - \frac{3\delta^2 \tanh\left(\frac{3xLf_0 + \sqrt{3}\sqrt{3+\delta^2}\sqrt{L}\sqrt{f_0}t}{3Lf_0}\right)}{(3+\delta^2)b}.$$

6. Conclusion

In this paper, the Lie group method is used to perform group classification of the partial differential equation that governs the nonlinear transmission line model. The algebraic properties of this model are given and in each case of the arbitrary function, we have reduced the initial partial differential equation to an ordinary differential equation. Finally, a wide variety of exact traveling wave solutions are obtained via the tanh-method. We see that this approach is effective for studying and analyzing nonlinear problems from engineering sciences and for determining the forms of the arbitrary function.

References

[1] E. Afshari, A. Hajimiri, *Nonlinear Transmission Lines for Pulse Shaping in Silicon*, IEEE J. Solid-State Circuits, **40** (2005), 744–752. 1

[2] T. Amtout, M. Er-Riani, M. El Jarroudi, *Applications of Lie Symmetry Analysis to the Natural Convection Flow of Boundary Layer Past an Inclined Surface*, Adv. Intell. Syst. Comput., **1418** (2022), 1053–1066. 1

[3] H. Biyadi, T. Amtout, M. Er-Riani, M. El Jarroudi, *Lie Symmetry Analysis of a Class of Thermal Conduction Models*, Appl. Math. Sci., **13** (2019), 1061–1067. 1

[4] G. W. Bluman, S. Kumei, *Symmetries and differential equations*, Springer-Verlag, New York, (1989). 1

[5] A. Chauhan, R. Arora, A. Tomar, *Lie Symmetry Analysis and traveling wave solutions of equal width wave equation*, Proyecciones, **39** (2020), 179–198. 1

[6] M. M. El-Borai, H. M. El-Owaidy, H. M. Ahmed, A. H. Arnous, *Exact and soliton solutions to nonlinear transmission line model*, Nonlinear Dyn., **87** (2017), 767–773. 1

[7] H. Gandhi, D. Singh, A. Tomar, *Explicit Solution of General Fourth Order Time Fractional KdV Equation by Lie Symmetry Analysis*, AIP Conference Proceedings, **2253** (2020), 14 pages. 1

[8] H. Gandhi, A. Tomar, D. Singh, *Conservation laws and exact series solution of fractional-order Hirota-Satsuma-coupled Korteveg-de Vries system by symmetry analysis*, Math. Methods Appl. Sci., **44** (2021), 14356–14370. 1

[9] N. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, RC Press, Boca Raton, (1995). 1

[10] M. A. Kayum, M. Ali Akbar, M. S. Osman, *Competent closed form soliton solutions to the nonlinear transmission and the low-pass electrical transmission lines*, Eur. Phys. J. Plus, **135** (2020), 20 pages.

[11] D. Kumar, A. R. Seadawy, M. R. Haque, *Multiple soliton solutions of the nonlinear partial differential equations describing the wave propagation in nonlinear low-pass electrical transmission lines*, Chaos Solitons Fractals, **115** (2018), 62–76. 1

[12] W. Malfliet, *Solitary wave solutions of nonlinear wave equation*, Amer. J. Phys., **60** (1992), 650–654. 5.2

[13] S. I. Mostafa, *Analytical study for the ability of nonlinear transmission lines to generate solitons*, Chaos Solitons Fractals, **39** (2009), 2125–2132. 1

[14] P. J. Olver, *Application of Lie Group to Differential Equation*, Springer-Verlag, New York, (1986). 1

[15] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, (1982). 1

[16] M. N. O. Sadiku, L. C. Agba, *A Simple Introduction to the Transmission-Line Modeling*, IEEE Trans. Circuits and Systems, **37** (1990), 991–999. 1

[17] A.-M. Wazwaz, *The* tanh-*method for traveling wave solutions of nonlinear equations*, Appl. Math. Comput., **154** (2004), 713–723. 5.2

[18] A.-M. Wazwaz, *Analytic study of the fifth order integrable nonlinear evolution equations by using the tanh method*, Appl. Math. Comput., **174** (2006), 289–299.

[19] A.-M. Wazwaz, *The tanh method for travelling wave solutions to the Zhiber-Shabat equation and other related equations*, Commun. Nonlinear Sci. Numer. Simul., **13** (2008), 584–592. 5.2