



## Some results on the approximate controllability of impulsive stochastic integro-differential equations with nonlocal conditions and state-dependent delay



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### Abstract

This paper presents approximate controllability results for impulsive stochastic integro-differential systems with state-dependent delay in a Hilbert space. The use of the resolvent operator in the sense of Grimmer, as well as stochastic analysis techniques, yields a new set of results. Finally, an example is given to show how the theory that has been worked out can be put into practice.

**Keywords:** Impulsive stochastic integrodifferential equations, state-dependent delay, mild solution, approximate controllability, semigroup theory, resolvent operator, fixed point theorem, nonlocal conditions.

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### 1. Introduction

Integro-differential equations can be used to describe a lot of natural phenomena arising from many areas such as physics, population dynamics, electrical engineering, finance, biology, ecology, sociology, and other areas of science and engineering. Most of these phenomena can not be described through classical differential equations. That is why in recent years they have attracted more attention of several mathematicians, physicists, and engineers. Qualitative properties such as existence, uniqueness, controllability, stability, and optimal control for various integro-differential equations have been extensively studied by many researchers with the help of resolvent operator theory, fixed point theorems, see for instance [5, 15, 19, 28, 50, 51]. To build more realistic models in economics, social sciences, chemistry, finance, physics, and other areas, stochastic effects need to be taken into account. Therefore, many real-world problems can be modeled by stochastic integrodifferential equations. The deterministic models often fluctuate due to noise, so we must move from deterministic control to stochastic control problems.

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In the last decades, many works have been done on the stochastic integrodifferential equations see for example [16, 19].

Impulsive differential equations are vital to study evolutionary processes in chemical reactor systems, electromagnetic waves, population growth models, and biological systems, etc. In such systems, the state changes abruptly at some instants during the evolution process in comparison to the duration of the whole process. Such sudden changes can be well approximated as being in the form of impulses and these processes are more appropriate to be modeled by the impulsive differential equations, (see [8, 24, 34, 43]). On the other hand, there are various real-world phenomena, for example, automatic control systems, heat conduction in materials with fading memory, inferred grinding models and neural networks etc, depending on the past states of the system and described by the delay differential equations (see [22, 35, 42]). In addition, the state-dependent delay differential equations arise in numerous practical models (see for example [4, 36], etc). But it appears that many authors considered the constancy of the time delay as an extra assumption, which makes the study more accessible. Several authors contributed towards the solvability and asymptotic analysis of these kinds of systems with state-dependent delays, and the details can be found in various articles, see for instance, [7, 26, 27], etc and the references therein.

The problem with nonlocal condition, which is a generalization of the problem of classical condition, was motivated by physical problem. The leading deal with nonlocal conditions due to Byszewski [10, 11]. Since it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problem. Stochastic differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained. For more details about nonlocal conditions see [3, 9, 31, 41].

One of the important fundamental concepts in mathematical control theory is controllability, it plays a crucial role in both deterministic and stochastic control systems. Since, the controllability notion has extensive industrial and biological applications, in the literature, there are many different notions of controllability, both for linear and nonlinear dynamical systems. Controllability of the deterministic and stochastic dynamical control systems in infinite-dimensional spaces is well-developed using a different kinds of approaches. In the case of infinite dimensions, the notions of exact and approximate controllability are to be distinguished. Approximate controllability enables to steer the system into an arbitrarily small neighborhood of the final state, whereas the exact controllability means that the system can be steered to the desired final state. It has been observed in the literature on the infinite dimensional control systems that the exact controllability rarely holds (see [6, 38, 46, 53], etc). Moreover, the approximately controllable systems are more prevalent and adequate in applications, see for example, [32, 38], etc. Therefore, it is important to investigate the problem of approximate controllability of nonlinear systems. Approximate controllability of nonlinear stochastic differential and integrodifferential systems with and without delay in infinite-dimensional spaces has been extensively studied (see [1, 2, 12, 17, 30, 41, 47] for example). Recently, Muthukumar and Rajivganthi in [41] proved the approximate controllability of control systems governed by a class of impulsive neutral stochastic functional differential systems with state-dependent delay in Hilbert spaces. Ahmed [1] studied the approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space. Using the Banach fixed point theorem, Chen [12] studied the approximate controllability for semilinear stochastic equations driven by fractional Brownian motion in Hilbert spaces. Very recently, Huang and Fu [30] focused on the approximate controllability for a class of semilinear stochastic integrodifferential equations. The approximate controllability of a new class of nonlocal and non-instantaneous impulsive Hilfer fractional neutral stochastic integrodifferential equations with fractional Brownian motion was investigated by Ahmed et al. [2]. The approximate controllability of Sobolev-type fractional control problems in Hilbert spaces without uniqueness was demonstrated by Vijayakumar et al. in [48] by applying the fixed point theorem to multivalued maps with nonconvex values. Very recently, authors in [18] established and proved a set of sufficient conditions for the approximate controllability of an Atangana-Baleanu fractional neutral stochastic system with infinite delay. They did this through the utilization of the theory of multivalued maps and the Bohnenblust-Karlin fixed point theorem. Sivasankar and Udhayakumar in [44]

provided a set of appropriate conditions for the approximate controllability of second-order impulsive neutral stochastic integro-differential evolution inclusions with infinite delay by employing concepts from the cosine function, sine function, and the fixed point method.

Over the course of many years, a large number of high-quality papers of paramount importance have been published. In contrast, the number of works involving state-dependent delay integro-differential equations with impulses is still limited, and there are only a few excellent and interesting works to be found in the literature. Taking into account the papers mentioned above, the limited number of works in the area, and the desire to present new results, we present in this paper results on the approximate controllability of impulsive stochastic integral-differential equations with state-dependent delay in a Hilbert space of the form

$$\begin{cases} dy(t) = \left[ Ay(t) + \int_0^t \Gamma(t-s)y(s)ds + \xi(t, y_{\sigma(t, y_t)}) + Cv(t) \right] dt \\ \quad + \zeta(t, y_{\sigma(t, y_t)})dW(t), \quad t \in J = [0, N], \quad t \neq t_i, \\ \Delta y(t_i) = I_i(y(t_i^-)), \quad i = 1, \dots, m, \\ y(0) + \mu(x) = y_0 = \varphi \in \mathcal{B}, \end{cases} \quad (1.1)$$

where the state variable  $y$  takes values in a Hilbert space  $\mathbb{X}$ ,  $\Delta y(t_i) = y(t_i^+) - y(t_i^-)$  represents the jump in the state  $y$  at time  $t_i$ ,  $0 < t_1 < t_2 < \dots < t_m < N$ . The history  $y_s$  represents the function defined by  $y_s : (-\infty, 0] \rightarrow \mathbb{X}$ ,  $y_s(\theta) = y(s + \theta)$  belongs to the some abstract phase space  $\mathcal{B}$  described axiomatically and  $\sigma : J \times \mathcal{B} \rightarrow (-\infty, N]$  is a continuous function.  $A$  is the infinitesimal generator of a compact  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on the Hilbert space  $\mathbb{X}$ , the control function  $v$  is given in  $L^2_{\mathbb{F}}(J, \mathbb{U})$ ,  $\mathbb{U}$  is a Hilbert space, and  $C$  is a bounded linear operator from  $\mathbb{U}$  to  $\mathbb{X}$ . Let  $\mathbb{Y}$  be another Hilbert space, suppose  $\{W(t)\}_{t \geq 0}$  is a given  $\mathbb{Y}$ -valued Brownian motion or Wiener process with a finite trace nuclear covariance operator  $Q \geq 0$  defined on a complete probability space  $(\Omega, \mathbb{F}, \mathbb{P})$ . Further, let  $PC(J, L^2(\Omega, \mathbb{X})) = \{y(t) \text{ is continuous everywhere except for some } t_i \text{ at which } y(t_i^-) \text{ and } y(t_i^+) \text{ exist and } y(t_i^-) = y(t_i)\}$  be the Banach space with the norm  $\|y\|_{PC} = \sup_{t \in J} |y(t)| < \infty$ . Denote by  $PC = PC(J, L^2)$  the closed subspace of  $PC(J, L^2(\Omega, \mathbb{X}))$  consisting of a measurable and  $\mathbb{F}_t$ -adapted  $\mathbb{X}$ -valued process  $y(\cdot) \in PC(J, L^2(\Omega, \mathbb{X}))$  with the norm  $\|y\|^2 = \sup_{t \in J} \mathbb{E}\|y(t)\|_{\mathbb{X}}^2$ . The functions  $\xi, \zeta, I_i, \mu$  are appropriate functions to be specified later.

To the best of our knowledge, up to now no work has reported on the approximate controllability of impulsive stochastic integral-differential equations with state-dependent delay and nonlocal initial condition in Hilbert spaces (equation (1.1)) using the resolvent operator approach. It has been an untreated topic in the literature, and this fact is the main aim and motivation of the present work.

The main contributions of this paper are summarized as follows.

- A new class of impulsive stochastic integro-differential equations with state-dependent delay and nonlocal conditions in Hilbert spaces is formulated.
- The discussions are based on stochastic analysis theory, Krasnoselskii's fixed point theorem and the theory of resolvent operator in the sense of Grimmer.
- The result is extended to study the approximate controllability of nonlinear impulsive stochastic integro-differential equations with state-dependent delay and nonlocal conditions in Hilbert spaces.
- Finally, an example is given to illustrate the proposed theoretical results.

We will firstly in Section 2 introduce some notations, concepts, and basic results which will be needed in the sequel. The approximate controllability results for the system(1.1) are established in Section 3 by using Krasnoselskii fixed point theorem. Finally, in Section 4 we apply the obtained results to (1.1) to illustrate the applications.

## 2. Preliminaries

In this section, we recall some fundamental definitions, notations, and results, which will be used throughout the work.

Let  $(\Omega, \mathbb{F}, \mathbb{P})$  be a complete probability space furnished with a complete family of right continuous increasing sub  $\sigma$ -algebras  $\{\mathbb{F}_t, t \in J\}$  satisfying  $\mathbb{F}_t \subset \mathbb{F}$ . An  $\mathbb{X}$ -valued random variable is an  $\mathbb{F}$  measurable function  $y(t) : \Omega \rightarrow \mathbb{X}$  and the collection of random variable  $S = \{y(t, \omega) : \Omega \rightarrow \mathbb{X} | t \in J\}$  is a stochastic process. Generally, we suppress the dependence on  $\omega \in \Omega$  and write  $y(t)$  instead of  $y(t, \omega)$  and  $y(t) : J \rightarrow \mathbb{X}$  in the place of  $S$ . Let  $(\gamma_k)_{k \geq 1}$  be the sequence of real valued independent Brownian motions. Set  $W(t) = \sum_{k=1}^{\infty} \sqrt{\delta_k} \gamma_k(t) \eta_k, t \geq 0$ , where  $\{\eta_k\}_{k \geq 1}$  is a complete orthonormal basis in  $\mathbb{Y}$  and  $\delta_k \geq 0, (k = 1, 2, \dots)$  are nonnegative real numbers.

Let  $Q \in L(\mathbb{Y}, \mathbb{Y})$  be an operator defined by  $Q\eta_k = \delta_k \eta_k$  with  $\text{Tr}(Q) = \sum_{k=1}^{\infty} \delta_k < \infty$  ( $\text{Tr}(Q)$  denotes trace of  $Q$ ). Then the above  $\mathbb{Y}$ -valued stochastic process is called a  $Q$ -Wiener process.  $\mathbb{F}_t = \sigma(W(s) : 0 \leq s \leq t)$  denotes the  $\sigma$ -algebra generated by  $W$  and  $\mathbb{F}_N = \mathbb{F}_t$ . Let  $\varphi \in L(\mathbb{Y}, \mathbb{X})$  and define  $\|\varphi\|_Q^2 = \text{Tr}(\varphi Q \varphi^*) = \sum_{k=1}^{\infty} \|\sqrt{\delta_k} \varphi \eta_k\|^2 < \infty$ , then  $\varphi$  is called a  $Q$ -Hilbert Schmidt operator and we denote its set as  $L_Q(\mathbb{Y}, \mathbb{X})$  with the norm  $\|\varphi\|_Q^2 = \langle \varphi, \varphi \rangle$ .

Further in this paper, we will employ an axiomatic definition for the phase space  $\mathcal{B}$  which is introduced by Hale and Kato [29], also see [25] for details. The axioms of the space  $\mathcal{B}$  are established for  $\mathbb{F}_0$ -measurable functions from  $(-\infty, 0]$  into  $\mathbb{X}$ , endowed with a seminorm  $\|\cdot\|_{\mathcal{B}}$ , which satisfies the following fundamental axioms.

(A<sub>1</sub>) If  $y : (-\infty, N) \rightarrow \mathbb{X}$  is continuous on  $[0, N)$  and  $y_0 \in \mathcal{B}$ , then for each  $t \in [0, N)$  the following conditions hold:

- (i)  $y_t \in \mathcal{B}$ ;
- (ii)  $\|y(t)\| \leq \Lambda_1 \|y_t\|_{\mathcal{B}}$ ;
- (iii)  $\|y_t\|_{\mathcal{B}} \leq \Lambda_2(t) \|y_0\|_{\mathcal{B}} + \Lambda_3(t) \sup\{\|y(s)\|; 0 \leq s \leq N\}$ ,

where  $\Lambda_1 > 0$  is a constant,  $\Lambda_2, \Lambda_3 : [0, \infty) \rightarrow [0, \infty)$ ,  $\Lambda_2$  is locally bounded,  $\Lambda_3$  is continuous and  $\Lambda_1, \Lambda_2, \Lambda_3$  are independent of  $y$ .

(A<sub>2</sub>) For the function  $y(\cdot)$  in (A<sub>1</sub>),  $y_t$  is a  $\mathcal{B}$ -valued continuous function on  $[0, N)$ .

(A<sub>3</sub>) The space  $\mathcal{B}$  is complete.

To prove the results, we mention the following essential properties.

**Lemma 2.1** ([52]). *Let  $y : (-\infty, N] \rightarrow \mathbb{X}$  be an  $\mathbb{F}_t$ -adapted measurable process such that we have the  $\mathbb{F}_0$ -adapted process  $y_0 = \varphi(t) \in L^2(\Omega, \mathcal{B})$  and  $y|_J \in PC(J, \mathbb{X})$ , then*

$$\mathbb{E}\|y_t\|_{\mathcal{B}}^2 \leq \bar{\Lambda}_2 \mathbb{E}\|\varphi\|_{\mathcal{B}}^2 + \bar{\Lambda}_3 \sup_{0 \leq s \leq N} \{\mathbb{E}\|y(s)\|^2\},$$

where  $\bar{\Lambda}_2 = \sup_{t \in J} \Lambda_2(t), \bar{\Lambda}_3 = \sup_{t \in J} \Lambda_3(t)$ .

The next lemma is proved using the phase space axioms.

**Lemma 2.2** ([26]). *Let  $\varphi \in \mathcal{B}$  and be such that  $\varphi_t \in \mathcal{B}$  for each  $t \in \mathbb{R}_-$ . Assume that there exists a locally bounded function  $J^\varphi : \mathbb{R}_- \rightarrow [0, \infty)$  such that  $\mathbb{E}\|\varphi_t\|_{\mathcal{B}}^2 \leq J^\varphi(t) \mathbb{E}\|\varphi\|_{\mathcal{B}}^2$  for  $t \in \mathbb{R}_-$ . Let  $y : (-\infty, N] \rightarrow \mathbb{X}$  be the function such that  $y_0 = \varphi$  and  $y \in PC(J, L^2)$ , then*

$$\mathbb{E}\|y_s\|_{\mathcal{B}}^2 \leq (\bar{\Lambda}_2 + J_0^\varphi) \mathbb{E}\|\varphi\|_{\mathcal{B}}^2 + \bar{\Lambda}_3 \sup\{\mathbb{E}\|y(\beta)\|^2; \beta \in [0, \max\{0, s\}]\}, \quad s \in (-\infty, N],$$

where  $J_0^\varphi = \sup_{t \in \mathbb{R}_-} J^\varphi(t), \bar{\Lambda}_2 = \sup_{t \in J} \Lambda_2(t), \bar{\Lambda}_3 = \sup_{t \in J} \Lambda_3(t)$ .

The theory of resolvent operator plays an important role in studying the existence of solutions of Eq. (1.1). Next, we collect definitions and some basic results about this theory.

Let  $\mathbb{X}$  and  $\mathbb{M}$  be Banach spaces. We denote by  $\mathcal{L}(\mathbb{X}, \mathbb{M})$  the Banach space of bounded linear operators from  $\mathbb{X}$  to  $\mathbb{M}$  endowed with the operator norm, and we abbreviate this notation to  $\mathcal{L}(\mathbb{X})$  when  $\mathbb{X} = \mathbb{M}$ .

In what follows,  $\mathbf{X}_1$  is a Banach space,  $A$  and  $B(t)$  are closed linear operators on  $\mathbf{X}_1$ ,  $\mathbf{X}_2$  is the Banach space  $D(A)$  endowed with the graph norm  $\|z\|_{\mathbf{X}_2} = \|Az\| + \|z\|$  for  $z \in \mathbf{X}_2$  and  $\mathcal{C}(\mathbb{R}^+, \mathbf{X}_2)$  denotes the space of continuous functions from  $\mathbb{R}^+$  into  $\mathbf{X}_2$ . For further purposes, let us consider the following system

$$\begin{cases} \gamma'(t) = A\gamma(t) + \int_0^t B(t-s)\gamma(s)ds & \text{for } t \geq 0, \\ \gamma(0) = \gamma_0 \in \mathbf{X}_1. \end{cases} \quad (2.1)$$

**Definition 2.3** ([21]). A bounded linear operator valued function  $R(t) \in \mathcal{L}(\mathbf{X}_1)$ ,  $t \geq 0$  is called the resolvent operator for system (2.1) if it satisfies the following conditions:

- (i)  $R(0) = \text{Id}$  and  $\|R(t)\|_{\mathcal{L}(\mathbf{X}_1)} \leq \mathcal{D}e^{\delta t}$  for some constants  $\mathcal{D}$  and  $\delta$ .
- (ii) For all  $x \in \mathbf{X}_1$ ,  $R(t)x$  is continuous for  $t \geq 0$ .
- (iii)  $R(t) \in \mathcal{L}(\mathbf{X}_2)$  for  $t \geq 0$ . For  $x \in \mathbf{X}_2$ ,  $R(\cdot)x \in \mathcal{C}^1(\mathbb{R}_+, \mathbf{X}_1) \cap \mathcal{C}(\mathbb{R}_+, \mathbf{X}_2)$  and

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds = R(t)Ax + \int_0^t R(t-s)B(s)xds, \quad t \geq 0.$$

In the sequel, we assume that the following assumptions hold:

- (C<sub>1</sub>)  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $\mathbf{X}_1$ .
- (C<sub>2</sub>) For all  $t \geq 0$ ,  $B(t)$  is a closed linear operator from  $\mathbf{X}_2$  to  $\mathbf{X}_1$  and  $B(t) \in \mathcal{L}(\mathbf{X}_2, \mathbf{X}_1)$ . For any  $x \in \mathbf{X}_1$ , the map  $t \mapsto B(t)x$  is bounded, differentiable and the derivative  $t \mapsto B'(t)x$  is bounded and uniformly continuous for  $t \geq 0$ . In addition, there is a function  $\vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which is integrable such that for each  $x \in \mathbf{X}_1$ , the map  $t \mapsto B(t)x$  belongs to  $W^{1,1}(\mathbb{R}^+, \mathbf{X}_1)$  and  $\left\| \frac{d}{dt} B(t)x \right\| \leq \vartheta(t)\|x\|$ ,  $x \in \mathbf{X}_1$ ,  $t \in \mathbb{R}^+$ .

**Theorem 2.4** ([23]). Assume that (C<sub>1</sub>) and (C<sub>2</sub>) hold. Then, Eq. (2.1) has a unique resolvent operator  $(R(t))_{t \geq 0}$ .

We have the following important estimate.

**Lemma 2.5** ([14]). Let (C<sub>1</sub>) and (C<sub>2</sub>) be satisfied. Then, for all  $t > 0$  there exists a constant  $\omega$  such that

$$\|R(t+k) - R(k)R(t)\|_{\mathcal{L}(\mathbf{X})} \leq \omega k \text{ for } 0 \leq k \leq t \leq N.$$

The following theorem establishes the equivalence between the operator-norm continuity of the  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  and the resolvent operator for integral equations.

**Theorem 2.6** ([20]). Let  $A$  be the infinitesimal generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  and let  $(B(t))_{t \geq 0}$  satisfy (C<sub>2</sub>). Then, the resolvent operator  $(R(t))_{t \geq 0}$  for Eq. (2.1) is operator-norm continuous (or continuous in the uniform operator topology) for  $t \geq 0$  if and only if  $(T(t))_{t \geq 0}$  is operator-norm continuous for  $t \geq 0$ .

Let  $y(N; y_0, \nu)$  be the state value of (1.1) at terminal time  $N$  corresponding to the initial value  $y_0 = \varphi \in \mathcal{B}$ . To define the notion of approximate controllability we introduce the following set:

$$\mathcal{R}(N, y_0) = \{y(N; y_0, \nu) : \nu \in L^2([0, N], \mathbb{U})\},$$

which is called the reachable set of system (1.1) at terminal time  $N$ . Its closure in  $\mathbb{X}$  is denoted by  $\overline{\mathcal{R}(N, y_0)}$ .

We next present the notion of approximate controllability.

**Definition 2.7** ([39]). System (1.1) is said to be approximately controllable on the interval  $[0, N]$  if  $\mathcal{R}(N, y_0)$  is dense in  $\mathbb{X}$ , i.e.,  $\overline{\mathcal{R}(N, y_0)} = \mathbb{X}$ .

To discuss the approximate controllability of system (1.1), we introduce the following operators:

- The controllability Grammian  $\Psi_0^N$  is defined as

$$\Psi_0^N = \int_0^N R(N-s)CC^*R^*(N-s)ds,$$

where  $R^*(t)$  and  $C^*$  denote the adjoints of the operators  $R(t)$  and  $C$ , respectively.

- $\Pi(\lambda, \Psi_0^N) = (\lambda I + \Psi_0^N)^{-1}$ .

In the sequel, we always assume that the operator  $\Pi(\lambda, \Psi_0^N)$  satisfies:

(H<sub>0</sub>)  $\lambda \Pi(\lambda, \Psi_0^N) \rightarrow 0$  as  $\lambda \rightarrow 0^+$  in the strong operator topology.

From [13], hypothesis (H<sub>0</sub>) is equivalent to the fact that the linear control system:

$$\begin{cases} y'(t) = Ay(t) + \int_0^t \Gamma(t-s)y(s)ds + Cv(t) & \text{for } t \in J, \\ y(0) = y_0 \in \mathcal{B}, \end{cases} \tag{2.2}$$

corresponding to Eq. (1.1), is approximately controllable on  $J$ .

Furthermore, we have that

**Theorem 2.8** ([6, 13]). *The following statement are equivalent.*

- (a) *The control system (2.2) is approximately controllable on  $[0, N]$ .*
- (b)  *$C^*R^*(t)y = 0$  for all  $t \in [0, N]$ ,  $\Rightarrow y = 0$ .*
- (c) *The condition (H<sub>0</sub>) holds.*

**Lemma 2.9** ([37]). *For any  $\bar{y}_N \in L^2(\Omega, \mathbb{X})$ , there exists  $\bar{\varphi} \in L^2(\Omega, L^2(J, L_Q(\mathbb{Y}, \mathbb{X})))$  such that  $\bar{y}_N = \mathbb{E}\bar{y}_N + \int_0^N \bar{\varphi}(s)dW(s)$ .*

Next, we give the following fixed point principle which will be used in the sequel to prove the existence of mild solutions of Eq. (1.1).

**Lemma 2.10** ([33, 45, Krasnoselskii fixed point theorem]). *Let  $D$  be a closed, convex, and nonempty subset of a Banach space  $\mathbb{X}$ . Let  $F, G$  be the operators such that*

- (i)  *$Fx + Gy \in D$ , wherever  $x, y \in D$ ;*
- (ii)  *$F$  is compact and continuous;*
- (iii)  *$G$  is a contraction mapping.*

*Then, there exists  $z \in D$  such that  $z = Fz + Gz$ .*

Finally, we end this section by presenting the mild solutions of system (1.1).

**Definition 2.11.** A stochastic process  $y : J \times \Omega \rightarrow \mathbb{X}$  is said to be a mild solution of Eq. (1.1) if

- (a)  $y(t)$  is measurable and  $\mathbb{F}_t$ -adapted for each  $t \in J$ ;
- (b)  $y_0(\cdot) = \varphi \in \mathcal{B}$  on  $(-\infty, 0]$  satisfying  $\|\varphi\|_{\mathcal{B}} < \infty$ ;
- (c)  $y(t) \in \mathbb{X}$  satisfies the following integral equation:

$$\begin{aligned} y(t) = & R(t)[\varphi(0) - \mu(y)] + \int_0^t R(t-s)\xi(s, y_{\sigma(s, y_s)})ds + \int_0^t R(t-s)Cv(s)ds \\ & + \int_0^t R(t-s)\zeta(s, y_{\sigma(s, y_s)})dW(s) + \sum_{0 < t_i < t} R(t-t_i)I_i(y(t_i^-)). \end{aligned}$$

### 3. Approximate controllability results

In this section, we prove the approximate controllability of system (1.1). To this end, we introduce the following hypotheses.

(H<sub>φ</sub>) The function  $t \mapsto \varphi_t$  is well defined from  $\Xi(\sigma^-) = \{\sigma(s, \phi); (s, \phi) \in J \times \mathcal{B}, \sigma(s, \phi) \leq 0\}$  into  $\mathcal{B}$  and there exists a continuous and bounded function  $J^\varphi : \Xi(\sigma^-) \rightarrow (0, \infty)$  such that  $\mathbb{E}\|\varphi_t\|_{\mathcal{B}}^2 \leq J^\varphi(t)\mathbb{E}\|\varphi\|_{\mathcal{B}}^2$  for every  $t \in \Xi(\sigma^-)$ .

(H<sub>1</sub>) The resolvent operator  $(R(t))_{t \geq 0}$  is compact and there exists a constant  $M_0 \geq 1$  such that  $\|R(t)\| \leq M_0$  for all  $t \geq 0$ .

(H<sub>2</sub>) The function  $\xi : J \times \mathcal{B} \rightarrow \mathbb{X}$  is continuous and there exist two constants  $T_\xi$  and  $S_\xi$  such that

$$\mathbb{E}\|\xi(t, y)\|^2 \leq T_\xi(1 + \|y\|_{\mathcal{B}}^2), \quad \mathbb{E}\|\xi(t, y) - \xi(t, z)\|^2 \leq S_\xi\|y - z\|_{\mathcal{B}}^2.$$

(H<sub>3</sub>) The function  $\zeta : J \times \mathcal{B} \rightarrow \mathbb{X}$  is continuous and there exist two constants  $T_\zeta$  and  $S_\zeta$  such that

$$\mathbb{E}\|\zeta(t, y)\|^2 \leq T_\zeta(1 + \|y\|_{\mathcal{B}}^2), \quad \mathbb{E}\|\zeta(t, y) - \zeta(t, z)\|^2 \leq S_\zeta\|y - z\|_{\mathcal{B}}^2.$$

(H<sub>4</sub>) The functions  $I_i : \mathbb{X} \rightarrow \mathbb{X}$  are continuous and there exist nondecreasing continuous functions  $T_{I_i} : [0, \infty) \rightarrow [0, \infty)$  such that for each  $y \in \mathbb{X}$ ,

$$\mathbb{E}\|I_i(y)\|^2 \leq T_{I_i}\mathbb{E}\|y\|^2 \text{ and } \lim_{\tau \rightarrow \infty} \frac{T_{I_i}(\tau)}{\tau} = \delta_i < \infty.$$

(H<sub>5</sub>)  $\mu$  is continuous and there exists a constant  $T_\mu$  satisfying

$$\mathbb{E}\|\mu(y)\|^2 \leq T_\mu(1 + \|y\|_{\mathcal{B}}^2).$$

(H<sub>6</sub>)  $C$  is a bounded linear operator from  $\mathbb{U}$  into  $\mathbb{X}$  such that  $\|C\| = M_C$ , for a constant  $M_C \geq 0$ .

(H<sub>7</sub>) The functions  $\xi$  and  $\zeta$  are bounded uniformly.

For any  $\bar{y}_N \in L^2(\mathbb{F}, \mathbb{X})$  and  $\lambda > 0$ , we define the control function as follows.

$$\begin{aligned} v^\lambda(t) = & C^*R^*(N-t)\Pi(\lambda, \Psi_0^N) \left[ \mathbb{E}\bar{y}_N + \int_0^N \bar{\varphi}(s)dw(s) - R(T)(\varphi(0) - \mu(y)) \right. \\ & \left. - \int_0^N R(N-s)\xi(s, \sigma(s, y_s))d(s) - \int_0^N R(N-s)\zeta(s, \sigma(s, y_s))dW(s) - \sum_{0 < t_i < t} R(t-t_i)I_i(y(t_i^-)) \right]. \end{aligned}$$

**Theorem 3.1.** Assume that hypotheses (C<sub>1</sub>), (C<sub>2</sub>), (H<sub>φ</sub>), (H<sub>1</sub>)-(H<sub>6</sub>) hold. Then, Eq. (1.1) with  $v = v^\lambda(t, y)$  has at least one mild solution provided that

$$6M_0^2 \left[ (T_\mu + N^2T_\xi + N^2T_\zeta)\bar{\Lambda}_3 + m \sum_{i=1}^m T_{I_i} \right] \times \left( 1 + \frac{7M_C^4 M_0^4 N^2}{\lambda^2} \right) < 1, \tag{3.1a}$$

$$S^* := 2M_0^2 N^2 (S_\xi + S_\zeta)\bar{\Lambda}_3 < 1. \tag{3.1b}$$

*Proof.* Let  $\mathbb{K} = \{y \in PC(J, L^2) : y(0) = \varphi(0)\}$  be the space endowed with the uniform convergence topology. Using the control  $v^\lambda(t)$ , we define the operator  $\Upsilon : \mathbb{K} \rightarrow \mathbb{K}$  by

$$\begin{aligned} \Upsilon y(t) &= R(t)[\varphi(0) - \mu(y)] + \int_0^t R(t-s)\xi(s, y_{\sigma(s, y_s)})ds + \int_0^t R(t-s)Cv^\lambda(s)ds \\ &\quad + \int_0^t R(t-s)\zeta(s, y_{\sigma(s, y_s)})dW(s) + \sum_{0 < t_i < t} R(t-t_i)I_i(y(t_i^-)). \end{aligned}$$

It is clear from the definition of  $\Upsilon$ , for  $\lambda > 0$ , that the fixed point of  $\Upsilon$  is a mild solution of system (1.1). Thus, to prove the existence of mild solutions, it suffices to prove that  $\Upsilon$  has a fixed point. For each  $\tau > 0$  let  $E_\tau = \{y \in \mathbb{K} : \mathbb{E}\|y\|^2 \leq \tau\}$ . Then,  $E_\tau$  is a bounded closed convex subset in  $\mathbb{K}$ . For convenience, we split the proof into a sequence of steps.

**Step 1:** We can claim that  $\Upsilon E_\tau \subset E_\tau$ . Assume that our claim is not true, then there exists a constant  $\lambda > 0$  such that for every  $\tau > 0$ , there exists a function  $y^\tau \in E_\tau$  and  $t^\tau \in J$  such that  $\mathbb{E}\|\Upsilon y^\tau(t^\tau)\|^2 > \tau$ . From Lemma 2.2, it follows that  $\mathbb{E}\|y_{\sigma(t^\tau, y_{t^\tau}^\tau)}^\tau\|^2 \leq (\bar{\Lambda}_2 + J_0^0)\mathbb{E}\|\varphi\|_{\mathbb{B}}^2 + \bar{\Lambda}_3\tau := \tau^*$ , then we obtain

$$\begin{aligned} \mathbb{E}\|v^\lambda(t^\tau, y^\tau)\|^2 &= \mathbb{E}\left\|C^*R^*(N-t)\Pi(\lambda, \Psi_0^N)\left[\mathbb{E}\bar{y}_N + \int_0^N \bar{\varphi}(s)dw(s) - R(T)(\varphi(0) - \mu(y))\right.\right. \\ &\quad \left.- \int_0^N R(N-s)\xi(s, \sigma(s, y_s))d(s) - \int_0^N R(N-s)\zeta(s, \sigma(s, y_s))dW(s)\right. \\ &\quad \left.- \sum_{0 < t_i < t} R(t-t_i)I_i(y(t_i^-))\right]\left\| \right. \\ &\leq \frac{7M_C^2 M_0^2}{\lambda^2} \left[ \mathbb{E}\|\bar{y}_N\|^2 + \int_0^N \mathbb{E}\|\bar{\varphi}(s)\|^2 ds + M_0^2 \mathbb{E}\|\varphi(0)\|^2 + M_0^2 \mathbb{E}\|\mu(y^\tau)\|^2 \right. \\ &\quad \left. + N \int_0^{t^\tau} \mathbb{E}\|R(t-s)\xi(s, y_{\sigma(s, y_s)^\tau}^\tau)\|^2 d(s) + \text{Tr}(Q)N \int_0^{t^\tau} \mathbb{E}\|R(t-s)\zeta(s, y_{\sigma(s, y_s)^\tau}^\tau)\|^2 d(s) \right. \\ &\quad \left. + M_0^2 m \sum_{i=1}^m \mathbb{E}\|I_i(y(t_i^-))\|^2 \right] \\ &\leq \frac{7M_C^2 M_0^2}{\lambda^2} \left[ \mathbb{E}\|\bar{y}_N\|^2 + \int_0^N \mathbb{E}\|\bar{\varphi}(s)\|^2 ds + M_0^2 \mathbb{E}\|\varphi(0)\|^2 + M_0^2 \mathbb{E}\|\mu(y^\tau)\|^2 \right. \\ &\quad \left. + M_0^2 N \int_0^{t^\tau} \mathbb{E}\|\xi(s, y_{\sigma(s, y_s)^\tau}^\tau)\|^2 d(s) + \text{Tr}(Q)M_0^2 N \int_0^{t^\tau} \mathbb{E}\|\zeta(s, y_{\sigma(s, y_s)^\tau}^\tau)\|^2 d(s) \right. \\ &\quad \left. + M_0^2 m \sum_{i=1}^m \mathbb{E}\|I_i(y(t_i^-))\|^2 \right] \\ &\leq \frac{7M_C^2 M_0^2}{\lambda^2} \left[ \mathbb{E}\|\bar{y}_N\|^2 + \int_0^N \mathbb{E}\|\bar{\varphi}(s)\|^2 ds + M_0^2 \mathbb{E}\|\varphi(0)\|^2 + M_0^2 T_\mu(1 + \tau^*) \right. \\ &\quad \left. + M_0^2 N^2 T_\xi(1 + \tau^*) + M_0^2 N^2 \text{Tr}(Q)T_\zeta(1 + \tau^*) + M_0^2 m \sum_{i=1}^m T_{I_i}\tau \right], \end{aligned}$$

and

$$\begin{aligned} \tau &< \mathbb{E}\|\Upsilon y^\tau(t^\tau)\|^2 \\ &\leq 6M_0^2(\mathbb{E}\|\varphi(0)\|^2 + \mathbb{E}\|\mu(y^\tau)\|^2) + 6\mathbb{E}\left\|\int_0^{t^\tau} R(t^\tau-s)Cv^\lambda(s)ds\right\|^2 \end{aligned}$$



$$\begin{aligned}
 &+ 6\mathbb{E} \left\| \int_0^{t^\tau} R(t^\tau - s) \xi(s, y_{\sigma(s, y_s^\tau)}) ds \right\|^2 + 6\mathbb{E} \left\| \int_0^{t^\tau} R(t^\tau - s) \zeta(s, y_{\sigma(s, y_s^\tau)}) dW(s) \right\|^2 \\
 &+ 6M_0^2 m \sum_{i=1}^m \mathbb{E} \|I_i(y(t_i^{\tau-}))\|^2 \\
 \leq &6M_0^2 [\mathbb{E} \|\varphi(0)\|^2 + T_\mu(1 + \tau^*)] + 6M_0^2 M_C^2 N \int_0^{t^\tau} \mathbb{E} \|\nu^\lambda(s)\|^2 ds + 6M_0^2 N \int_0^{t^\tau} \mathbb{E} \|\xi(s, y_{\sigma(s, y_s^\tau)})\|^2 ds \\
 &+ 6M_0^2 N \text{Tr}(Q) \int_0^{t^\tau} \mathbb{E} \|\zeta(s, y_{\sigma(s, y_s^\tau)})\|^2 ds + 6M_0^2 m \sum_{i=1}^m T_{I_i} \tau \\
 \leq &6M_0^2 [\mathbb{E} \|\varphi(0)\|^2 + T_\mu(1 + \tau^*)] + 6M_0^2 M_C^2 N^2 \mathbb{E} \|\nu^\lambda(t^\tau)\|^2 \\
 &+ 6M_0^2 N^2 T_\xi(1 + \tau^*) + 6M_0^2 N^2 T_\zeta(1 + \tau^*) + 6M_0^2 m \sum_{i=1}^m T_{I_i} \tau.
 \end{aligned}$$

Dividing both sides by  $\tau$  and taking the limit as  $\tau \rightarrow \infty$ , we get

$$1 \leq 6M_0^2 \left[ (T_\mu + N^2 T_\xi + N^2 T_\zeta) \bar{\Lambda}_3 + m \sum_{i=1}^m T_{I_i} \right] \times \left( 1 + \frac{7M_C^4 M_0^4 N^2}{\lambda^2} \right),$$

and this contradicts (3.1a). Hence, for each  $\tau > 0$ , there exists some positive number  $\tau$  such that  $\Upsilon E_\tau \subset E_\tau$ . Next, we set  $\Upsilon := \Upsilon_1 + \Upsilon_2$ , where

$$\begin{aligned}
 \Upsilon_1 y(t) &= \int_0^t R(t-s) \xi(s, y_{\sigma(s, y_s)}) ds + \int_0^t R(t-s) \zeta(s, y_{\sigma(s, y_s)}) dW(s), \\
 \Upsilon_2 y(t) &= R(t)[\varphi(0) - \mu(y)] + \int_0^t R(t-s) C \nu^\lambda(s) ds + \sum_{0 < t_i < t} R(t-t_i) I_i(y(t_i^-)).
 \end{aligned}$$

**Step2:**  $\Upsilon_1(y)$  is a contraction. Let  $y, z \in E_\tau$ , then

$$\begin{aligned}
 \mathbb{E} \|\Upsilon_1 y(t) - \Upsilon_1 z(t)\|^2 &\leq 2\mathbb{E} \left\| \int_0^t R(t-s) [\xi(s, y_{\sigma(s, y_s)}) - \xi(s, z_{\sigma(s, z_s)})] ds \right\|^2 \\
 &\quad + 2\mathbb{E} \left\| \int_0^t R(t-s) [\zeta(s, y_{\sigma(s, y_s)}) - \zeta(s, z_{\sigma(s, z_s)})] dW(s) \right\|^2 \\
 &\leq 2M_0^2 N \int_0^t \mathbb{E} \|\xi(s, y_{\sigma(s, y_s)}) - \xi(s, z_{\sigma(s, z_s)})\|^2 ds \\
 &\quad + 2M_0^2 N \int_0^t \mathbb{E} \|\zeta(s, y_{\sigma(s, y_s)}) - \zeta(s, z_{\sigma(s, z_s)})\|^2 ds \\
 &\leq 2M_0^2 N^2 S_\xi \|y_{\sigma(s, y_s)} - z_{\sigma(s, z_s)}\|^2 + 2M_0^2 N^2 S_\zeta \|y_{\sigma(s, y_s)} - z_{\sigma(s, z_s)}\|^2 \\
 &\leq 2M_0^2 N^2 (S_\xi + S_\zeta) M_3 \sup_{0 \leq s \leq N} \|y(s) - z(s)\|^2 \\
 &\leq 2M_0^2 N (S_\xi + S_\zeta) M_3 \|y - z\|_{PC}^2 = S^* \|y - z\|_{PC}^2.
 \end{aligned}$$

Therefore, by (3.1b), we conclude that  $\Upsilon_1$  is a contraction mapping on  $E_\tau$ .

**Step3:**  $\Upsilon_2$  maps bounded sets to bounded sets in  $E_\tau$ . Let  $y \in E_\tau$ , we have

$$\begin{aligned} \mathbb{E}\|\Upsilon_2 y(t)\|^2 &\leq 3\mathbb{E}\|R(t)[\varphi(0) - \mu(y)]\|^2 + 3\mathbb{E}\left\|\int_0^t R(t-s)Cv^\lambda(s)ds\right\|^2 + 3\mathbb{E}\left\|\sum_{i=1}^m R(t-t_i)I_i(y(t_i^{\tau-}))\right\|^2 \\ &\leq 6M_0^2[\mathbb{E}\|\varphi(0)\|^2 + \mathbb{E}\|\mu(y)\|^2] + 3M_0^2M_C^2N \int_0^t \mathbb{E}\|v^\lambda(s)\|^2 ds + 3M_0^2m \sum_{i=1}^m \mathbb{E}\|I_i(y(t_i^{\tau-}))\|^2 \\ &\leq 6M_0^2[\mathbb{E}\|\varphi(0)\|^2 + T_\mu(1 + \tau^*)] + 3M_0^2M_C^2N^2\mathbb{E}\|v^\lambda\|^2 + 3M_0^2m \sum_{i=1}^m T_{I_i}\tau = T^* < \infty. \end{aligned}$$

Consequently,  $\Upsilon_2 E_\tau \subset \Upsilon_2$

**Step 4:**  $\Upsilon_2$  is continuous. In fact let  $0 < t < t + \varepsilon \leq N$ , where  $t$ ,  $t + \varepsilon$  and  $|\varepsilon|$  is sufficiently small, we have

$$\begin{aligned} \mathbb{E}\|\Upsilon_2 y(t + \varepsilon) - \Upsilon_2 y(t)\|^2 &\leq 4\mathbb{E}\|[R(t + \varepsilon) - R(t)][\varphi(0) - \mu(y)]\|^2 + 4\mathbb{E}\left\|\sum_{i=1}^m [R(t + \varepsilon - t_i) - R(t - t_i)]I_i(y(t_i^{\tau-}))\right\|^2 \\ &\quad + 4\mathbb{E}\left\|\int_0^t [R(t + \varepsilon - s) - R(t - s)]Cv^\lambda(s)ds\right\|^2 + 4\mathbb{E}\left\|\int_t^{t+\varepsilon} R(t + \varepsilon - s)Cv^\lambda(s)ds\right\|^2 \\ &\leq 8\|[R(t + \varepsilon) - R(t)]\|^2[\mathbb{E}\|\varphi(0)\|^2 + T_\mu(1 + \tau^*)] + 4 \sum_{i=1}^m \|R(t + \varepsilon - t_i) - R(t - t_i)\|^2 T_{I_i} \tau \\ &\quad + \int_0^t \|R(t + \varepsilon - s) - R(t - s)\|^2 \mathbb{E}\|Cv^\lambda(s)\|^2 ds + \int_t^{t+\varepsilon} \|R(t + \varepsilon - s)\|^2 \mathbb{E}\|Cv^\lambda(s)\|^2 ds. \end{aligned}$$

By the continuity of  $(R(t))_{t \geq 0}$  in the operator-norm topology and the dominated convergence Theorem, we see that the right hand side of the above inequality tends to zero as  $\varepsilon \rightarrow 0$ . Thus,  $\Upsilon_2(E_\tau)$  is equicontinuous on  $J$ .

**Step 5:**  $F(t) = \{\Upsilon_2 y(t), y \in E_\tau\}$  is relatively compact in  $E_\tau$ . We decompose  $\Upsilon_2$  as  $P_1 + P_2$  where  $P_1$  and  $P_2$  are the operators on  $E_\tau$  defined, respectively, by

$$P_1 y(t) = R(t)[\varphi(0) - \mu(y)] + \int_0^t R(t-s)Cv^\lambda(s)ds$$

and

$$P_2 y(t) = \sum_{0 < t_i < t} R(t-t_i)I_i(y(t_i^-)).$$

We now prove that  $F_1(t) = \{Gy(t), y \in E_\tau\}$ , where  $Gy(t) = \int_0^t R(t-s)Cv^\lambda(s)ds$ . For this purpose, for any  $\eta \in (0, 1)$  we define the following operators

$$G^\eta y(t) = R(\eta) \int_0^{t-\eta} R(t-s-\eta)Cv^\lambda(s)ds, \quad G^{*\eta} y(t) = \int_0^{t-\eta} R(t-s)Cv^\lambda(s)ds.$$

It follows from the compactness of  $R(\eta)$  that the set  $\{G^\eta y(t), y \in E_\tau\}$  is relatively compact in  $E_\tau$ . Moreover,

using Lemma 2.5 and Hölder inequality for each  $y \in E_\tau$ , we have

$$\begin{aligned} \mathbb{E}\|G^\eta y(t) - G^{*\eta} y(t)\|^2 &= \mathbb{E}\left\| \int_0^{t-\eta} R(\eta)R(t-s-\eta)Cv^\lambda(s)ds - \int_0^{t-\eta} R(t-s)Cv^\lambda(s)ds \right\|^2 \\ &\leq M_C^2 t \int_0^{t-\eta} \|R(\eta)R(t-\eta-s) - R(t-s)\|^2 \mathbb{E}\|v^\lambda(s)\|^2 ds \\ &\leq M_C^2 t(L\eta)^2 \int_0^{t-\eta} \mathbb{E}\|v^\lambda(s)\|^2 ds \rightarrow 0 \text{ as } \eta \rightarrow 0. \end{aligned}$$

Therefore, the set  $\{G^{*\eta} y(t), y \in E_\tau\}$  is relatively compact in  $E_\tau$  by using the total boundedness. Applying this method again, we get

$$\begin{aligned} \mathbb{E}\|Gy(t) - G^{*\eta} y(t)\|^2 &= \mathbb{E}\left\| \int_0^t R(t-s)Cv^\lambda(s)ds - \int_0^{t-\eta} R(t-s)Cv^\lambda(s)ds \right\|^2 \\ &= \mathbb{E}\left\| \int_{t-\eta}^t R(t-s)Cv^\lambda(s)ds \right\|^2 \leq M_0^2 M_C^2 \eta \int_{t-\eta}^t \mathbb{E}\|v^\lambda(s)\|^2 ds \rightarrow 0 \text{ as } \eta \rightarrow 0. \end{aligned}$$

Hence,  $G^{*\eta}(y)$  converges uniformly to  $G(y)$  and then  $F_1(t)$  is relatively compact in  $E_\tau$ .

To prove the compactness of  $P_2$ , note that

$$P_2 y(t) = \sum_{0 < t_i < t} R(t-t_i)I_i(y(t_i^-)) = \begin{cases} 0, & t \in [0, t_1], \\ R(t-t_1)I_1(y(t_1^-)), & t \in (t_1, t_2], \\ \vdots \\ \sum_{i=1}^m R(t-t_i)I_i(y(t_i^-)), & t \in (t_i, N], \end{cases}$$

and that the interval  $[0, N]$  is divided into finite subintervals by  $t_i, i = 1, 2, \dots, m$ . Thus, we only need to prove that

$$\Theta(t) = \{R(t-t_1)I_1(y(t_1^-)), t \in (t_1, t_2], y \in E_\tau\}$$

is relatively compact in  $E_\tau$ , as the cases for other subintervals are the same.

Indeed, from  $(H_1)$  and  $(H_4)$ , it follows that the set  $\{R(t-t_1)I_1(y(t_1^-)), y \in E_\tau\}$  is relatively compact in  $E_\tau$  for all  $t \in [t_1, t_2]$ . Then,  $F(t) = \{\Upsilon_2 y(t), y \in E_\tau\}$  is relatively compact in  $E_\tau$ . By the Arzelà-Ascoli theorem,  $\Upsilon_2$  is completely continuous. Finally, by means of the Krasnoselskii fixed point theorem, the operator  $\Upsilon$  has a fixed point, which is a mild solution of system (1.1). The proof is complete.  $\square$

We are now on the position to prove the approximate controllability of system (1.1).

**Theorem 3.2.** Assume that  $(H_0)$ ,  $(H_7)$ , and the assumptions of Theorem 3.1 are satisfied. Then, system (1.1) is approximately controllable on  $J$ .

*Proof.* Let  $y^\lambda$  be a solution of (1.1), then it is not difficult to see that

$$\begin{aligned} y^\lambda(N) &= \bar{y}_N - \lambda \Pi(\lambda, \Psi_0^N) \left[ \mathbb{E}\bar{y}_N + \int_0^N \bar{\varphi}(s)dw(s) - R(N)[\varphi(0) - \mu(y^\lambda(N))] \right. \\ &\quad + \int_0^N R(N-s)\xi(s, y_{\sigma(s, y_s^\lambda)}^\lambda)ds + \int_0^N R(N-s)\zeta(s, y_{\sigma(s, y_s^\lambda)}^\lambda)dW(s) \\ &\quad \left. + \sum_{0 < t_i < N} R(t-t_i)I_i(y(t_i^-)) \right]. \end{aligned} \tag{3.2}$$

By the uniform boundedness of  $\xi$  and  $\zeta$ , there exist subsequences still denoted by  $\{\xi(s, y_{\sigma(s, y_s^\lambda)}^\lambda)\}_\lambda$  and  $\{\zeta(s, y_{\sigma(s, y_s^\lambda)}^\lambda)\}_\lambda$ , which converge weakly to  $\xi(s)$  and  $\zeta(s)$ , respectively. Therefore, by (3.2),  $(H_0)$ , the Lebesgue dominated convergence theorem, and the compactness of  $R(t)$ , it follows that

$$\begin{aligned} \mathbb{E}\|y^\lambda(N) - \bar{y}_N\|^2 &= \mathbb{E}\left\| \lambda\Pi(\lambda, \Psi_0^N) \left[ \mathbb{E}\bar{y}_N + \int_0^N \bar{\varphi}(s) d\omega(s) - R(N)[\varphi(0) - \mu(y^\lambda(N))] \right. \right. \\ &\quad \left. \left. + \int_0^N R(N-s)\xi(s, y_{\sigma(s, y_s^\lambda)}^\lambda) ds + \int_0^N R(N-s)\zeta(s, y_{\sigma(s, y_s^\lambda)}^\lambda) dW(s) \right. \right. \\ &\quad \left. \left. + \sum_{0 < t_i < N} R(t-t_i)I_i(y(t_i^-)) \right] \right\|^2 \\ &\leq 6\mathbb{E}\left\| \lambda\Pi(\lambda, \Psi_0^N) \left[ \mathbb{E}\bar{y}_N + \int_0^N \bar{\varphi}(s) d\omega(s) - R(N)[\varphi(0) - \mu(y^\lambda(N))] \right] \right\|^2 \\ &\quad + 6\mathbb{E}\left( \lambda\Pi(\lambda, \Psi_0^N) \int_0^N \|R(N-s)[\xi(s, y_{\sigma(s, y_s^\lambda)}^\lambda) - \xi(s)]\| ds \right)^2 \\ &\quad + 6\mathbb{E}\left( \lambda\Pi(\lambda, \Psi_0^N) \int_0^N \|R(N-s)[\zeta(s, y_{\sigma(s, y_s^\lambda)}^\lambda) - \zeta(s)]\| dW(s) \right)^2 \\ &\quad + 6\mathbb{E}\|\lambda\Pi(\lambda, \Psi_0^N) \sum_{0 < t_i < N} R(N-t_i)I_i(y^\lambda(t_i^-))\|^2 \rightarrow 0, \text{ as } \lambda \rightarrow 0^2. \end{aligned}$$

Hence,  $y^\lambda(N) \rightarrow \bar{y}_N$  holds in  $\mathbb{X}$  and then we obtain the approximate controllability of system (1.1).  $\square$

#### 4. Example

To illustrate our main results, we consider the following control system governed the stochastic partial integro-differential equation with state-dependent delay

$$\left\{ \begin{aligned} dz(t, x) &= \left[ \frac{\partial^2}{\partial x^2} z(t, x) + \int_0^t \gamma(t-s) \frac{\partial^2}{\partial x^2} z(s, x) ds + \lambda \mathcal{S}(t, x) \right] dt \\ &\quad + \frac{1}{25} \int_{-\infty}^t \sin(s-t) e^{s-t} z(s-t) \frac{1+2\|z(t)\|}{1+4\|z(t)\|} ds dt \\ &\quad + \frac{1}{36} \int_{-\infty}^t e^{s-t} z(s-t) \frac{1+2\|z(t)\|}{1+4\|z(t)\|} ds d\omega(t), \quad t \in [0, N] \setminus \{t_1, t_2, \dots, t_m\}, \\ z(t, 0) &= z(t, \pi) = 0, \\ z(\theta, x) + \sum_{j=1}^n \cos(z(t_j, x)) &= \varphi(\theta, x), \quad \theta \in (-\infty, 0], \quad x \in [0, \pi], \\ \Delta z(t_i, x) &= \frac{1}{18} \int_{-\infty}^{t_i} e^{s-t_i} z(s, x) ds, \quad i = 1, 2, \dots, m, \end{aligned} \right. \tag{4.1}$$

where  $0 < t_1 < \dots < t_m < N$  are prefixed numbers,  $\omega(t)$  is a one-dimensional standard Brownian motion defined on the filtered probability space  $(\Omega, \mathbb{F}, \mathbb{P})$  and  $\gamma$  is bounded and  $C^1$ -function such that  $\gamma'$  is bounded and uniformly continuous.

Let  $\mathbb{X} = \mathbb{U} = L^2([0, \pi])$  and define the operator  $A : D(A) \subset \mathbb{X} \rightarrow \mathbb{X}$  as

$$\begin{cases} D(A) = H^2([0, \pi]) \cap H_0^1([0, \pi]), \\ A\kappa = \frac{\partial^2}{\partial x^2} \kappa. \end{cases}$$

Then,  $A\kappa = -\sum_{n=1}^{\infty} n^2 \langle \kappa, s_n \rangle s_n$ , where  $s_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ ,  $n = 1, 2, \dots$  is the orthogonal basis of eigenvectors of  $A$ .

**Theorem 4.1** (Theorem 4.1.2, p.79 of [49]). *A is the infinitesimal generator of a  $C_0$ -semigroup on  $L^2([0, \pi])$ .*

$A$  generates a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $L^2([0, \pi])$ . Hence,  $(C_1)$  holds. Furthermore, the above  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  is compact for  $t \geq 0$  and then it is operator-norm continuous for  $t \geq 0$ .

Next we define the operator  $\Gamma : D(A) \subset \mathbb{X} \rightarrow \mathbb{X}$  by

$$\Gamma(t)z = \gamma(t)A \text{ for } t \geq 0 \text{ and } z \in D(A).$$

Let  $\tau \geq 0$ ,  $q \geq 1$  and let  $p : (-\infty, -\tau] \rightarrow \mathbb{R}$  be a non-negative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of Hino et al. [29]. Briefly, this means that  $p$  is locally integrable and there is a non-negative, locally bounded function  $\sigma$  on  $(-\infty, 0]$  such that  $p(\xi + \delta) \leq \sigma(\xi)p(\delta)$  for all  $\xi \leq 0$  and  $\delta \in (-\infty, -\tau] \setminus N_\theta$ , where  $N_\theta \subseteq (-\infty, -\tau]$  is a set whose Lebesgue measure equals zero. We denote by  $PC_\tau \times L^q(p, \mathbb{X})$  the set consists of all classes of functions  $\varphi : (-\infty, 0] \rightarrow \mathbb{X}$  such that  $\varphi|_{[-\tau, 0]} \in PC([-\tau, 0], \mathbb{X})$ ,  $\varphi(\cdot)$  is Lebesgue measurable on  $(-\infty, -\tau)$ , and  $p\|\varphi\|^\delta$  is Lebesgue integrable on  $(-\infty, -\tau)$ . The seminorm is given by

$$\|\varphi\|_{\mathcal{B}} = \sup_{-\tau \leq \delta \leq 0} \|\varphi(\delta)\| + \left( \int_{-\infty}^{-\tau} p(\delta) \|\varphi\|^q d\delta \right)^{1/q}.$$

The space  $\mathcal{B} = PC_\tau \times L^q(p, \mathbb{X})$  satisfies axioms  $(A_1)$ - $(A_3)$ . Moreover, when  $\tau = 0$  and  $q = 2$ , we can take  $\Lambda_1 = 1, \Lambda_2(t) = \sigma(-t)^{1/2}$  and  $\Lambda_3(t) = 1 + \left( \int_{-t}^0 p(\delta) d\delta \right)^{1/2}$  for  $t \geq 0$  (see [29, Theorem 1.3.8] for details).

To transform system (4.1) in the abstract form, we define

$$\begin{aligned} y(t)(x) &= z(t, x) \text{ for } t \geq 0 \text{ for } x \in [0, \pi], & \varphi(\theta)(\delta) &= y_0(\theta, x) \text{ for } \theta \in (-\infty, 0] \text{ and } x \in [0, \pi], \\ \xi(t, \vartheta)(x) &= \frac{1}{25} \int_{-\infty}^0 \sin(s) e^{s\vartheta} ds, & \sigma(t, \vartheta) &= t \frac{1 + 2\|\vartheta(0)\|}{1 + 4\|\vartheta(0)\|}, \\ \zeta(t, \vartheta)(x) &= \frac{1}{36} \int_{-\infty}^0 e^{s\vartheta} ds, & I_i(\vartheta)(x) &= \frac{1}{18} \int_{-\infty}^0 e^{s\vartheta} ds, \\ \mu(\vartheta(t, x)) &= \mu(\vartheta)(x) = \sum_{j=1}^n \cos(\vartheta(t_j, x)). \end{aligned}$$

Let  $C : \mathbb{U} \rightarrow \mathbb{X}$  be defined by  $Cv(t)(x) = S(t, x)$ ,  $0 \leq x \leq \pi, v \in \mathbb{U}$ , where  $S : J \times [0, \pi] \rightarrow \mathbb{X}$  is continuous. Therefore, under the above definitions, we can represent the system (4.1) in the abstract form

$$\begin{cases} dy(t) = \left[ Ay(t) + \int_0^t \Gamma(t-s)y(s) ds + \xi(t, y_{\sigma(t, y_t)}) + Cv(t) \right] dt \\ \quad + \zeta(t, y_{\sigma(t, y_t)}) dW(t), \quad t \in J = [0, N], \quad t \neq t_i, \\ \Delta y(t_i) = I_i(y(t_i^-)), \quad i = 1, \dots, m, \\ y(0) + \mu(x) = y_0 = \varphi \in \mathcal{B}. \end{cases}$$

Moreover,  $\Gamma(t)$  fulfils  $(C_2)$ . Consequently, in virtue of Theorem 2.4, Eq. (2.1) has a unique resolvent operator  $(R(t))_{t \geq 0}$  on  $\mathbb{X}$ , which is also operator-norm continuous for  $t \geq 0$  thanks to Theorem 2.6.

For any  $(t, y) \in [0, N] \times \mathcal{B}$ , we compute

$$\begin{aligned} \mathbb{E}\|\xi(t, y)\|_{\mathbb{X}}^2 &\leq \mathbb{E}\left[\int_0^\pi \left(\int_{-\infty}^0 \left\|\frac{\sin(s)e^s}{25}y\right\| ds\right)^2 dx\right] \\ &\leq \frac{\pi^2}{25^2} \mathbb{E}\left[\left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right)^{1/2} \left(\int_{-\infty}^0 p(s)\|y\|^2 ds\right)^{1/2}\right]^2 \\ &\leq \frac{\pi^2}{625} \left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right) \mathbb{E}\left(\int_{-\infty}^0 p(s)\|y\|^2 ds\right) \leq T_\xi(1 + \|y\|_{\mathcal{B}}^2), \end{aligned}$$

where  $T_\xi = \frac{\pi^2}{625} \int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds,$

$$\begin{aligned} \mathbb{E}\|\zeta(t, y)\|_{\mathbb{X}}^2 &\leq \mathbb{E}\left[\int_0^\pi \left(\int_{-\infty}^0 \left\|\frac{e^s}{36}y\right\| ds\right)^2 dx\right] \leq \frac{\pi^2}{36^2} \mathbb{E}\left[\left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right)^{1/2} \left(\int_{-\infty}^0 p(s)\|y\|^2 ds\right)^{1/2}\right]^2 \\ &\leq \frac{\pi^2}{1296} \left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right) \mathbb{E}\left(\int_{-\infty}^0 p(s)\|y\|^2 ds\right) \leq T_\zeta(1 + \|y\|_{\mathcal{B}}^2), \end{aligned}$$

where  $T_\zeta = \frac{\pi^2}{1296} \int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds,$

$$\begin{aligned} \mathbb{E}\|I_i(t, y)\|_{\mathbb{X}}^2 &\leq \mathbb{E}\left[\int_0^\pi \left(\int_{-\infty}^0 \left\|\frac{e^s}{18}y\right\| ds\right)^2 dx\right] \leq \frac{\pi^2}{36^2} \mathbb{E}\left[\left(\int_{-\infty}^0 e^{2s} ds\right)^{1/2} \left(\int_{-\infty}^0 \|y\|^2 ds\right)^{1/2}\right]^2 \\ &\leq \frac{\pi^2}{1296} \left(\int_{-\infty}^0 e^{2s} ds\right) \left(\int_{-\infty}^0 \mathbb{E}\|y\|^2 ds\right) \leq T_{I_i} \mathbb{E}\|y\|^2, \end{aligned}$$

where  $T_{I_i} = \frac{\pi^2}{1296} \int_{-\infty}^0 e^{2s} ds.$

On the other hand, for all  $(t, y_i) \in [0, N] \times \mathcal{B}$ ,  $i = 1, 2$ , we estimate:

$$\begin{aligned} \mathbb{E}\|\xi(t, y_1) - \xi(t, y_2)\|_{\mathbb{X}}^2 &\leq \mathbb{E}\left[\int_0^\pi \left(\int_{-\infty}^0 \frac{\sin(s)e^s}{25} \|y_1 - y_2\| ds\right)^2 dx\right] \\ &\leq \frac{\pi^2}{25^2} \mathbb{E}\left[\left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right)^{1/2} \left(\int_{-\infty}^0 p(s)\|y_1 - y_2\|^2 ds\right)^{1/2}\right]^2 \\ &\leq \frac{\pi^2}{625} \left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right) \mathbb{E}\left(\int_{-\infty}^0 p(s)\|y_1 - y_2\|^2 ds\right) \leq S_\xi \|y_1 - y_2\|_{\mathcal{B}}^2, \end{aligned}$$

where  $S_\xi = \frac{\pi^2}{625} \int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds,$

$$\begin{aligned} \mathbb{E}\|\zeta(t, y_1) - \zeta(t, y_2)\|_{\mathbb{X}}^2 &\leq \mathbb{E}\left[\int_0^\pi \left(\int_{-\infty}^0 \frac{e^s}{36} \|y_1 - y_2\| ds\right)^2 dx\right] \\ &\leq \frac{\pi^2}{36^2} \mathbb{E}\left[\left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right)^{1/2} \left(\int_{-\infty}^0 p(s)\|y_1 - y_2\|^2 ds\right)^{1/2}\right]^2 \\ &\leq \frac{\pi^2}{1296} \left(\int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds\right) \mathbb{E}\left(\int_{-\infty}^0 p(s)\|y_1 - y_2\|^2 ds\right) \leq S_\zeta \|y_1 - y_2\|_{\mathcal{B}}^2, \end{aligned}$$

where  $S_\zeta = \frac{\pi^2}{1296} \int_{-\infty}^0 \frac{e^{2s}}{p(s)} ds$ . Hence, the functions  $\xi$ ,  $\zeta$ , and  $I_i$  satisfy hypotheses (H<sub>2</sub>), (H<sub>3</sub>), and (H<sub>4</sub>), respectively. Similarly we can show that the function  $\mu$  satisfies (H<sub>5</sub>). Also,  $\xi$  and  $\zeta$  are bounded linear operators,  $\|\xi\| \leq T_\xi$ ,  $\|\zeta\| \leq T_\zeta$  and therefore (H<sub>7</sub>) is verified. It remains now to check that (H<sub>0</sub>) is fulfilled. To this end, we have the following result:

**Lemma 4.2** ([40]). *Let  $\gamma(t) \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$  with primitive  $\Theta(t) \in L^1_{loc}(\mathbb{R}^+)$  such that  $\Theta(t)$  is non-positive, non-decreasing and  $\Theta(0) = -1$ . If operator  $A$  is self-adjoint and positive semi-definite, then the resolvent operator  $R(t)$  associated to (1.1) is self-adjoint as well.*

By Lemma 4.2 above, the resolvent operator  $R(t)$  of Eq. (4.1) is self-adjoint. Therefore it follows that

$$C^*R^*(t)x = R(t)x, \text{ for any } x \in \mathbb{X}.$$

Let now  $C^*R^*(t)x = 0$  for every  $t \in [0, N]$ . Then,  $R(t)x = 0$ , for any  $t \in [0, N]$ . Since  $R(0) = \text{Id}$ , we deduce that  $x = 0$ . Hence, from [13, Theorem 4.1.7], we conclude that the linear control system corresponding to Eq. (4.1) is approximately controllable on  $[0, N]$ , and then (H<sub>0</sub>) is satisfied. Thus, all the conditions of Theorem 3.2 hold, and so we conclude that the system (4.1) is approximately controllable on  $[0, N]$ .

## 5. Conclusion

In this manuscript, we investigated the approximate controllability for a class of impulsive stochastic integro-differential equations with nonlocal conditions and state-dependent delays. Using the resolvent operator in the sense of Grimmer, stochastic analysis theory, and fixed point techniques (Krasnoselskii's fixed point theorem), we were able to accomplish the proposed results. In conclusion, an example is given to illustrate how our findings can be applied to real-world situations. In addition, fractional Brownian motion can be used to model the behavior of certain phenomena that occur in economics and the financial markets. There are two direct issues that call for additional research to be done. In the first step of this process, we will investigate the approximation of controllability for stochastic neutral integro-differential equations that have state-dependent delay and non-instantaneous impulses. In the second step of this process, we are going to look into whether or not there are optimal controls for stochastic neutral integro-differential equations that have state-dependent delays and nonlocal conditions.

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