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On the one spectral relation for the analytic function of operator

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Abstract

In this work, some estimates for the difference number between operator norm and spectral radius of analytic functions of linear bounded Hilbert space operators via difference numbers of powers of corresponding Hilbert space operators have been obtained. Firstly, these evaluations for the polynomial functions of the linear bounded Hilbert space operator have been established. Using previous results, this question was later investigated for the exponential, sine, and cosine functions of a given operator. Finally, starting from obtained results, this subject for the analytic functions of the linear bounded Hilbert space operator has been generalized.

Keywords: Operator norm, spectral radius, analytic functions of operator.

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1. Introduction

In the spectral theory of operators, the precise determination of the geometric place in complex plane of the resolvent sets or spectrum of the operator is one of the most important points. Remember that outside of the origin-centered and norm-radius circles are completely in the resolvent set of each linear bounded operator. However, there are also the cases some points on and inside this circle such that are in the resolvent set, although it is not the whole plane for linear bounded operators. Whether these points belong to the resolvent set is one of the most significant questions in the theory of operators. It is clear that in case when spectral radius is strongly smaller than the operator norm, then all points which lines outside of the circle origin-centered with radius equal to spectral radius are in the resolvent set also. As it is known that the classical Gelfand formula [7] is the only way used to compute the spectral radius of the linear bounded operators as well as often presenting technical difficulties. Unfortunately, a more practical method for calculating the spectral radius has not been found so far. Although it is not possible to give the exact geometry of the spectrum set of a linear bounded operator, it is of great importance in practice whether the zone between the spectral radius and the operator norm in the complex plane is a resolvent set.

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Let $B(H)$ be the class of linear bounded operators in Hilbert space H, $\|A\|$, $r(A)$ and $gap(A) := \|A\| - r(A)$, $A \in B(H)$ be the operator norm, spectral radius and spectral type gap of the operator A, respectively. It is known that $gap(A) = 0$ for any normal operator $A \in B(H)$. Generally, $gap(A) \geqslant 0$ for any operator $A \in B(H)$. For example, for $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$, $\|A\| = \sqrt{5}$, $r(A) = 1$.

This research problem means that the bounded solvability property of the operator function $A(\lambda) : A - \lambda I : B \to B, \lambda \in \mathbb{C}$ can extend into circle $C(0, \|A\|)$, where B is any Banach space. It is indisputable that the problem is of great importance in the theory of operator questions. As a matter of fact, this problem continues to be one of the most up-to-date questions that needs to be clarified in analysis, algebra and applied mathematics.

In [8] some spectral radius inequalities for 2×2 block operator matrix, sum, product, and commutators of two linear bounded Hilbert space operators have been examined. In [1] some estimates for the spectral radius of the Frobenius companion matrix have been obtained.

In [2] some estimates for spectral radius have been obtained for the product, sum, commutator and anticommutor of two Hilbert space operators.

And also some algebraic inequalities for the spectral and local spectral radii for the sum and multiplicity of certain two linear bounded operators in some Banach spaces have been investigated in works [9–11]. In addition, in [12] the several inequalities for the spectral radius of positive commutators of two linear bounded operators in a Banach space ordered by a normal and generating cone have been obtained.

The open problem posed by Demuth in 2015 had an important effect on the formation of the subject discussed in this study (see [4]).

In works [5, 6] it is investigated multivariable holomorphic functional calculus over Frechet modules. Later on, the Taylor joint spectrum of an algebra variety over an algebraically closed field is considered.

This investigation is organized as following scheme. In Section 2, some important evaluations for the spectral type gap for polynomial functions of linear bounded operator in Hilbert space via the spectral type gap of powers of given operator are given. Using these results, this question for exponential, sine and cosine functions of linear bounded Hilbert space operator has been researched in Section 3. In this Section 3, again this subject for analytic functions of linear bounded Hilbert space operator has been generalized.

2. Gaps between norm and spectral radius for polynomials of operator

Theorem 2.1. *For any linear bounded operator A in any complex Hilbert space H and any $n \in \mathbb{N}$ it is true that*

$$gap(A^n) \leqslant \sum_{k=1}^{n} \|A^n\|^{\frac{n-k}{n}} r^{k-1}(A) gap(A).$$

In general for $\alpha \in \mathbb{C}$,

$$gap((A - \alpha I)^n) \leqslant \sum_{k=1}^{n} \|(A - \alpha I)^n\|^{\frac{n-k}{n}} r^{k-1}(A - \alpha I) gap(A - \alpha I)$$

is true.

Proof. Indeed, in this case for any $n \geqslant 1$ and [8] it is obtained that

$$gap(A^n) = \|A^n\| - r(A^n) = (\|A^n\|^{\frac{1}{n}})^n - r^n(A) \leqslant (\|A\| - r(A)) \sum_{k=1}^{n} \|A^n\|^{\frac{n-k}{n}} r^{k-1}(A)$$

and similarly

$$gap((A - \alpha I)^n) \leqslant \sum_{k=1}^{n} \|(A - \alpha I)^n\|^{\frac{n-k}{n}} r^{k-1}(A - \alpha I) gap(A - \alpha I). \qquad \square$$

Corollary 2.2. *Under the conditions of above theorem*

$$\operatorname{gap}(A^n) \leqslant n\|A\|^{n-1}\operatorname{gap}(A)$$

and for any $\alpha \in \mathbb{C}$

$$\operatorname{gap}((A-\alpha I)^n) \leqslant n\|A-\alpha I\|^{n-1}\operatorname{gap}(A-\alpha I)$$

are true.

Theorem 2.3. *Let us* A *be linear bounded operator in any complex Hilbert space* H*. In this case for any polynomial of* A $P(A) = \sum_{k=0}^{n} a_k A^k,\ a_k \in \mathbb{C},\ n \geqslant 0,$

$$\left|\operatorname{gap}(P(A)) - \sum_{k=1}^{n} |a_k|\operatorname{gap}(A^k)\right| \leqslant 2 \max_{0\leqslant k\leqslant n-1} |a_k| \sum_{k=0}^{n-1} \|A^k\|$$

is true.

Proof. In this case from [3] it is obtained that

$$\begin{aligned}
\operatorname{gap}(P(A)) &= \|P(A)\| - r(P(A)) \\
&= \|P(A)\| - \sup_{\lambda\in\sigma(A)} |P(\lambda)| \\
&= \inf_{\lambda\in\sigma(A)} (\|P(A)\| - |P(\lambda)|) \\
&\leqslant \inf_{\lambda\in\sigma(A)} \Big[\big(|a_n|\|A^n\| + |a_{n-1}|\|A^{n-1}\| + \cdots + |a_1|\|A\| + |a_0|\big) \\
&\quad - \big(|a_n||\lambda|^n - |a_{n-1}||\lambda|^{n-1} - \cdots - |a_1||\lambda| + |a_0|\big) \Big] \\
&= \inf_{\lambda\in\sigma(A)} \Big[(|a_n|\|A^n\| - |a_n||\lambda|^n) + \cdots + (|a_1|\|A\| - |a_1||\lambda|) \Big] + 2 \inf_{\lambda\in\sigma(A)} \sum_{k=0}^{n-1} |a_k||\lambda|^k \\
&\leqslant \sum_{k=0}^{n-1} |a_k|\operatorname{gap}(A^k) + 2 \max_{0\leqslant k\leqslant n-1} |a_k| \sum_{k=0}^{n-1} \|A^k\|.
\end{aligned}$$

On the other hand from the results in [8] we have

$$\begin{aligned}
\operatorname{gap}(P(A)) &= \|P(A)\| - r(P(A)) \\
&\geqslant \sum_{k=0}^{n} |a_k| \big(\|A^k\| - r(A^k)\big) - 2\sum_{k=0}^{n-1} |a_k|\|A^k\| \\
&= \sum_{k=0}^{n} |a_k|\operatorname{gap}(A^k) - 2\sum_{k=0}^{n-1} |a_k|\|A^k\| \\
&\geqslant \sum_{k=1}^{n} |a_k|\operatorname{gap}(A^k) - 2 \max_{0\leqslant k\leqslant n-1} |a_k| \sum_{k=0}^{n-1} \|A^k\|.
\end{aligned}$$

This is sufficient to complete the proof. □

Using Theorems 2.1 and 2.3 we have the following.

Corollary 2.4. *Under the conditions of the last theorem, it is true*

$$\mathrm{gap}(P(A)) \leqslant \sum_{k=1}^{n} c_k|a_k|\mathrm{gap}(A) + 2\max_{0\leqslant k\leqslant n-1}|a_k|\sum_{k=0}^{n-1}\|A^k\|,$$

where $c_k = \sum\limits_{m=1}^{k}\|A^k\|^{\frac{k-m}{k}}r^{m-1}(A), 1\leqslant k\leqslant n$.

Corollary 2.5. *With the conditions of the Theorem 2.3 in case when* $\|A\| = 1$, *it is true that*

$$\left|\mathrm{gap}(P(A)) - \sum_{k=1}^{n}|a_k|\mathrm{gap}(A^k)\right| \leqslant 2n\max_{0\leqslant k\leqslant n-1}|a_k|.$$

But in case when $\|A\| \neq 1$ *it is true*

$$\left|\mathrm{gap}(P(A)) - \sum_{k=1}^{n}|a_k|\mathrm{gap}(A^k)\right| \leqslant \max_{0\leqslant k\leqslant n-1}|a_k|\frac{2\left(1-\|A\|^n\right)}{1-\|A\|}.$$

3. Gaps between norm and spectral radius for analytic functions of operator

Theorem 3.1. *Let us* A *be linear bounded operator in any complex Hilbert space* H. *Then*

$$\left|\mathrm{gap}(e^A) - \sum_{k=1}^{\infty}\frac{1}{k!}\mathrm{gap}(A^k)\right| \leqslant 2\sum_{k=0}^{\infty}\|A^k\|$$

is true.

Proof. First of all, denote $P_n = \sum\limits_{k=0}^{n}\frac{1}{k!}A^k$ and $Q_n = \sum\limits_{k=n+1}^{\infty}\frac{1}{k!}A^k$. Then from result of [8] for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathrm{gap}(e^A) &= \|e^A\| - r(e^A)\\ &= \|P_n + Q_n\| - r(P_n + Q_n)\\ &\leqslant (\|P_n\| - r(P_n)) + \|Q_n\| + (r(P_n) - r(P_n + Q_n))\\ &\leqslant (\|P_n\| - r(P_n)) + (\|Q_n\| + r(Q_n)).\end{aligned}$$

Since for any $A \in B(H)$ the series $\sum\limits_{k=0}^{\infty}\frac{1}{k!}\|A\|^k$ is convergent, then $\lim\limits_{n\to\infty}\|Q_n\| = 0$. Therefore $\lim\limits_{n\to\infty} r(Q_n) = 0$. From this and last relation it is obtained that

$$\mathrm{gap}(e^A) \leqslant \lim_{n\to\infty}\mathrm{gap}(P_n).$$

From this inequality and Theorem 2.3 it implies that

$$\mathrm{gap}(e^A) \leqslant \sum_{k=1}^{\infty}\frac{1}{k!}\mathrm{gap}(A^k) + 2\sum_{k=0}^{\infty}\|A^k\|.$$

On the other hand, for any $A \in B(H)$ we have

$$\begin{aligned}\mathrm{gap}(e^A) = \|e^A\| - r(e^A) &\geqslant \|P_n + Q_n\| - r(P_n + Q_n)\\ &\geqslant (\|P_n\| - \|Q_n\|) - (r(P_n) + r(Q_n)) = (\|P_n\| - r(P_n)) - \|Q_n\| - r(Q_n).\end{aligned}$$

Hence from this it is established that

$$\operatorname{gap}(e^{A}) \geqslant \lim_{n\to\infty} \operatorname{gap}(P_n).$$

In this case using the Theorem 2.3 again,

$$\operatorname{gap}(e^{A}) \geqslant \sum_{k=1}^{\infty} \frac{1}{k!}\operatorname{gap}(A^k) - 2\sum_{k=0}^{\infty} \|A^k\|$$

is true. □

Corollary 3.2. *With the conditions of the Theorem 3.1, if* $\|A\| < 1$, *then it is true*

$$\left|\operatorname{gap}(e^{A}) - \sum_{k=0}^{\infty} \frac{1}{k!}\operatorname{gap}(A^k)\right| \leqslant \frac{2}{1-\|A\|}.$$

It is known that from literature the operators $\sin A$ and $\cos A$ for any linear bounded operator A in any Banach space $\mathfrak{X}$ are denoted by

$$\sin A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}A^{2k+1}, \quad \cos A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}A^{2k}.$$

By a similar method, the validity of the following results can be proved.

Theorem 3.3. *For any linear bounded operator* A *in Hilbert space* H,

$$\left|\operatorname{gap}(\sin A) - \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\operatorname{gap}(A^{2k+1})\right| \leqslant 2\sum_{k=0}^{\infty} \|A^{2k+1}\|,$$
$$\left|\operatorname{gap}(\cos A) - \sum_{k=0}^{\infty} \frac{1}{(2k)!}\operatorname{gap}(A^{2k})\right| \leqslant 2\sum_{k=0}^{\infty} \|A^{2k}\|,$$

are true.

Proof. From Theorems 2.1, 3.1, and 3.3 it is clear. □

Corollary 3.4. *For any* $A \in B(H)$ *the following evaluations are true*

$$\operatorname{gap}(e^{A}) \leqslant \sum_{k=1}^{\infty} \frac{c_k}{k!}\operatorname{gap}(A) + 2\sum_{k=0}^{\infty} \|A^k\|,$$
$$\operatorname{gap}(\sin A) \leqslant \sum_{k=0}^{\infty} \frac{c_k}{(2k+1)!}\operatorname{gap}(A) + 2\sum_{k=0}^{\infty} \|A^{2k+1}\|,$$
$$\operatorname{gap}(\cos A) \leqslant \sum_{k=0}^{\infty} \frac{c_k}{(2k)!}\operatorname{gap}(A) + 2\sum_{k=0}^{\infty} \|A^{2k}\|,$$

where $c_k = \sum_{m=1}^{k} \|A^k\|^{\frac{k-m}{k}} r^{m-1}(A), k \geqslant 1.$

Now, Let A be a linear bounded operator in any complex Hilbert space H. The sets $\rho(A)$ and $\sigma(A)$ are resolvent and spectrum sets of the operator A, respectively.

Suppose that $f(\cdot)$ is a analytic function defined in a neighbourhood of the spectrum $\sigma(A)$ of the operator A. The family of all functions of this type will be denoted by $\mathscr{A} = \mathscr{A}(\sigma(A))$ [3].

Definition 3.5 ([3])**.** For $f \in \mathscr{A}$ the operator $f(A) \in B(H)$ will be denoted by

$$f(A) = \frac{-1}{2\pi i} \oint_{\Gamma} f(z) R_z(A) dz,$$

where $\Gamma \subset \operatorname{dom}(f)$ is a closed contour, composed of finitely many restifable Jordan curves oriented to the positive direction and $R_z(A)$ is a resolvent operator of A in the point $z \in \rho(A)$.

Now prove the next result.

Theorem 3.6. *For any $A \in B(H)$ and $f \in \mathscr{A}$ the following is true*

$$\left| \operatorname{gap}(f(A)) - \sum_{k=1}^{\infty} \left| \frac{f^k(a)}{k!} \right| \operatorname{gap}((A - aI)^k) \right| \leqslant 2 \max_{0 \leqslant k} \left| \frac{f^k(a)}{k!} \right| \sum_{k=0}^{\infty} \left\| (A - aI)^k \right\|,$$

where $a \in \rho(A)$.

Proof. For fixed point $a \in \rho(A)$ and $f \in \mathscr{A}$,

$$f(A) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!} (A - aI)^k$$

is true. For any $n = 0, 1, 2, \ldots$, denote

$$P_n = \sum_{k=0}^{n} \frac{f^k(a)}{k!} (A - aI)^k, \quad Q_n = \sum_{k=n+1}^{\infty} \frac{f^k(a)}{k!} (A - aI)^k.$$

Later on, using the scheme of the proof of Theorems 3.1 and 3.3, it can be easily shown that the proposition of the theorem is true. □

From the last result and Theorem 2.1 the following proposition is obtained.

Corollary 3.7. *Under the conditions of last theorem it is true that*

$$\operatorname{gap}(f(A)) \leqslant \sum_{k=1}^{\infty} c_k \left| \frac{f^k(a)}{k!} \right| \operatorname{gap}(A - aI) + 2 \max_{0 \leqslant k} \left| \frac{f^k(a)}{k!} \right| \sum_{k=0}^{\infty} \left\| (A - aI)^k \right\|,$$

where $c_k = \sum\limits_{m=1}^{k} \|A^k\|^{\frac{k-m}{k}} r^{m-1}(A), k \geqslant 1.$

4. Concluding remarks and observations

In this article, firstly, a correlation between the spectral gaps of the powers of a linear bounded Hilbert space operator and the operator itself is made. Next, this question is investigated for the relationship between the spectral gaps of the polynomials of the mentioned operators and the spectral gaps of its power operators. The results obtained here are applied to the exponential, sine and cosine functions of these operators. Finally, the problem under consideration is examined for analytical functions of linear bounded Hilbert space operators.

In fact, Demuth's open problem in 2015 had a great impact on the emergence and shaping of the subject examined in this article (see [4]).

The purpose of this problem is to clarify whether there are spectral points on the circle with a radius appropriate to the norm of the given linear bounded operator. It is indisputable that these results are of great importance in applications. That is, the existence and boundedness of the inverse operator at points above the circle with the appropriate norm radius in the plane has always been, and still is, a matter of debate.

It is a matter of curiosity what changes will occur in the non-analytic functions of the linear bounded Hilbert space operator given this problem.

Similar problems can also be a free subject of research for the difference between the norm and the numerical radius. The benefits of the results to be obtained are indisputable in applied sciences.

References

[1] A. Abu-Omar, F. Kittaneh, *Estimates for the numerical radius and spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials*, Ann. Funct. Anal., **5** (2014), 56–62. 1

[2] A. Abu-Omar, F. Kittaneh, *Notes on some spectral radius and numerical radius inequalities*, Studia Math., **227** (2015), 97–109. 1

[3] Y. M. Berezansky, Z. G. Sheftel, G. H. Us, *Functional Analysis*, Birkhäuser Verlag, Basel, (1996). 2, 3, 3.5

[4] M. Demuth, *Mathematical aspect of physics with non-selfadjoint operators: List of open problem*, American Institute of Mathematics Workshop, **8** (2015), 12 pages. 1, 4

[5] A. Dosi, *Functional calculus on Noetherian schemes*, C. R. Math. Acad. Sci. Paris, **353** (2015), 57–61. 1

[6] A. Dosi, *The spectrum of a module along scheme morphism and multi-operator functional calculus*, Mosc. Math. J., **21** (2021), 287–323. 1

[7] I. M. Gelfand, *Normierte ringe*, (German) Rec. Math. [Mat. Sbornik] N. S., **9** (1941), 3–24. 1

[8] F. Kittaneh, *Spectral radius inequalities for Hilbert space operators*, Proc. Amer. Math. Soc., **134** (2006), 385–390. 1, 2, 2, 3

[9] M. Zima, *A theorem on the spectral radius of the sum of two operators and its application*, Bull. Austral. Math. Soc., **48** (1993), 427–434. 1

[10] M. Zima, *On the local spectral radius in partially ordered Banach spaces*, Czechoslovak Math. J., **49** (1999), 835–841.

[11] M. Zima, *On the local spectral radius of positive operators*, Proc. Amer. Math. Soc., **131** (2003), 845–850. 1

[12] M. Zima, *Spectral radius inequalities for positive commutators*, Czechoslovak Math. J., **64** (2014), 1–10. 1