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Two inertial CQ-algorithms for generalized split inverse problem of infinite family of demimetric mappings

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Abstract

In this paper, two new inertial CQ-algorithms with strong convergence results are constructed to approximate the solution of the generalized split common fixed point problem: given as a task of finding a point that belongs to the intersection of an infinite family of fixed point sets of demimetric mappings such that its image under an infinite number of linear transformations belongs to the intersection of another infinite family of fixed point sets of demimetric mappings in the image space. The algorithms are established based on the CQ-projection method with inertial effect and step-size selection technique so that the implementation of the proposed algorithms does not need any prior information about the operator norms. The proposed methods improve, complement, and generalize many of the important results in the literature.

Keywords: Split common fixed-point problem, κ-demimetric mapping, inertial term, CQ-algorithm, Hilbert space, strong convergence.

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1. Introduction

Let E be a nonempty set and $U : E \to E$ is a nonlinear mapping. A point $x \in E$ with $Ux = x$ is called a fixed point of U. The notation $F(U)$ describes the set of all fixed points of the operator U, i.e., $F(U) = \{x \in E : U(x) = x\}$.

For a real Hilbert space H the mapping $U : H \to H$ is called

(1) 2-generalized hybrid mapping [22] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|U^2x - Uy\|^2 + \alpha_2\|Ux - Uy\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Uy\|^2 \\ \leqslant \beta_1\|U^2x - y\|^2 + \beta_2\|Ux - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2, \quad \forall(x, y) \in H \times H;\end{aligned}$$

(2) κ-strict pseudocontraction [3] if there exists a $\kappa \in [0, 1)$ such that

$$\|Ux - Uy\|^2 \leqslant \|x - y\|^2 + \kappa\|(I - U)x - (I - U)y\|^2, \quad \forall(x, y) \in H \times H;$$

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(3) firmly nonexpansive if

$$\|Ux - Uy\|^2 \leqslant \|x - y\|^2 - \|(I - U)x - (I - U)y\|^2, \ \forall(x, y) \in H \times H;$$

(4) μ-demicontractive mapping [20] if $F(U) \neq \emptyset$ and there exists a $\mu \in [0, 1)$ such that

$$\|Ux - \bar{x}\|^2 \leqslant \|x - \bar{x}\|^2 + \mu\|x - Ux\|^2, \ \forall(x, \bar{x}) \in H \times F(U);$$

(5) directed [10] if $F(U) \neq \emptyset$ and

$$\|Ux - \bar{x}\|^2 \leqslant \|x - \bar{x}\|^2 - \|x - Ux\|^2, \ \forall(x, \bar{x}) \in H \times F(U).$$

The class of 2-generalized hybrid mappings contains several classes of mappings, for example, the classes of nonexpansive mappings, nonspreading mappings, hybrid mappings, and generalized hybrid mappings. However, the class of 2-generalized hybrid mappings does not contain the class of κ-strict pseudocontractions and the class of κ-strict pseudocontractions does not contain the class of 2-generalized hybrid mappings by the fact that κ-strict pseudo-contractions are continuous and 2-generalized hybrid mappings are not continuous; see [22, 25] and the reference therein. Motivated by this, recently, Takahashi [34] introduced a broad class of nonlinear mappings called κ-demimetric mapping in a smooth Banach space. To be precise, for a smooth Banach space E the mapping $U : E \to E$ is called κ-demimetric if $F(U) \neq \emptyset$ and there exists $\kappa \in (-\infty, 1)$ such that

$$\langle x - \bar{x}, J(x - Ux)\rangle \geqslant \frac{1 - \kappa}{2}\|x - Ux\|^2, \ \forall(x, \bar{x}) \in E \times F(U),$$

where J is the duality mapping on E. The definition of κ-demimetric in the context of real Hilbert space H ($E = H$, where H is a real Hilbert space) is reduced to

$$\langle x - \bar{x}, x - Ux\rangle \geqslant \frac{1 - \kappa}{2}\|x - Ux\|^2, \ \forall(x, \bar{x}) \in H \times F(U). \tag{1.1}$$

It is clear that (1.1) is equivalent to the following:

$$\|Ux - \bar{x}\|^2 \leqslant \|x - \bar{x}\|^2 + \kappa\|x - Ux\|^2, \ \forall(x, \bar{x}) \in H \times F(U).$$

The class of κ-demimetric mappings in Hilbert space contains the classes of κ-strict pseudocontractions, 2-generalized hybrid mappings, firmly quasi-nonexpansive mappings, quasi-nonexpansive mappings, μ-demicontractive and directed mappings, see [22] and the reference therein. Moreover, several types of mappings appearing in optimization belong to the class of κ-demimetric, see [21, 35] and references therein. Due to this, recently, there is a growing research interest about fixed point existence and iterative approximation of demimetric mapping, see for example [21, 34, 35].

Let $\Lambda \subset \mathbb{R}$ be an index set, IP1 and IP2 are two inverse problems installed in spaces X and Y, respectively. Then Generalized Split Inverse Problem (GSIP) is stated as follows:

$$\begin{cases} \text{find } x^* \in X \text{ that solves IP1} \\ \text{such that} \\ y^* = A_k(x^*) \in Y, \ \forall k \in \Lambda \text{ and } y^* \text{ solves IP2,} \end{cases}$$

where A_k is a linear transformation from X to Y for each $k \in \Lambda$. If $A = A_k$ for all $k \in \Lambda$, then GSIP will be reduced to Split Inverse Problem (SIP) [9]. There is a considerable investigation of different types of problems in the framework of SIP due to its several applications, for instance, in image restoration, computer tomograph, radiation therapy treatment planning, sensor networks, resolution enhancement, in optics and neural networks, see [4, 7]. The well-known problem in the framework of SIP is split common fixed-point problem (SCFP), which is first introduced by Censor and Segal [10], formulated as a problem of finding:

$$\bar{x} \in F(U) \text{ such that } A\bar{x} \in F(T),$$

where H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is nonzero bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are directed operators. After Censor and Segal [10], several studies has been done to solve SCFP for different class of mappings (see, for example, [5, 6, 27, 28, 37] and the references therein).

In this paper, we consider GSIP type of problem which generalizes several studies considered in literature. The problem under consideration in this paper is the *generalized split system of common fixed point problem* (in short GSSCFP), formulated as a problem of finding

$$\bar{x} \in \bigcap_{i=1}^{\infty} F(U_i) \text{ such that } A_k(\bar{x}) \in \bigcap_{i=1}^{\infty} F(T_i), \quad \forall k \in \mathbb{N}, \tag{1.2}$$

where H_1 and H_2 are two real Hilbert spaces, $A_k : H_1 \to H_2$ is linear transformation for all $k \in \mathbb{N}$, $U_i : H_1 \to H_1$ and $T_i : H_2 \to H_2$ are nonlinear mappings for all $i \in \mathbb{N}$.

If $A_k = A$ for all $k \in \mathbb{N}$, then GSSCFP will be reduced to the problem considered by Eslamian [13] and Abkar and Shahrosvand [1] for the class of demicontractive mappings U_i and T_i. To be precise, Eslamian [13] introduced the following strong convergence Theorem.

Theorem 1.1. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $U_i : H_1 \to H_1$ is ζ-demicontractive mappings and $I - U_i$ is demiclosed for all $i \in \mathbb{N}$, $T_i : H_2 \to H_2$ is μ-demicontractive mappings and $I - T_i$ is demiclosed for all $i \in \mathbb{N}$. Suppose $\Omega = \{\bar{x} \in \bigcap_{i=1}^{\infty} F(U_i) : A(\bar{x}) \in \bigcap_{i=1}^{\infty} F(T_i)\} \neq \emptyset$. Assume that f is a contraction of H_1 into itself with constant $b \in (0,1)$ and B be a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\bar{\gamma} < 1$ and $0 < \gamma < \frac{\bar{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H_1$ and by*

$$\begin{cases} y_n = x_n - \sum\limits_{i=1}^{\infty} \beta_{(i,n)} \eta \beta A^*(T_i - I)Ax_n, \\ z_n = y_n - \sum\limits_{i=1}^{\infty} \alpha_{(i,n)} \frac{1-\zeta}{2}(U_i - I)y_n, \\ x_{n+1} = \delta_n \gamma f(x_n) + (I - \delta_n B)z_n, \end{cases}$$

where $\beta \in (0,1)$ and $\eta \in (0, \frac{1-\mu}{\lambda\beta})$ with λ being the spectral radius of the operator A^*A and $\{\alpha_{(i,n)}\}$, $\{\beta_{(i,n)}\}$, and $\{\delta_n\}$ satisfy the following conditions:

(i) $\liminf\limits_{n\to\infty} \alpha_{(i,n)} > 0$ *and* $\liminf\limits_{n\to\infty} \beta_{(i,n)} > 0$;

(ii) $\sum\limits_{i=1}^{\infty} \alpha_{(i,n)} = \sum\limits_{i=1}^{\infty} \beta_{(i,n)} = 1$, $\lim\limits_{n\to\infty} \delta_n = 0$, *and* $\sum\limits_{n=1}^{\infty} \delta_n = \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Omega$.

The main drawback of Eslamian [13] and several other results in literature concerning SIP (see, for example, [1, 6, 13] and the references therein) is that the step-size selection are dependent on operator norm value $\|A\|$. This means that in order to implement the algorithms, one has to first compute (or, at least, estimate) operator norm $\|A\|$, which is not always an easy task, see Theorem of Hendrickx and Olshevsky in [19]. Motivated and inspired by the works in literature, and by the ongoing research in these directions, the purpose of this paper is to introduce two new accelerated CQ-algorithms to solve GSSCFP for the broad class of nonlinear mappings called κ-demimetric, such that the implementation of the proposed algorithms does not need any prior information about the operator norms. The proposed methods use the idea of CQ-iterative scheme with adaptive technique combining the inertial extrapolation term $\theta(x_n - x_{n+1})$, which is the procedure of speeding up the convergence (see, [31]). In many practical applications speeding up the convergence the sequence generated by iterative method is needed, see, for example, [23, 24].

We propose two accelerated CQ-algorithms for solving the GSCFP (1.2) by making use of the following four standard assumptions:

(A1) $A_k : H_1 \to H_2$ is nonzero bounded linear operator for all $k \in \mathbb{N}$;
(A2) $U_i : H_1 \to H_1$ is η_i-demimetric mapping and $I - U_i$ is demiclosed at 0 for all $i \in \mathbb{N}$;
(A3) $T_i : H_2 \to H_2$ is β_i-demimetric mapping and $I - T_i$ is demiclosed at 0 for all $i \in \mathbb{N}$;
(A4) Γ denotes the solution set of the GSCFP (1.2) and Γ is nonempty.

The rest of this paper is organized as follows. In Sec. 2, some useful facts and tools are given. In Sec. 3, the two inertial CQ-algorithms and the proof of their strong convergence theorem are presented. Finally, in Sec. 4, we give some applications, where we show some of the applications that follow from our main result.

2. Preliminary

In this section, we give notations and recall some useful definitions and results in a Hilbert space H that are useful in our main result.

The notation '$\to$' denotes the strong convergence and '$\rightharpoonup$' denotes the weak convergence, and for a sequence $\{x_n\}$ in a real Hilbert space H, $\omega_w(x_n)$ stands for the set of cluster points in the weak topology, that is,

$$\omega_w(x_n) = \{p : \exists \{x_{n_l}\} \subset \{x_n\} \text{ such that } x_{n_l} \rightharpoonup p\}.$$

The following equality is well-known in a real Hilbert space H:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle.$$

Let C be a nonempty closed convex subset of H. The metric projection $P_C : H \to C$ is defined by $P_C(x) = \arg\min\{\|y - x\| : y \in C\}$. Note that for $x \in H$ and a point $z \in C$, then $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leqslant 0, \ \forall y \in C$.

Definition 2.1. Let $U : H \to H$ be a mapping. Then the mapping $I - U$ is called demiclosed at 0 if, for a sequence $\{x_n\}$ in H with $x_n \rightharpoonup \bar{x}$ and $x_n - Ux_n \to 0$, implies $\bar{x} \in F(U)$.

Lemma 2.2 ([34])**.** *The set of all fixed points $F(U)$ of κ-demimetric mapping $U : H \to H$ is closed and convex subset of H.*

Lemma 2.3 ([11])**.** *Let $\{x_n\}$ be a sequence in H and let $\{\alpha_n\}$ be a sequence real numbers with $0 < \alpha_n < 1$ for all $n \geqslant 1$ and $\sum_{n=1}^{\infty} \alpha_n = 1$. Then for any positive integers m, r with $m < r$, we have*

$$\Big\|\sum_{n=1}^{\infty} \alpha_n x_n\Big\|^2 \leqslant \sum_{n=1}^{\infty} \alpha_n \|x_n\|^2 - \alpha_m \alpha_r \|x_m - x_r\|^2.$$

Lemma 2.4 ([26])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H, $u \in H$ and let $\bar{y} = P_C(u)$. If the sequence $\{x_n\}$ satisfies the following two conditions:*

(i) $\omega_w(x_n) \subset C$;
(ii) $\|x_n - u\| \leqslant \|u - \bar{y}\|$ *for all* $n \geqslant 1$,

then, $x_n \to \bar{y}$.

Let **Condition I** refers to, the following real parameter sequence restrictions.

Condition I: Suppose $\{\theta_n\}$, $\{\delta_n^{(i)}\}_{n=1}^{\infty}$ $(i \in \mathbb{N})$, and $\{\sigma_n^{(k)}\}_{n=1}^{\infty}$ $(k \in \mathbb{N})$ are real sequences satisfying the conditions:

(C1) $0 \leqslant \theta_n \leqslant \theta$ for all $n \geqslant 1$, for some $\theta > 0$;

(C2) $0 < \liminf_{n\to\infty} \delta_n^{(i)} \leqslant \limsup_{n\to\infty} \delta_n^{(i)} < 1$ ($i \in \mathbb{N}$) and $\sum_{i=1}^{\infty} \delta_n^{(i)} = 1$;

(C3) $0 < \liminf_{n\to\infty} \sigma_n^{(k)} \leqslant \limsup_{n\to\infty} \sigma_n^{(k)} < 1$ ($k \in \mathbb{N}$) and $\sum_{k=1}^{\infty} \sigma_n^{(k)} = 1$.

3. Main results

In this section, we propose the two inertial CQ-algorithms (using Parallel and Sequential computing method) by combining the CQ-projection scheme with inertial extrapolation and self-adaptive step-size selection technique, and we prove the strong convergence of the algorithms to solution point of GSCFP (1.2) under the given suitable assumptions **(A1)-(A4)**.

Algorithm 3.1 (Parallel-computing inertial CQ-algorithm)**.** Let $\{\theta_n\}$, $\{\delta_n^{(i)}\}_{n=1}^{\infty}$ ($i \in \mathbb{N}$) and $\{\sigma_n^{(k)}\}_{n=1}^{\infty}$ ($k \in \mathbb{N}$) be real sequences satisfying **Condition I**, and let $\{\rho_{(i,n)}\}_{n=1}^{\infty}$ ($i \in \mathbb{N}$) be a real sequence such that

(C4) $0 < \rho_{(i,n)} < \lambda_i = \min\{1-\mu_i, 1-\beta_i\}$ and $\liminf_{n\to\infty} \rho_{(i,n)}(\lambda_i - \rho_{(i,n)}) > 0$.

Choose $x_0, x_1 \in H_1$ arbitrarily and follow the following iterative steps.

STEP 1. Evaluate $z_n = x_n + \theta_n(x_n - x_{n-1})$.

STEP 2. Evaluate $t_n^{(i)} = (I - U_i)z_n$ and $y_n^{(i,k)} = (I - T_i)A_k z_n$. Let $\Psi_n = \{(i,k) \in \mathbb{N}\times\mathbb{N} : \|A_k^*(y_n^{(i,k)}) + t_n^{(i)}\|\} \neq 0\}$. If $\Psi_n = \emptyset$, then **STOP**. Otherwise, go to **STEP 3**.

STEP 3. Compute

$$s_n = z_n - \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \left(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\right),$$

where $\mu_{(i,k)}(z_n) = \|A_k^*(y_n^{(i,k)}) + t_n^{(i)}\|^2$ if $(i,k) \in \Psi_n$, $\mu_{(i,k)}(z_n) = 1$ otherwise.

STEP 4. Evaluate $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, where C_n and Q_n are half-spaces given by

$$C_n = \left\{ z \in H_1 : \|s_n - z\|^2 \leqslant \|z_n - z\|^2 - \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)}(\lambda_i - \rho_{(i,n)}) \frac{(\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)} \right\}$$

and $Q_n = \left\{ z \in H_1 : \langle x_n - z, x_n - x_0 \rangle \leqslant 0 \right\}$.

STEP 5. Set $n := n+1$ and go to **STEP 1**.

Lemma 3.2. *The stopping condition (in STEP 2) of Algorithm 3.1 is satisfied ($\Psi_n = \emptyset$ for some $n \in \mathbb{N}$) iff $z_n \in \Gamma$.*

Proof. Suppose $\|A_k^*(I - T_i)A_k z_n + (I - U_i)z_n\| = 0$ for all $(i,k) \in \mathbb{N}\times\mathbb{N}$. Now, for $p \in \Gamma$, we have

$$\begin{aligned} 0 &= \|A_k^*(I - T_i)A_k z_n + (I - U_i)z_n\|\|z_n - p\| \\ &\geqslant \langle A_k^*(I - T_i)A_k z_n + (I - U_i)z_n, z_n - p\rangle \\ &= \langle A_k^*(I - T_i)A_k z_n, z_n - p\rangle + \langle (I - U_i)z_n, z_n - p\rangle \\ &= \langle (I - T_i)A_k z_n, A_{k_x} z_n - A_{k_x} p\rangle + \langle (I - U_i)z_n, z_n - p\rangle \\ &\geqslant \frac{1-\eta_i}{2}\|(I - T_i)A_k z_n\|^2 + \frac{1-\beta_i}{2}\|(I - U_i)z_n\|^2. \end{aligned}$$

This implies $\|(I - T_i)A_k z_n\| = \|(I - U_i)z_n\| = 0$ for all $(i,k) \in \mathbb{N}\times\mathbb{N}$. Therefore, z_n solves GSCFP (1.2).

The converse is straightforward. □

Theorem 3.3. *The sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to* $\bar{x} \in \Gamma$*, where* $\bar{x} = P_\Gamma(x_0)$.

Proof. Let $\bar{x} \in \Gamma$. Now, using the definition of s_n, we have

$$\begin{aligned} \|s_n - \bar{x}\|^2 &= \Big\|z_n - \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big) - \bar{x}\Big\|^2 \\ &= \|z_n - \bar{x}\|^2 + \Big\| \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big)\Big\|^2 \\ &\quad - 2\Big\langle \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big), z_n - \bar{x}\Big\rangle. \end{aligned} \tag{3.1}$$

Now,

$$\begin{aligned} &\Big\| \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big)\Big\|^2 \\ &\leqslant \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \Big\|\rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big)\Big\|^2 \\ &= \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \Big(\rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)}\Big)^2 \big\|\big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big)\big\|^2 \\ &= \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)}^2 \frac{(\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}^2(z_n)} \big\|\big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big)\big\|^2 \\ &\leqslant \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)}^2 \frac{(\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)}, \end{aligned} \tag{3.2}$$

and using the definition of $y_n^{(i,k)}$ and $t_n^{(i)}$ and since U_i and T_i are demimetric, we get

$$\begin{aligned} &\Big\langle \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(A_k^*(y_n^{(i,k)}) + t_n^{(i)}\big), z_n - \bar{x}\Big\rangle \\ &= \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \langle A_k^*(y_n^{(i,k)}) + t_n^{(i)}, z_n - \bar{x}\rangle \\ &= \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(\langle y_n^{(i,k)}, A_k z_n - A_k\bar{x}\rangle + \langle t_n^{(i)}, z_n - \bar{x}\rangle\big) \\ &\geqslant \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \Big(\frac{1-\beta_i}{2}\|y_n^{(i,k)}\|^2 + \frac{1-\eta_i}{2}\|t_n^{(i)}\|^2\Big) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \lambda_i \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} \big(\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2\big) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \lambda_i \frac{(\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)}. \end{aligned} \tag{3.3}$$

By virtue of (3.1), (3.2), and (3.3), we deduce

$$\|s_n - \bar{x}\|^2 \leqslant \|z_n - \bar{x}\|^2 + \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)}^2 \frac{(\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)}$$

$$
\begin{aligned}
&-\sum_{k=1}^{\infty}\sigma_n^{(k)}\sum_{i=1}^{\infty}\delta_n^{(i)}\rho_{(i,n)}\lambda_i\frac{(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)} \qquad (3.4)\\
&\leqslant \|z_n-\bar{x}\|^2-\sum_{k=1}^{\infty}\sigma_n^{(k)}\sum_{i=1}^{\infty}\delta_n^{(i)}\rho_{(i,n)}(\lambda_i-\rho_{(i,n)})\frac{(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)}.
\end{aligned}
$$

From (3.4), we have $\bar{x}\in C_n$, and this implies that $\Gamma\subset C_n$ for all $n\geqslant 0$. Next, we show by induction $\Gamma\subset Q_n$ for all $n\geqslant 0$. Now, for $n=0$, we have $\Gamma\subset H_1=Q_0$. Assume that $\Gamma\subset Q_n$. From $x_{n+1}=P_{C_n\cap Q_n}(x_0)$, we have

$$\langle x_{n+1}-z, x_{n+1}-x_0\rangle\leqslant 0,\ \ \forall z\in C_n\cap Q_n.$$

Thus, by $Q_{n+1}=\{z\in H_1:\langle x_{n+1}-z,x_{n+1}-x_0\rangle\leqslant 0\}$ we have $C_n\cap Q_n\subset Q_{n+1}$. From $\Gamma\subset C_n$ and the assumption $\Gamma\subset Q_n$, we get $\Gamma\subset C_n\cap Q_n\subset Q_{n+1}$. Hence, $\Gamma\subset Q_n$ for all $n\geqslant 0$.

Let $\bar{y}=P_\Gamma(x_0)$. Now from the definition of Q_n we can see that $x_n=P_{Q_n}(x_0)$. It then follows from $x_{n+1}=P_{C_n\cap Q_n}(x_0)$ that

$$\|x_n-x_0\|=\|P_{Q_n}(x_0)-x_0\|\leqslant\|P_\Gamma(x_0)-x_0\|=\|\bar{y}-x_0\|.$$

This shows that $\{\|x_n-x_0\|\}$ is $\{x_n\}$ is bounded, and thus $\omega_w(x_n)\neq\emptyset$. Note that since $\{x_n\}$ is bounded and $\|z_n\|\leqslant(1+\theta)\|x_n\|+\theta\|x_{n-1}\|$ we see that $\{z_n\}$ is bounded. Moreover, in view of (3.4) and the conditions (C2)-(C4), we obtain $\|s_n-\bar{x}\|\leqslant\|z_n-\bar{x}\|$, and hence $\{z_n\}$ is also bounded.
Again, from $x_n=P_{Q_n}(x_0)$ and $x_{n+1}\in Q_n$, we have $\langle x_{n+1}-x_n,x_n-x_0\rangle\geqslant 0$ and hence

$$\|x_{n+1}-x_n\|^2\leqslant\|x_{n+1}-x_n\|^2+2\langle x_{n+1}-x_n,x_n-x_0\rangle=\|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2.$$

This implies $\sum_{l=0}^{n}\|x_{l+1}-x_l\|^2\leqslant\|x_{n+1}-x_0\|^2\leqslant\|P_\Gamma(x_0)-x_0\|^2$ for all $n\geqslant 0$, and hence $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2\leqslant$ $\|P_\Gamma(x_0)-x_0\|^2$. That is, $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2<\infty$ and hence

$$\|x_{n+1}-x_n\|\to 0,\ \ n\to\infty. \qquad (3.5)$$

By the definition of z_n and condition (C1), we get

$$\|z_n-x_n\|=\theta_n\|x_n-x_{n-1}\|\leqslant\theta\|x_n-x_{n-1}\|\to 0,\ \ n\to\infty. \qquad (3.6)$$

Since $x_{n+1}\in C_n$, employing (3.5) and (3.6), we get

$$
\begin{aligned}
\|s_n-z_n\|&=\|s_n-x_{n+1}\|+\|z_n-x_{n+1}\|\leqslant 2\|z_n-x_{n+1}\|\\
&\leqslant 2(\|z_n-x_n\|+\|x_n-x_{n+1}\|)\to 0,\ \ n\to\infty.
\end{aligned}
$$

Noting that $\{z_n\}$ and $\{s_n\}$ are bounded, from (3.4), we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sigma_n^{(k)}\sum_{i=1}^{\infty}\delta_n^{(i)}\rho_{(i,n)}(\lambda_i-\rho_{(i,n)})\frac{(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)} \qquad (3.7)\\
&\leqslant\|z_n-\bar{x}\|^2-\|s_n-\bar{x}\|^2\leqslant\|z_n-s_n\|(\|z_n-\bar{x}\|+\|s_n-\bar{x}\|)\to 0,\ \ n\to\infty.
\end{aligned}
$$

Using the conditions (C2)-(C4), we have from (3.7) that

$$\frac{(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)}\to 0,\ \ n\to\infty, \qquad (3.8)$$

for all $(i,k) \in \mathbb{N}\times\mathbb{N}$. Now, if $(i,k) \in \Psi_n$, we have

$$\begin{aligned} \frac{\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2}{\max\{\|A_k\|^2,1\}} &= \frac{(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\max\{\|A_k^*\|^2,1\}(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)} \\ &\leqslant \frac{(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\|A_k^*\|^2\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2} \\ &\leqslant \frac{(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\|A_k^*(y_n^{(i,k)})\|^2+\|t_n^{(i)}\|^2} \\ &\leqslant \frac{2(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\|A_k^*(y_n^{(i,k)})+t_n^{(i)}\|^2} = \frac{2(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)}. \end{aligned} \tag{3.9}$$

Moreover, if $(i,k) \notin \Psi_n$, then $\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2 = 0$ and thus

$$\frac{\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2}{\max\{\|A_k\|^2,1\}} = 0 = \frac{0}{\mu_{(i,k)}(z_n)} = \frac{2(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)}{\mu_{(i,k)}(z_n)}. \tag{3.10}$$

Hence, by (3.9) and (3.10), for all $n \in \mathbb{N}$, we get

$$\frac{\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2}{\max\{\|A_k\|^2,1\}} \leqslant \frac{2(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2)}{\mu_{(i,k)}(z_n)},$$

for all $(i,k) \in \mathbb{N}\times\mathbb{N}$, and so this together with (3.8) gives

$$\lim_{n\to\infty}(\|y_n^{(i,k)}\|^2+\|t_n^{(i)}\|^2) = 0 \Leftrightarrow \lim_{n\to\infty}\|y_n^{(i,k)}\| = \lim_{n\to\infty}\|t_n^{(i)}\| = 0,$$

for all $(i,k) \in \mathbb{N}\times\mathbb{N}$. Therefore,

$$\lim_{n\to\infty}\|(I-T_i)A_kz_n\| = \lim_{n\to\infty}\|(I-U_i)z_n\| = 0, \tag{3.11}$$

for all $(i,k) \in \mathbb{N}\times\mathbb{N}$.

Next we show $\omega_w(x_n) \subset \Gamma$. Let $p \in \omega_w(x_n)$ and a subsequence $\{x_{n_l}\}$ and $\{x_n\}$ with $x_{n_l} \rightharpoonup p$. Using (3.6), we get that $z_{n_l} \rightharpoonup p$ and $A_k(z_{n_l}) \rightharpoonup p$ ($\forall k \in \mathbb{N}$), and hence by demiclosedness at 0 assumptions of of $I-U_i$ and $I-T_i$ together with (3.11), we obtain $p \in \Gamma$. That is, $\omega_w(x_n) \subset \Gamma$. Therefore, for $\bar{y} = P_\Gamma(x_0)$ and the sequence $\{x_n\}$ generated by Algorithm 3.1, we obtained the inequality (3.5) and $\omega_w(x_n) \subset \Gamma$, and hence Lemma 2.4, gives $x_n \to P_\Gamma(x_0)$. This completes the proof. □

In following sequential type inertial CQ-algorithm we assume that $\eta_i < 0$ for all $i \in \mathbb{N}$ in assumption **(A2)**.

Algorithm 3.4 (Sequential-computing inertial CQ-algorithm)**.** Let $\{\theta_n\}$, $\{\delta_n^{(i)}\}_{n=1}^\infty$ ($i \in \mathbb{N}$) and $\{\sigma_n^{(k)}\}_{n=1}^\infty$ ($k \in \mathbb{N}$) be real sequences satisfying **Condition I**, and let $\{\rho_{(i,n)}\}_{n=1}^\infty$ ($i \in \mathbb{N}$) be a real sequence such that

(C4) $0 < \rho_{(i,n)} < 1-\beta_i$ and $\liminf_{n\to\infty}\rho_{(i,n)}(1-\beta_i-\rho_{(i,n)}) > 0$.

Choose $x_0, x_1 \in H_1$ arbitrarily and follow the following iterative steps.

STEP 1. Evaluate $z_n = x_n + \theta_n(x_n - x_{n-1})$.

STEP 2. Evaluate $y_n^{(i,k)} = (I-T_i)A_kz_n$. Let $\Psi_n = \{(i,k) \in \mathbb{N}\times\mathbb{N} : \|A_k^*(y_n^{(i,k)})\| \neq 0\}$.

STEP 3. Compute

$$s_n = \sum_{i=1}^{\infty} \delta_n^{(i)} U_i\Big(z_n - \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)})\Big),$$

where $\mu_{(i,k)}(z_n) = \|A_k^*(y_n^{(i,k)})\|^2$ if $(i,k) \in \Psi_n$, $\mu_{(i,k)}(z_n) = 1$ otherwise.

STEP 4. Evaluate $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ where C_n and Q_n are half-spaces given by

$$C_n = \Big\{z \in H_1 : \|s_n - z\|^2 \leqslant \|z_n - \bar{x}\|^2 - \sum_{i=1}^{\infty} \delta_n^{(i)} \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)}(1-\beta_i-\rho_{(i,n)}) \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)}\Big\}$$

and $Q_n = \big\{z \in H_1 : \langle x_n - z, x_n - x_0\rangle \leqslant 0\big\}$.

STEP 5. Set $n := n+1$ and go to **STEP 1**.

Theorem 3.5. *The sequence $\{x_n\}$ generated by Algorithm 3.4 converges strongly to $\bar{x} \in \Gamma$, where $\bar{x} = P_\Gamma(x_0)$.*

Proof. Let $\bar{x} \in \Gamma$ and let $t_{(i,n)} = z_n - \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)})$. Now since T_i is β_i-demimetric mapping, we have

$$\begin{aligned}
&\|t_{(i,n)} - \bar{x}\|^2 \\
&= \Big\|z_n - \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)}) - \bar{x}\Big\|^2 \\
&= \|z_n - \bar{x}\|^2 + \Big\|\sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)})\Big\|^2 - 2\Big\langle \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)}), z_n - \bar{x}\Big\rangle \\
&= \|z_n - \bar{x}\|^2 + \sum_{k=1}^{\infty} \sigma_n^{(k)} \Big(\rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)}\Big)^2 \|A_k^*(y_n^{(i,k)})\|^2 - 2\sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} \langle A_k^*(y_n^{(i,k)}), z_n - \bar{x}\rangle \\
&\leqslant \|z_n - \bar{x}\|^2 + \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)}^2 \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)} - 2\sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} \langle y_n^{(i,k)}, A_k(z_n) - A_k\bar{x}\rangle \\
&\leqslant \|z_n - \bar{x}\|^2 + \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)}^2 \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)} - \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)}(1-\beta_i) \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)} \\
&= \|z_n - \bar{x}\|^2 - \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)}(1-\beta_i-\rho_{(i,n)}) \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)}.
\end{aligned}$$

Noting U_i is η_i-demimetric mapping with $\eta_i < 0$ and using the definition of y_n, we have

$$\begin{aligned}
&\|s_n - \bar{x}\|^2 \\
&= \Big\|\sum_{i=1}^{\infty} \delta_n^{(i)} U_i(t_{(i,n)}) - \bar{x}\Big\|^2 \\
&\leqslant \sum_{i=1}^{\infty} \delta_n^{(i)} \|U_i(t_{(i,n)}) - \bar{x}\|^2 \\
&\leqslant \sum_{i=1}^{\infty} \delta_n^{(i)} \Big(\|t_{(i,n)} - \bar{x}\|^2 + \eta_i \|U_i(t_{(i,n)}) - t_{(i,n)}\|^2\Big) \\
&\leqslant \|z_n - \bar{x}\|^2 - \sum_{i=1}^{\infty} \delta_n^{(i)} \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)}(1-\beta_i-\rho_{(i,n)}) \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)} + \sum_{i=1}^{\infty} \delta_n^{(i)} \eta_i \|U_i(t_{(i,n)}) - t_{(i,n)}\|^2 \qquad (3.12)
\end{aligned}$$

$$\leqslant \|z_n - \bar{x}\|^2 - \sum_{i=1}^{\infty} \delta_n^{(i)} \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} (1 - \beta_i - \rho_{(i,n)}) \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)}. \tag{3.13}$$

In view of (3.13), we have $\bar{x} \in C_n$, and this implies that $\Gamma \subset C_n$ for all $n \geqslant 0$.

Now, by similar procedure as in the proof of Theorem 3.3, we obtain that

(i) $\Gamma \subset Q_n$ for all $n \geqslant 0$;

(ii) for $\bar{y} = P_\Gamma(x_0)$, we have $\|x_n - x_0\| \leqslant \|\bar{y} - x_0\|$, consequently, $\{x_n\}$ is bounded, and hence $\{s_n\}$ and $\{z_n\}$ are also bounded;

(iv) $\|x_{n+1} - x_n\| \to 0$, $\|z_n - x_n\| \to 0$, and $\|z_n - s_n\| \to 0$ as $n \to \infty$.

Since $\{z_n\}$ and $\{s_n\}$ are bounded and $\|s_n - z_n\| \to 0$, we have

$$\begin{aligned} &\sum_{i=1}^{\infty} \delta_n^{(i)} \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} (1 - \beta_i - \rho_{(i,n)}) \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)} \\ &\qquad \leqslant \|z_n - \bar{x}\|^2 - \|s_n - \bar{x}\|^2 \leqslant \|z_n - s_n\|(\|z_n - \bar{x}\| + \|s_n - \bar{x}\|) \to 0, \quad n \to \infty. \end{aligned} \tag{3.14}$$

From (3.14) and the conditions (C2)-(C4), we have

$$\frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)} \to 0, \quad n \to \infty, \tag{3.15}$$

for all $(i,k) \in \mathbb{N} \times \mathbb{N}$. Now for $(i.k) \in \Psi_n$, we have

$$\frac{\|y_n^{(i,k)}\|^2}{\|A_k\|^2} = \frac{\|y_n^{(i,k)}\|^4}{\|A_k^*\|^2 \|y_n^{(i,k)}\|^2} \leqslant \frac{\|y_n^{(i,k)}\|^4}{\|A_k^*(y_n^{(i,k)})\|^2} \leqslant \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)}. \tag{3.16}$$

If $(i,k) \notin \Psi_n$ and $p \in \Gamma$, we have

$$\begin{aligned} 0 = \|A_k^*(y_n^{(i,k)})\| \|z_n - p\| &\geqslant \langle A_k^*(y_n^{(i,k)}), z_n - p \rangle \\ &= \langle A_k^*(I - T_i) A_k z_n, z_n - p \rangle \\ &= \langle (I - T_i) A_k z_n, A_k z_n - A_k p \rangle \\ &\geqslant \frac{1 - \beta_i}{2} \|(I - T_i) A_k z_n\|^2 = \frac{1 - \beta_i}{2} \|y_n^{(i,k)}\|^2, \end{aligned}$$

which implies $\|(I - T_i) A_k z_n\| = \|y_n^{(i,k)}\| = 0$. Hence, for $(i,k) \notin \Psi_n$, we have

$$\frac{\|y^{(i,k)}\|^2}{\|A_k\|^2} = 0 = \frac{0}{1} = \frac{0}{\mu_{(i,k)}(z_n)}. \tag{3.17}$$

From (3.16) and (3.17), we get

$$\frac{\|y^{(i,k)}\|^2}{\|A_{k_n}\|^2} \leqslant \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)}, \tag{3.18}$$

for all $(i,k) \in \mathbb{N} \times \mathbb{N}$. Therefore, (3.15) and (3.18) yield

$$\lim_{n \to \infty} \|y^{(i,k)}\| = \lim_{n \to \infty} \|(I - T_i) A_k z_n\| = 0,$$

for all $(i,k) \in \mathbb{N} \times \mathbb{N}$. Note that the conditions (C3) and (C4) together with (3.15) give

$$\|t_{(i,n)} - z_n\|^2 \leqslant \Big\| \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)}) \Big\|^2$$

$$
\begin{aligned}
&\leqslant \sum_{k=1}^{\infty} \sigma_n^{(k)} \Big\| \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)}) \Big\|^2 \\
&= \sum_{k=1}^{\infty} \sigma_n^{(k)} \rho_{(i,n)}^2 \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}^2(z_n)} \|A_k^*(y_n^{(i,k)})\|^2 \\
&\leqslant (1-\beta_i)^2 \sum_{k=1}^{\infty} \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)} \to 0, \ \ n \to \infty,
\end{aligned}
$$

for all $i \in \mathbb{N}$. Thus, employing $\|t_{(i,n)} - z_n\| \to 0$ and $\|z_n - x_n\| \to 0$, we get

$$
\|t_{(i,n)} - x_n\| \leqslant \|t_{(i,n)} - z_n\| + \|z_n - x_n\| \to 0, \ \ n \to \infty,
$$

for all $i \in \mathbb{N}$. Using the definition of demimetric mapping, condition (C4), and (3.12), we have

$$
\sum_{i=1}^{\infty} \delta_n^{(i)}(-\eta_i)\|U_i(t_{(i,n)}) - t_{(i,n)}\|^2 \leqslant \|z_n - \bar{x}\|^2 - \|s_n - \bar{x}\|^2 \leqslant \|z_n - s_n\|(\|z_n - \bar{x}\| + \|s_n - \bar{x}\|). \tag{3.19}
$$

In view of (3.19) and boundedness of $\{s_n\}$ and $\{z_n\}$ together with $\|z_n - s_n\| \to 0$, we obtain

$$
\sum_{i=1}^{\infty} \delta_n^{(i)}(-\eta_i)\|U_i(t_{(i,n)}) - t_{(i,n)}\|^2 \to 0, \ \ n \to \infty,
$$

and so noting that $\eta_i < 0$ for all $i \in \mathbb{N}$, we obtain that

$$
\|U_i(t_{(i,n)}) - t_{(i,n)}\| \to 0, \ \ n \to \infty,
$$

for all $i \in \mathbb{N}$.

By similar procedure as in the proof of Theorem 3.3, i.e., using $\|t_{(i,n)} - x_n\| \to 0$ ($\forall i \in \mathbb{N}$) and $\|z_n - x_n\| \to 0$ together with demiclosedness at 0 assumptions of $I - U_i$ and $I - T_i$, we get $\omega_w(x_n) \subset \Gamma$, and hence, applying Lemma 2.4, we obtain $x_n \to P_\Gamma(x_0)$. □ □

Remark 3.6. As a direct consequence of our result, we can have several new algorithms for different class of mappings; for example, for κ-strict pseudocontractions, 2-generalized hybrid mappings, firmly quasi-nonexpansive mappings, quasi-nonexpansive mappings, μ-demicontractive and directed mappings. Note that κ-strict pseudocontractive mapping is κ-demimetric mapping, and firmly nonexpansive mapping is 1-demimetric mapping. Moreover, both κ-strict pseudocontractive and firmly nonexpansive mappings satisfy demiclosedness at 0 condition, see [30].

Remark 3.7. Consider the problem in [14, 15], i.e., finding a point

$$
\bar{x} \in \bigcap_{i=1}^{N} F(U_i) \text{ such that } A_k(\bar{x}) \in \bigcap_{j=1}^{M} F(T_i), \ \ \forall k \in \{1, \dots, R\}, \tag{3.20}
$$

where H_1 and H_2 are two real Hilbert spaces, $A_k : H_1 \to H_2$ is nonzero bounded linear operator for all $k \in \{1, \dots, R\}$, and $U_i : H_1 \to H_1$ and $T_j : H_2 \to H_2$ are demimetric mappings for all $i \in \{1, \dots, N\}$, $j \in \{1, \dots, M\}$.

It is worth to point out that the results in this paper can solve the problem (3.20) by extending it to GSSCFP (1.2) as follows:

(i) setting $U_i = I$ for all $i > N$, or for some $i_0 \in \{1, \dots, N\}$ set $U_i = U_{i_0}$ for all $i > N$, or use the mod function $[i]_1 = i(\text{mod } N)$ for $i \in \mathbb{N}$;

(ii) setting $T_i = I$ for all $i > M$, or for some $j_0 \in \{1, \dots, M\}$ set $T_i = T_{j_0}$ for all $i > M$, or using the mod function $[i]_2 = i(\text{mod } M)$ for $i \in \mathbb{N}$;

(iii) for some $k_0 \in \{1, \dots, R\}$ set $A_k = A_{k_0}$ for all $k > R$, or use the mod function $[k]_3 = i(\text{mod } R)$ for $k \in \mathbb{N}$.

Therefore, from Remarks 3.6 and 3.7, we can see that our results improve and extend several known results in the literature, see for example [1, 6, 10, 13–15, 27, 32, 33] and the reference therein.

Now we would like to point the following observation given as an example.

Consider the mapping $U : \mathbb{R}^p \to \mathbb{R}^p$ given by

$$U : x = (x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(p)}) \mapsto \left(a_1 x^{(2)}, a_2 x^{(2)}, \dots, a_p x^{(p)}\right),$$

where $a_t \in \mathbb{R}$ for all $t \in G = \{1, \dots, p\}$. It is clear to see that if $a_t = 1$ $\forall t \in G$, then $F(U) = \mathbb{R}^p$ and U is 0-demimetric mapping.

Example 3.8. We consider the mapping U for the case $a_t \neq 1$ for some $t \in G$. For this case $F(U) = \{0\}$. Let $D = \{t \in G : a_t = 1\}$. Then for $\bar{x} \in F(U)$, we have

$$\begin{aligned}
\|Ux - \bar{x}\|^2 = \|Ux\|^2 = \|\left(a_1 x^{(2)}, a_2 x^{(2)}, \dots, a_p x^{(p)}\right)\|^2 &= \sum_{t=1}^{p} a_t^2 (x^{(t)})^2 \\
&= \sum_{t=1}^{p} (x^{(t)})^2 + \sum_{t \in G \setminus D} (a_t^2 - 1)(x^{(t)})^2 \\
&= \|x\|^2 + \sum_{t \in G \setminus D} (a_t^2 - 1)(x^{(t)})^2 \\
&= \|x - \bar{x}\|^2 + \sum_{t \in G \setminus D} \frac{(a_t^2 - 1)}{(1 - a_t)^2} (x^{(t)} - a_t x^{(t)})^2 \\
&\leqslant \|x - \bar{x}\|^2 + \xi \sum_{t \in G \setminus D} (x^{(t)} - a_t x^{(t)})^2 + 0 \sum_{t \in D} (x^{(t)} - a_t x^{(t)})^2,
\end{aligned} \tag{3.21}$$

where $\xi = \max\left\{\frac{a_t^2 - 1}{(a_t - 1)^2} : t \in G \setminus D\right\}$. Observe that, if $\xi < 1$, then U is demimetric mapping. But,

$$\xi < 1 \iff \frac{a_t^2 - 1}{(a_t - 1)^2} < 1, \ \forall t \in G \setminus D \iff a_t < 1, \ \forall t \in G \setminus D.$$

Therefore, we have the following for $\xi < 1$ ($a_t < 1$, $\forall t \in G \setminus D$).

Case 1: $D = \emptyset$. Thus, from (3.21), we get

$$\|Ux - \bar{x}\|^2 \leqslant \|x - \bar{x}\|^2 + \xi \|x - Ux\|^2, \ \ \forall (x, \bar{x}) \in \mathbb{R}^p \times F(U),$$

and hence, U is η-demimetric mapping with $\xi \leqslant \eta < 1$. To be more precise,

(a) if $0 \leqslant a_t < 1$, $\forall t \in G$, then $\xi \leqslant -1$;

(b) if $-1 \leqslant a_t < 0$, $\forall t \in G$, then $-1 < \xi \leqslant 0$;

(c) if $a_t \leqslant -1$, $\forall t \in G$, then $0 \leqslant \xi < 1$.

Case 2: $D \neq \emptyset$. Thus, from (3.21), we have

$$\|Ux - \bar{x}\|^2 \leqslant \|x - \bar{x}\|^2 + \sigma \|x - Ux\|^2, \ \ \forall (x, \bar{x}) \in \mathbb{R}^p \times F(U),$$

where $\sigma = \max\{\xi, 0\}$. Therefore, U is η-demimetric with $0 \leqslant \sigma \leqslant \eta < 1$.

One can notice here that if $a_t > 1$ for some $t \in G$, then $F(U) = \{0\}$ and U is not demimetric mapping.

4. Application

As applications, we can obtain several new algorithms to solve problems that can be converted to the fixed point problem of demimetric mappings.

Let H_1 and H_2 be two real Hilbert spaces and $A_k : H_1 \to H_2$ is a nonzero bounded linear operator for each $k \in \mathbb{N}$.

4.1. Generalized split system of minimization problem

Let H be a real Hilbert space and let $f : H \to \mathbb{R} \cup \{+\infty\}$ proper, lower semicontinuous convex function. The proximal operator of the function f with scaling parameter $\lambda > 0$ is a mapping $\text{prox}_{\lambda f} : H \to H$ given by

$$\text{prox}_{\lambda f}(x) = \arg\min_{y \in H}\{f(y) + \frac{1}{2\lambda}\|x - y\|^2\}.$$

Proximal operators are firmly nonexpansive, and the point $\bar{x}$ minimizes f if and only if $\text{prox}_{\lambda f}(\bar{x}) = \bar{x}$, see [2].

Consider the *generalized split system of minimization problem* given by

$$\bar{x} \in \bigcap_{i=1}^{\infty}(\arg\min f_i) \text{ such that } A_k(\bar{x}) \in \bigcap_{i=1}^{\infty}(\arg\min g_i), \ \ \forall k \in \mathbb{N}, \tag{4.1}$$

where $f_i : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g_i : H_2 \to \mathbb{R} \cup \{+\infty\}$ are proper, lower semicontinuous convex functions for $i \in \mathbb{N}$. Let Ω be the solution set of (4.1), and assume that Ω is nonempty.

Now, by taking $T_i = \text{prox}_{\lambda f_i}$ and $U_i = \text{prox}_{\lambda g_i}$ in (1.2), we can have following two strong convergence Theorem to approximate the solution of (4.1).

Algorithm 4.1. Let $\{\theta_n\}$, $\{\delta_n^{(i)}\}_{n=1}^{\infty}$ ($i \in \mathbb{N}$) and $\{\sigma_n^{(k)}\}_{n=1}^{\infty}$ ($k \in \mathbb{N}$) be real sequences satisfying **Condition I**, and let $\{\rho_{(i,n)}\}_{n=1}^{\infty}$ ($i \in \mathbb{N}$) be a real sequence such that

(C4) $0 < \rho_{(i,n)} < 2$ and $\liminf_{n\to\infty} \rho_{(i,n)}(2 - \rho_{(i,n)}) > 0$.

Choose $x_0, x_1 \in H_1$ arbitrarily and follow the following iterative steps.

STEP 1. Evaluate $z_n = x_n + \theta_n(x_n - x_{n-1})$.

STEP 2. Evaluate $t_n^{(i)} = (I - \text{prox}_{\lambda f_i})z_n$ and $y_n^{(i,k)} = (I - \text{prox}_{\lambda g_i})A_k z_n$. Let $\Psi_n = \{(i, k) \in \mathbb{N} \times \mathbb{N} : \|A_k^*(y_n^{(i,k)}) + t_n^{(i)}\|\} \neq 0\}$. If $\Psi_n = \emptyset$, then **STOP**. Otherwise, go to **STEP 3**.

STEP 3. Compute

$$s_n = z_n - \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2}{\mu_{(i,k)}(z_n)} (A_k^*(y_n^{(i,k)}) + t_n^{(i)}),$$

where $\mu_{(i,k)}(z_n) = \|A_k^*(y_n^{(i,k)}) + t_n^{(i)}\|^2$ if $(i, k) \in \Psi_n$, $\mu_{(i,k)}(z_n) = 1$ otherwise.

STEP 4. Evaluate $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, where

$$C_n = \Big\{z \in H_1 : \|s_n - z\|^2 \leqslant \|z_n - z\|^2 - \sum_{k=1}^{\infty} \sigma_n^{(k)} \sum_{i=1}^{\infty} \delta_n^{(i)} \rho_{(i,n)}(2 - \rho_{(i,n)}) \frac{(\|y_n^{(i,k)}\|^2 + \|t_n^{(i)}\|^2)^2}{\mu_{(i,k)}(z_n)}\Big\}$$

and $Q_n = \big\{z \in H_1 : \langle x_n - z, x_n - x_0\rangle \leqslant 0\big\}$.

STEP 5. Set $n := n + 1$ and go to **STEP 1**.

Theorem 4.2. *The sequence* $\{x_n\}$ *generated by Algorithm 4.1 converges strongly to* $\bar{x} \in \Omega$*, where* $\bar{x} = P_\Omega(x_0)$*.*

Algorithm 4.3. Let $\{\theta_n\}$, $\{\delta_n^{(i)}\}_{n=1}^\infty$ $(i \in \mathbb{N})$ and $\{\sigma_n^{(k)}\}_{n=1}^\infty$ $(k \in \mathbb{N})$ be real sequences satisfying **Condition I**, and let $\{\rho_{(i,n)}\}_{n=1}^\infty$ $(i \in \mathbb{N})$ be a real sequence such that

(C4) $0 < \rho_{(i,n)} < 2$ and $\liminf_{n\to\infty} \rho_{(i,n)}(2-\rho_{(i,n)}) > 0$.

Choose $x_0, x_1 \in H_1$ arbitrarily and follow the following iterative steps.

STEP 1. Evaluate $z_n = x_n + \theta_n(x_n - x_{n-1})$.

STEP 2. Evaluate $y_n^{(i,k)} = (I - \text{prox}_{\lambda g_i})A_k z_n$. Let $\Psi_n = \{(i,k) \in \mathbb{N}\times\mathbb{N} : \|A_k^*(y_n^{(i,k)})\| \neq 0\}$.

STEP 3. Compute

$$s_n = \sum_{i=1}^\infty \delta_n^{(i)} \text{prox}_{\lambda f_i}\Big(z_n - \sum_{k=1}^\infty \sigma_n^{(k)} \rho_{(i,n)} \frac{\|y_n^{(i,k)}\|^2}{\mu_{(i,k)}(z_n)} A_k^*(y_n^{(i,k)})\Big),$$

where $\mu_{(i,k)}(z_n) = \|A_k^*(y_n^{(i,k)})\|^2$ if $(i,k) \in \Psi_n$, $\mu_{(i,k)}(z_n) = 1$ otherwise.

STEP 4. Evaluate $x_{n+1} = P_{C_n\cap Q_n}(x_0)$, where

$$C_n = \Big\{z \in H_1 : \|s_n - z\|^2 \leqslant \|z_n - \bar{x}\|^2 - \sum_{i=1}^\infty \delta_n^{(i)} \sum_{k=1}^\infty \sigma_n^{(k)} \rho_{(i,n)}(2-\rho_{(i,n)}) \frac{\|y_n^{(i,k)}\|^4}{\mu_{(i,k)}(z_n)}\Big\}$$

and $Q_n = \big\{z \in H_1 : \langle x_n - z, x_n - x_0\rangle \leqslant 0\big\}$.

STEP 5. Set $n := n+1$ and go to **STEP 1**.

Theorem 4.4. *The sequence* $\{x_n\}$ *generated by Algorithm 4.3 converges strongly to* $\bar{x} \in \Omega$*, where* $\bar{x} = P_\Omega(x_0)$*.*

4.2. Generalized multiple-set split feasibility problem

The generalized multiple-set split feasibility problem (GMSSFP) is the problem of finding a point

$$\bar{x} \in \bigcap_{i=1}^\infty C_i \text{ such that } A_k(\bar{x}) \in \bigcap_{i=1}^\infty Q_i, \quad \forall k \in \mathbb{N}, \tag{4.2}$$

where C_i $(i \in \mathbb{N})$ and Q_i $(i \in \mathbb{N})$ are nonempty closed convex subsets of H_1 and H_2, respectively. The GMSSFP (4.2) is a special case of (4.1), i.e., take $f_i = \delta_{C_i}$ and $g_i = \delta_{Q_i}$ (the indicator functions) in (4.1).

4.3. Generalized split system of inclusion problem

For a real Hilbert space H and maximal monotone set-valued mapping $T : H \to 2^H$, the resolvent operator J_λ^T associated with T and $\lambda > 0$ is

$$J_\lambda^T(x) = (I + \lambda T)^{-1}(x), \quad x \in H.$$

The resolvent operator J_λ^T is single-valued and firmly nonexpansive. Moreover, $0 \in T(\bar{x})$ if and only if $\bar{x}$ is a fixed point of J_λ^T for all $\lambda > 0$; see [33].

Let $T_i : H_1 \to 2^{H_1}$ and $U_i : H_2 \to 2^{H_2}$ be maximal monotone mappings for all $i \in \mathbb{N}$. The *generalized split system of inclusion problem* is to find $\bar{x} \in H_1$ such that

$$\begin{cases} 0 \in T_i(\bar{x}), & \forall i \in \mathbb{N}, \\ 0 \in U_i(A_k(\bar{x})), & \forall (i,k) \in \mathbb{N}\times\mathbb{N}. \end{cases} \tag{4.3}$$

Replacing U_i by $J_\lambda^{T_i}$ and T_i by $J_\lambda^{U_i}$ in Algorithms 3.1 and 3.4, we obtain the following two strong convergence Theorems for solving (4.3).

4.4. Generalized split system of equilibrium problems

Let $h : H \times H \to \mathbb{R}$ be a bifunction. Then, we say that h satisfies Condition CO on H if the following assumptions are satisfied:

(A1) $h(u, u) = 0$, for all $u \in H$;
(A2) h is monotone on H, i.e., $h(u, v) + h(v, u) \leqslant 0$, for all $u, v \in H$;
(A3) for each $u, v, w \in H$, $\limsup_{\alpha \downarrow 0} h(\alpha w + (1-\alpha)u, v) \leqslant h(u, v)$;
(A4) $h(u, .)$ is convex and lower semicontinuous on H for each $u \in H$.

Lemma 4.5 ([12]). *If* h *satisfies Condition* CO *on* H*, then for each* $r > 0$ *and* $u \in H$*, the mapping (called resolvant of* h*) given by*

$$T_r^h(u) = \{w \in H : h(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geqslant 0,\ \forall v \in H\},$$

satisfies the following conditions:

(1) T_r^h *is single-valued and* T_r^h *is a firmly nonexpansive;*
(2) $\mathrm{Fix}(T_r^h) = \{\bar{x} \in H : h(\bar{x}, y) \geqslant 0, \forall y \in H\}$;
(3) $\{\bar{x} \in H : h(\bar{x}, y) \geqslant 0, \forall y \in H\}$ *is closed and convex.*

Let $f_i : H_1 \times H_1 \to \mathbb{R}$ and $g_i : H_2 \times H_2 \to \mathbb{R}$ be bifunctions, where $i \in \mathbb{N}$. Assume each bifunction f_i and g_i satisfies Condition CO on H_1 and H_2, respectively. The *generalized split system of equilibrium problem* (GSSEP) is to find $\bar{x} \in H_1$ such that

$$\begin{cases} f_i(\bar{x}, x) \geqslant 0, & \forall x \in H_1, \forall i \in \mathbb{N}, \\ g_i(A_k(\bar{x}), u) \geqslant 0, & \forall u \in H_2, , \forall (i, k) \in \mathbb{N} \times \mathbb{N}. \end{cases} \tag{4.4}$$

The problem (4.4) includes as special cases, split variational inequalities, minimax problems and Nash equilibrium problems in noncooperative games (see, for example, [29]).

Similarly, for r>0, replacing U_i by $T_r^{f_i}$ and T_i by $T_r^{g_i}$ in Algorithms 3.1 and 3.4, we obtain two strong convergence Theorems (Theorems 3.3 and 3.5) for approximation of solution of consistent GSSEP (4.4).

Remark 4.6. The results mentioned as an application improves and generalizes several well-known results in literature, for example [6, 8, 16–18, 33, 36–38] and the reference therein.

5. Conclusions and remark

In this paper, we proposed two novel inertial CQ-algorithms with operator norm independent self-adaptive step size section technique for approximating a solution of the generalized split system of common fixed point problem (1.2) for demimetric mappings in framework of a real Hilbert spaces. In view of the previous results in the literature, the contribution of this paper is twofold, firstly, it gives strong convergence Theorems with convergence is accelerated using inertial extrapolation, secondly, it generalizes several problems considered by researchers in framework of GSIP. Moreover, the constructed iterative methods used extended variable step-sizes generated by the algorithms at each iteration, based on previously evaluated iterations so that the implementation of our algorithm does not need any prior information about the operator norms A_k ($k \in \mathbb{N}$). Our future research work project aim is to establish a computationally easy iterative algorithm and to extend this result from Hilbert spaces to more general reflexive Banach spaces.

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