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# On Gould-Hopper based fully degenerate Type2 poly-Bernoulli polynomials with a q-parameter

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**Abstract**

In this paper, the Gould-Hopper based fully degenerate type2 poly-Stirling polynomials of the first kind with a q parameter are considered and some of their diverse identities and properties are investigated. Then, the Gould-Hopper based fully degenerate type2 poly-Bernoulli polynomials with a q parameter are introduced and some of their properties are analyzed and derived. Furthermore, several formulas and relations covering implicit summation formulas, recurrence relations and symmetric property are attained.

**Keywords:** Gould-Hopper polynomials, Bernoulli polynomials, poly-Bernoulli polynomials, degenerate Bernoulli function, Stirling numbers of the first kind.

**2020 MSC:** 33C45, 11B68, 11B73.



## 1. Introduction

Special polynomials possess an crucial role in mathematics such as solutions of the numerical problems, solutions of the difference equations, describe the trajectory of projectiles, combinatorics relations, determine the composition of certain molecules and compounds, determining pressure in applications of fluid dynamics, cost analysis in economics, approximation theory and so on, see [1–15] and the Bernoulli polynomials (see [2, 14, 16]. Two of the significant families of polynomials are the Gould-Hopper polynomials (see [5–7]) and the Bernoulli polynomials (see [2, 14, 16]). In the recent years, the aforesaid polynomials with their several generalizations have been worked and investigated by several physicists and mathematicians, see [1–16]. Motivated and inspired the mentioned works, here we consider a new family of polynomials covering the Gould-Hopper polynomials and Stirling polynomials of the first kind by using the degenerate exponential function and degenrate logarithm function and investigate some of their properties. Moreover, we consider another new family of polynomials including the Gould-Hopper

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polynomials and Bernoulli polynomials by utilizing the degenerate exponential function and polyexponential function and attain some of their diverse basic relations and properties covering symmetric property, recurrence relations and summation formulas.

The $\lambda$-falling factorial $(\sigma)_{\rho,\lambda}$ is provided, for $\lambda \in \mathbb{C}$, as follows (see [5, 7, 8, 10–12])

$$(\sigma)_{\rho,\lambda} = \begin{cases} \sigma(\sigma-\lambda)(\sigma-2\lambda)\cdots(\sigma-(\rho-1)\lambda), & \rho = 1,2,\ldots \\ 1, & \rho = 0. \end{cases} \tag{1.1}$$

Upon setting $\lambda = 1$, $(\sigma)_{\rho,1} := (\sigma)_\rho$ termed the usual falling factorial given by

$$(\sigma)_{\rho,1} := (\sigma)_\rho = \sigma(\sigma-1)\cdots(\sigma-\rho+1) \text{ and } (\sigma)_0 = 1.$$

The difference operator $\Delta_{\lambda,\sigma}$ with respect to $\sigma$ is provided as follows (cf. [3, 6])

$$\Delta_{\lambda,\sigma} f(\sigma) = \frac{1}{\lambda}(f(\sigma+\lambda) - f(\sigma)), \quad \lambda \neq 0. \tag{1.2}$$

The degenerate version of $e^\sigma$ is given by (see [5, 7, 8, 10–12])

$$(\lambda z + 1)^{\frac{\sigma}{\lambda}} = e_\lambda^\sigma(z) \text{ and } e_\lambda^1(z) = e_\lambda(z). \tag{1.3}$$

Note that $\lim_{\lambda\to 0} e_\lambda^\sigma(z) = e^{\sigma z}$. It is readily observed from (1.1), (1.2), and (1.3) that

$$e_\lambda^\sigma(z) = \sum_{\rho=0}^{\infty} (\sigma)_{\rho,\lambda} \frac{z^\rho}{\rho!} \quad \text{and} \quad \Delta_{\lambda,\sigma} e_\lambda^\sigma(z) = z e_\lambda^\sigma(z).$$

The degenerate form of logarithm function is defined as follows

$$\log_\lambda(1+z) := \frac{(1+z)^\lambda - 1}{\lambda} = \frac{1}{\lambda}\sum_{\ell=1}^{\infty} (\lambda)_\ell \frac{z^\ell}{\ell!},$$

which holds the following relations with the degenerate exponential function:

$$\log_\lambda(e_\lambda(1+z)) = 1+z \text{ and } e_\lambda\left(\log_\lambda(1+z)\right) = 1+z.$$

The Bernoulli $B_\rho(\sigma)$ polynomials and the degenerate Bernoulli $B_{\rho,\lambda}(\sigma)$ polynomials are given as follows:

$$\frac{z}{e^z-1} e^{\sigma z} = \sum_{\rho=0}^{\infty} B_\rho(\sigma) \frac{z^\rho}{\rho!} \quad \text{and} \quad \frac{z}{e_\lambda(z)-1} e_\lambda^\sigma(z) = \sum_{\rho=0}^{\infty} B_{\rho,\lambda}(\sigma) \frac{z^\rho}{\rho!}.$$

One can look at the references [1–16] to see the several studies and applications of these polynomials.

The Gould-Hopper polynomials are provided as follows (see [5–7])

$$\sum_{\rho=0}^{\infty} GH_\rho^{(j)}(\sigma,\vartheta) \frac{z^\rho}{\rho!} = e^{\sigma z + \vartheta z^j}, \tag{1.4}$$

where $j \in \mathbb{N}$ in conjunction with $2 \leqslant j$. Letting $j = 2$ in (1.4) yields the usual Hermite polynomials $GH_\rho^{(2)}(\sigma,\vartheta)$ (see [8, 9, 13–15]).

The Stirling numbers $S_1(\rho,\upsilon)$ of the first kind and the Stirling numbers $S_2(\rho,\upsilon)$ of the second kind are provided by ([2, 5–7, 10–12, 16])

$$\frac{(\log(1+z))^\upsilon}{\upsilon!} = \sum_{\rho=0}^{\infty} S_1(\rho,\upsilon) \frac{z^\rho}{\rho!} \text{ and } \frac{(e^z-1)^\upsilon}{\upsilon!} = \sum_{\rho=0}^{\infty} S_2(\rho,\upsilon) \frac{z^\rho}{\rho!}. \tag{1.5}$$

From (1.5), we get the following relations for $\rho \geqslant 0$:

$$(\sigma)_\rho = \sum_{\upsilon=0}^{\rho} S_1(\rho,\upsilon)\,\sigma^\upsilon \quad \text{and} \quad \sigma^\rho = \sum_{\upsilon=0}^{\rho} S_2(\rho,\upsilon)\,(\sigma)_\upsilon .$$

The degenerate form of poly-Bernoulli polynomials are considered by (see [10])

$$\sum_{\rho=0}^{\infty} \beta_{\rho,\lambda}^{(\upsilon)}(\sigma)\frac{z^\rho}{\rho!} = \mathrm{Ei}_\upsilon(\log(1+z))\frac{e_\lambda^\sigma(z)}{e_\lambda(z)-1}, \tag{1.6}$$

where the polyexponential function is provided by

$$\mathrm{Ei}_\upsilon(z) = \sum_{\rho=1}^{\infty}\frac{z^\rho}{(\rho-1)!\rho^\upsilon}$$

as inverse function to the polylogarithm function

$$\mathrm{Li}_\upsilon(z) = \sum_{\rho=1}^{\infty}\frac{z^\rho}{\rho^\upsilon} \qquad (|z|<1;\upsilon\in\mathbb{Z}).$$

Upon setting $\sigma = 0$ in (1.6), $\beta_{\rho,\lambda}^{(\upsilon)}(0) := \beta_{\rho,\lambda}^{(\upsilon)}$ are termed the degenerate poly-Bernoulli numbers. For details about these polynomials see [10]. Since $\mathrm{Ei}_1(z) = e^z - 1$, we note that

$$\beta_{\rho,\lambda}^{(1)}(\sigma) := B_{\rho,\lambda}(\sigma).$$

The degenerate form of the Stirling numbers $S_{1,\lambda}(\rho,\upsilon)$ of the first kind is provided by

$$\frac{(\log_\lambda(1+z))^\psi}{\psi!} = \sum_{\rho=\psi}^{\infty} S_{1,\lambda}(\rho,\psi)\frac{z^\rho}{\rho!}.$$

Let $\rho, j \in \mathbb{Z}$ with $\rho \geqq 0$, $j > 0$ and $q \in \mathbb{R}/\{0\}$ with $q \neq 0$. The fully degenerate Gould-Hopper polynomials with a q parameter are provided by

$$\sum_{\rho=0}^{\infty} GH_{\rho,\lambda;q}^{(j)}(\sigma,\vartheta)\frac{z^\rho}{\rho!} = e_\lambda^\sigma(qz)\,e_\lambda^\vartheta\left(qz^j\right). \tag{1.7}$$

When $q \to 1$ and $\lambda \to 0$, the polynomials in (1.7) reduce to the Gould-Hopper polynomials denoted by $GH_\rho^{(j)}(\sigma,\vartheta)$ ([5, 7]).

The degenerate form of Gould-Hopper polynomials with a q parameter satisfy the following summation formula

$$GH_{\rho,\lambda;q}^{(j)}(\sigma,\vartheta) = \rho!\sum_{\upsilon=0}^{\lfloor\rho/j\rfloor}\frac{(\sigma)_{\rho-j\upsilon,\lambda}(\vartheta)_{\upsilon,\lambda}}{(\rho-j\upsilon)!\upsilon!}q^{\rho-(j-1)\upsilon},$$

where we used the following elementary series manipulation

$$\sum_{\rho=0}^{\infty}\sum_{\upsilon=0}^{\infty}A(\upsilon,\rho) = \sum_{\rho=0}^{\infty}\sum_{\upsilon=0}^{\lfloor\rho/j\rfloor}A(\upsilon,\rho-j\upsilon).$$

Also note that the following difference rules hold ([5, 7])

$$\Delta_{\lambda,\sigma}GH_{\rho,\lambda;q}^{(j)}(\sigma,\vartheta) = q\rho\; GH_{\rho-1,\lambda;q}^{(j)}(\sigma,\vartheta), \quad \Delta_{\lambda,\vartheta}GH_{\rho,\lambda;q}^{(j)}(\sigma,\vartheta) = q\,(\rho)_j\; GH_{\rho-j,\lambda;q}^{(j)}(\sigma,\vartheta).$$

## 2. The Gould-Hopper based fully degenerate type2 poly-Bernoulli polynomials with a q parameter

In this section, we deal with the Gould-Hopper based fully degenerate Stirling polynomials of the first kind with a q parameter and the Gould-Hopper based fully degenerate type2 poly-Bernoulli polynomials with a q parameter. Then, we investigate their diverse relations and properties.

**Definition 2.1.** Let $\rho, \psi \in \mathbb{N}_0$ and $j \in \mathbb{N}$ and let $q \in \mathbb{R}/\{0\}$. The Gould-Hopper based fully degenerate Stirling polynomials of the first kind with a q parameter are introduced by

$$\sum_{\rho=\psi}^{\infty} S_{1,\lambda;q}^{(j)}(\rho,\psi:\sigma,\vartheta)\frac{z^{\rho}}{\rho!}=\frac{\left(\log_{\lambda}(1+qz)\right)^{\psi}}{\psi!}e_{\lambda}^{\sigma}(qz)\,e_{\lambda}^{\vartheta}\left(qz^{j}\right). \tag{2.1}$$

*Remark* 2.2.

1. Upon setting $\lambda \to 0$, we attain the Gould-Hopper based Stirling polynomials of the first kind with a q parameter represented by $S_{1;q}^{(j)}(\rho,\psi:\sigma,\vartheta)$ ([6]).
2. Letting $\lambda \to 0$ and $q \to 1$, we acquire the Gould-Hopper based Stirling polynomials of the first kind represented by $S_{1}^{(j)}(\rho,\psi:\sigma,\vartheta)$ ([6, 13]).
3. Upon setting $\lambda \to \vartheta = 0$ and $q \to 1$, we acquire the Stirling polynomials of the first kind represented by $S_1(\rho,\psi:\sigma)$ ([16]).
4. Letting $\lambda \to \vartheta = \sigma = 0$ and $q \to \alpha = 1$, we acquire the usual Stirling numbers of the first kind represented by $S_1(\rho,\psi)$ (see [2, 5–7, 10–12, 16]).

**Theorem 2.3.** *The following summation formulae*

$$S_{1,\lambda;q}^{(j)}(\rho,\psi:\sigma,\vartheta)=\sum_{s=0}^{\rho}\binom{\rho}{s}S_{1,\lambda;q}(s,\psi)\,GH_{\rho-s,\lambda;q}^{(j)}(\sigma,\vartheta),$$
$$S_{1,\lambda;q}^{(j)}(\rho,\psi:\sigma,\vartheta)=\sum_{s=0}^{\rho}\binom{\rho}{s}\lambda^{s}q^{s}\left(\frac{\sigma}{\lambda}\right)_{s}S_{1,\lambda;q}^{(j)}(\rho-s,\psi:0,\vartheta),$$
$$S_{1,\lambda;q}^{(j)}(\rho,\psi:\sigma,\vartheta)=\rho!\sum_{s=0}^{\lfloor\rho/j\rfloor}\frac{\lambda^{s}q^{s}}{s!\,(\rho-js)!}\left(\frac{\vartheta}{\lambda}\right)_{s}S_{1,\lambda;q}^{(j)}(\rho-js,\psi:\sigma)$$

*hold for* $j \in \mathbb{N}$ *and* $q \in \mathbb{R}/\{0\}$.

*Proof.* The proofs can be completed just by utilizing Cauchy product to the generating function (2.1). Therefore, we omit the details. □

**Theorem 2.4.** *The following difference rules*

$$\Delta_{\lambda,\sigma}S_{1,\lambda;q}^{(j)}(\rho,\psi:\sigma,\vartheta)=q\rho S_{1,\lambda;q}^{(j)}(\rho-1,\psi:\sigma,\vartheta)\quad\textit{and}\quad\Delta_{\lambda,\vartheta}S_{1,\lambda;q}^{(j)}(\rho,\psi:\sigma,\vartheta)=q\,(\rho)_{j}\,S_{1,\lambda;q}^{(j)}(\rho-j,\psi:\sigma,\vartheta)$$

*are valid.*

*Proof.* The proofs can be completed just by utilizing Cauchy product to the generating function (2.1). Therefore, we omit the details. □

**Definition 2.5.** Let $\rho, \upsilon, j \in \mathbb{Z}$ with $\rho \geqq 0$, $\upsilon, j > 0$ and $q \in \mathbb{R}/\{0\}$ with $q \neq 0$. The Gould-Hopper based fully degenerate type2 poly-Bernoulli ${}_{GH}\beta_{\rho,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)$ polynomials with a q parameter are considered as follows

$$q\mathrm{Ei}_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)\frac{e_{\lambda}^{\sigma}(qz)\,e_{\lambda}^{\vartheta}\left(qz^{j}\right)}{1-e_{\lambda}^{-1}(qz)}=\sum_{\rho=0}^{\infty}{}_{GH}\beta_{\rho,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}. \tag{2.2}$$

Upon setting $\sigma = 0 = \vartheta$, we then get ${}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(0,0) := {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}$ termed the Gould-Hopper based fully degenerate type2 poly-Bernoulli ${}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}$ numbers with a q parameter, see [11].

Here are some examinations of specific circumstances of ${}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)$.

*Remark* 2.6.

1. Upon setting $\lambda \to 0$, we attain the Gould-Hopper based type2 poly-Bernoulli polynomials with a q parameter represented by ${}_{GH}\beta^{(\upsilon,j)}_{\rho;q}(\sigma,\vartheta)$ ([6]).
2. Setting $\lambda \to 0$ and $q \to 1$, we attain the Gould-Hopper based type2 poly-Bernoulli polynomials represented by ${}_{GH}\beta^{(\upsilon,j)}_{\rho}(\sigma,\vartheta)$ (see [6, 9, 14]).
3. Letting $\lambda \to 0$ and $\upsilon = 1$, we attain the Gould-Hopper based Bernoulli polynomials with a q parameter represented by ${}_{GH}\beta^{(\upsilon,j)}_{\rho}(\sigma,\vartheta)$ (see [4, 6]).
4. Letting $\upsilon = q \to 1$ and $\vartheta = 0$, we attain the fully degenerate Bernoulli polynomials represented by $\beta_{\rho,\lambda}(\sigma)$ (see [6, 8, 10–12]).
5. Upon setting $\upsilon = q \to 1$, and $\lambda \to 0$, we acquire the Gould-Hopper based Bernoulli polynomials represented by ${}_{GH}\beta_{\rho}(\sigma,\vartheta)$ ([14]).
6. Letting $\upsilon = q \to 1$, $\lambda \to 0$ and $\vartheta = 0$, we attain the classical Bernoulli polynomials represented by $B_{\rho}(\sigma)$ (see [2, 14, 16]).

Here are some properties of ${}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)$.

**Theorem 2.7** (Summation formulas)**.** *We have*

$$ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta) = \sum_{s=0}^{\rho} \binom{\rho}{s} \beta^{(\upsilon)}_{s,\lambda;q} GH^{(j)}_{\rho-s,\lambda;q}(\sigma,\vartheta) $$

*and*

$$ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta) = \rho! \sum_{s=0}^{\lfloor \rho/j \rfloor} \binom{\frac{\vartheta}{\lambda}}{s} \frac{\lambda^s q^s}{s!(\rho-js)!} \beta^{(\upsilon)}_{\rho-js,\lambda;q}(\sigma). $$

*Proof.* Indeed, by (2.2), we get

$$ \begin{aligned} \sum_{\rho=0}^{\infty} {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^\rho}{\rho!} &= q\mathrm{Ei}_{\upsilon}\left(\frac{\log_\lambda(1+qz)}{q}\right)\frac{e^{\sigma}_{\lambda}(qz)\, e^{\vartheta}_{\lambda}(qz^j)}{1-e^{-1}_{\lambda}(qz)} \\ &= \sum_{\rho=0}^{\infty}\left(\sum_{s=0}^{\rho}\binom{\rho}{s}\beta^{(\upsilon)}_{s,\lambda;q} GH^{(j)}_{\rho-s,\lambda;q}(\sigma,\vartheta)\right)\frac{z^\rho}{\rho!} \end{aligned} $$

and

$$ \begin{aligned} \sum_{\rho=0}^{\infty} {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^\rho}{\rho!} &= q\mathrm{Ei}_{\upsilon}\left(\frac{\log_\lambda(1+qz)}{q}\right)\frac{e^{\sigma}_{\lambda}(qz)\, e^{\vartheta}_{\lambda}(qz^j)}{1-e^{-1}_{\lambda}(qz)} \\ &= \left(\sum_{\rho=0}^{\infty}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma)\frac{z^\rho}{\rho!}\right)\left(\sum_{\rho=0}^{\infty}\binom{\frac{\vartheta}{\lambda}}{s} q^\rho\lambda^\rho\frac{z^{j\rho}}{\rho!}\right) \\ &= \sum_{\rho=0}^{\infty}\left(\sum_{s=0}^{\lfloor \rho/j \rfloor}\beta^{(\upsilon,j)}_{\rho-js,\lambda;q}(\sigma)\frac{z^{\rho-js}}{(\rho-js)!}\binom{\frac{\vartheta}{\lambda}}{s} q^s\lambda^s\frac{z^{js}}{s!}\right) \\ &= \sum_{\rho=0}^{\infty}\left(\sum_{s=0}^{\lfloor \rho/j \rfloor}\beta^{(\upsilon,j)}_{\rho-js,\lambda;q}(\sigma)\binom{\frac{\vartheta}{\lambda}}{s}\frac{q^s\lambda^s\rho!}{s!(\rho-js)!}\right)\frac{z^\rho}{\rho!}, \end{aligned} $$

which give the desired results. □

**Theorem 2.8.** *$\lambda$-Difference Rules for* ${}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)$ *are*

$$\Delta_{\lambda,\sigma}\ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)=\rho q\ {}_{GH}\beta^{(\upsilon,j)}_{\rho-1,\lambda;q}(\sigma,\vartheta)$$

*and*

$$\Delta_{\lambda,\vartheta}\ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)=q(\rho)_j\ {}_{GH}\beta^{(\upsilon,j)}_{\rho-j,\lambda;q}(\sigma,\vartheta).$$

*Proof.* The proofs can be completed just by utilizing Cauchy product to the generating function (2.2). Therefore, we omit the details. □

Here, we give multifarious connection formulas by the following theorems.

**Theorem 2.9.** *The following relation holds*

$$\sum_{\gamma=0}^{\rho}\binom{\rho}{\gamma}\sum_{\psi=0}^{\gamma}\frac{q^{\rho-\psi}}{\lambda(\psi+1)^{\upsilon}}GH^{(j)}_{\rho-\gamma,\lambda;q}(\sigma,\vartheta)\left(\sum_{\ell=0}^{\gamma}\binom{\gamma}{l}S_{1,\lambda}(\ell,\psi)(\lambda)_{\gamma-\ell}-S_{1,\lambda}(\gamma,\psi)\right)$$
$$={}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)-\sum_{\ell=0}^{\rho}\binom{\rho}{l}(-1)_{\ell,\lambda}q^{\ell}\ {}_{GH}\beta^{(\upsilon,j)}_{\rho-l,\lambda;q}(\sigma,\vartheta).$$

*Proof.* We observe that

$$\left(1-e_{\lambda}^{-1}(qz)\right)\left(\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}\right)=qEi_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)e_{\lambda}^{\sigma}(qz)e_{\lambda}^{\vartheta}\left(qz^{j}\right). \tag{2.3}$$

Let RHS and LHS be the right hand-side and the left hand-side of (2.3), respectively. It is observed that

$$\begin{aligned}
\mathrm{RHS}&=qe_{\lambda}^{\vartheta}\left(qz^{j}\right)Ei_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)e_{\lambda}^{\sigma}(qz)\\
&=q\sum_{\psi=1}^{\infty}\frac{\left(\log_{\lambda}(1+qz)\right)^{\psi}.q^{-\psi}}{(\psi-1)!.\psi^{\upsilon}}.e_{\lambda}^{\sigma}(qz)e_{\lambda}^{\vartheta}\left(qz^{j}\right)\\
&=\sum_{\psi=0}^{\infty}\frac{1}{\lambda}\left(\sum_{\rho=0}^{\infty}(\lambda)_{\rho}\frac{q^{\rho}z^{\rho}}{\rho!}-1\right)\left(\sum_{\rho=\psi}^{\infty}S_{1,\lambda}(\rho,\psi)\frac{q^{\rho}z^{\rho}}{\rho!}\right)\frac{q^{-\psi}}{(\psi+1)^{\upsilon}}e_{\lambda}^{\sigma}(qz)e_{\lambda}^{\vartheta}\left(qz^{j}\right)\\
&=\sum_{\psi=0}^{\infty}\frac{1}{\lambda}\sum_{\rho=0}^{\infty}\frac{q^{\rho}z^{\rho}}{\rho!}\frac{q^{-\psi}}{(\psi+1)^{\upsilon}}\left(\sum_{\ell=0}^{\rho}\binom{\rho}{l}S_{1,\lambda}(\ell,\psi)(\lambda)_{\rho-\ell}-S_{1,\lambda}(\rho,\psi)\right)e_{\lambda}^{\sigma}(qz)e_{\lambda}^{\vartheta}\left(qz^{j}\right)\\
&=\sum_{\rho=0}^{\infty}\frac{q^{\rho-\psi}}{(\psi+1)^{\upsilon}}\frac{z^{\rho}}{\rho!}\left(\frac{1}{\lambda}\sum_{\psi=0}^{\rho}\sum_{\ell=0}^{\rho}\binom{\rho}{l}S_{1,\lambda}(\ell,\psi)(\lambda)_{\rho-\ell}-S_{1,\lambda}(\rho,\psi)\right)\sum_{\rho=0}^{\infty}GH^{(j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}\\
&=\sum_{\rho=0}^{\infty}\sum_{\gamma=0}^{\rho}\binom{\rho}{\gamma}\sum_{\psi=0}^{\gamma}GH^{(j)}_{\rho-\gamma,\lambda;q}(\sigma,\vartheta)\left(\sum_{\ell=0}^{\gamma}\binom{\gamma}{l}S_{1,\lambda}(\ell,\psi)(\lambda)_{\gamma-\ell}-S_{1,\lambda}(\gamma,\psi)\right)\frac{q^{\rho-\psi}}{\lambda(\psi+1)^{\upsilon}}\frac{z^{\rho}}{\rho!}.
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{LHS}&=\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}-\sum_{\rho=0}^{\infty}(-1)_{\rho,\lambda}\frac{q^{\rho}z^{\rho}}{\rho!}\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}\\
&=\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}-\sum_{\rho=0}^{\infty}\left(\sum_{\ell=0}^{\rho}{}_{GH}\beta^{(\upsilon,j)}_{\rho-l,\lambda;q}(\sigma,\vartheta)\binom{\rho}{l}(-1)_{\ell,\lambda}q^{\ell}\right)\frac{z^{\rho}}{\rho!}\\
&=\sum_{\rho=0}^{\infty}\left({}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)-\sum_{\ell=0}^{\rho}{}_{GH}\beta^{(\upsilon,j)}_{\rho-l,\lambda;q}(\sigma,\vartheta)\binom{\rho}{l}(-1)_{\ell,\lambda}q^{\ell}\right)\frac{z^{\rho}}{\rho!}.
\end{aligned}$$
□

**Theorem 2.10.** *For* $\left|e_{\lambda}^{-1}(qz)\right| < 1$, *we have*

$$ {}_{GH}\beta_{\rho,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta) = \sum_{\psi=1}^{\rho}\sum_{\gamma=0}^{\infty}\frac{S_{1,\lambda;q}^{(j)}(\rho,\psi;\sigma-\gamma,\vartheta)}{\psi^{\upsilon-1}}q^{-\psi+1}. $$

*Proof.* By (2.2), we get

$$ \begin{aligned} \sum_{\rho=0}^{\infty} {}_{GH}\beta_{\rho,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!} &= \sum_{\gamma=0}^{\infty} e_{\lambda}^{-\gamma}(qz)\, e_{\lambda}^{\sigma}(qz)\, e_{\lambda}^{\vartheta}\left(qz^{j}\right) q\mathrm{Ei}_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right) \\ &= \sum_{\gamma=0}^{\infty} e_{\lambda}^{\vartheta}\left(qz^{j}\right) e_{\lambda}^{\sigma-\gamma}(qz)\sum_{\psi=0}^{\infty}\frac{\left(\log_{\lambda}(1+qz)\right)^{\psi-1}q^{-\psi+1}}{(\psi-1)!\psi^{\upsilon}} \\ &= \sum_{\psi=1}^{\infty}\sum_{\gamma=0}^{\infty}\frac{q^{-\psi+1}}{\psi^{\upsilon-1}}\frac{\left(\log_{\lambda}(1+qz)\right)^{\psi}}{\psi!}e_{\lambda}^{\vartheta}(qz^{j})e_{\lambda}^{\sigma-\gamma}(qz) \\ &= \sum_{\gamma=0}^{\infty}\sum_{\psi=1}^{\infty} e_{\lambda}^{\sigma-\gamma}(qz)\, e_{\lambda}^{\vartheta}\left(qz^{j}\right)\left(\sum_{\rho=\psi}^{\infty} S_{1,\lambda}(\rho,\psi)\frac{q^{\rho}z^{\rho}}{\rho!}\right)\frac{q^{-\psi+1}}{\psi^{\upsilon-1}} \\ &= \sum_{\psi=1}^{\infty}\sum_{\gamma=0}^{\infty}\sum_{\rho=\psi}^{\infty}\frac{q^{-\psi+1}}{\psi^{\upsilon-1}}S_{1,\lambda;q}^{(j)}(\rho,\psi;\sigma-\gamma,\vartheta)\, q^{\rho}\frac{z^{\rho}}{\rho!} \\ &= \sum_{\rho=\psi}^{\infty}\left(\sum_{\psi=1}^{\rho}\sum_{\gamma=0}^{\infty}S_{1,\lambda;q}^{(j)}(\rho,\psi;\sigma-\gamma,\vartheta)\frac{q^{-\psi+1}}{\psi^{\upsilon-1}}\right)\frac{z^{\rho}}{\rho!}, \end{aligned} $$

which is the desired result. □

**Theorem 2.11.** *We have*

$$ {}_{GH}\beta_{\rho,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta) - \sum_{\ell=0}^{\rho}\binom{\rho}{l}(-1)_{\ell,\lambda}\, q^{\ell}\ {}_{GH}\beta_{\rho-l,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta) = \sum_{\psi=1}^{\infty}\frac{q^{1-\psi}}{\psi^{\upsilon-1}}S_{1,\lambda;q}^{(j)}(\rho,\psi;\sigma,\vartheta). $$

*Proof.* Let LHS and RHS be the left hand-side and the right hand-side of (2.3), respectively. So, it is observed that

$$ \mathrm{LHS} = \sum_{\rho=0}^{\infty}\left({}_{GH}\beta_{\rho,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta) - \sum_{\ell=0}^{\rho}{}_{GH}\beta_{\rho-l,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\binom{\rho}{l}(-1)_{\ell,\lambda}\, q^{\ell}\right)\frac{z^{\rho}}{\rho!} $$

and

$$ \begin{aligned} \mathrm{RHS} &= \sum_{\psi=1}^{\infty}\frac{\log_{\lambda}(1+qz)^{\psi}}{\psi!}\frac{q^{1-\psi}}{\psi^{\upsilon-1}}e_{\lambda}^{\sigma}(qz)\, e_{\lambda}^{\vartheta}\left(qz^{j}\right) \\ &= \sum_{\psi=1}^{\infty}\sum_{\rho=\psi}^{\infty}S_{1,\lambda;q}^{(j)}(\rho,\psi;\sigma,\vartheta)\frac{q^{1-\psi}}{\psi^{\upsilon-1}}\frac{z^{\rho}}{\rho!} = \sum_{\rho=\psi}^{\infty}\left(\sum_{\psi=1}^{\infty}S_{1,\lambda;q}^{(j)}(\rho,\psi;\sigma,\vartheta)\frac{q^{1-\psi}}{\psi^{\upsilon-1}}\right)\frac{z^{\rho}}{\rho!}, \end{aligned} $$

which means the desired result. □

We give the following result.

**Theorem 2.12.** *We have*

$$ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)=\frac{1}{\rho+1}\sum_{\gamma=0}^{\rho+1}\binom{\rho+1}{\gamma}\left(\sum_{\psi=1}^{\gamma}S_{1,\lambda}(\gamma,\psi)\frac{q^{\gamma-1-\psi}}{\psi^{\upsilon-1}}\right){}_{GH}\beta^{(j)}_{\rho+1-\gamma,\lambda;q}(\sigma+1,\vartheta), $$

*where* ${}_{GH}\beta^{(j)}_{\rho,\lambda;q}(\sigma,\vartheta)$ *represents the Gould-Hopper based degenerate Bernoulli polynomials with a* $q$ *parameter provided by*

$$ \frac{ze^{\sigma}_{\lambda}(qz)\,e^{\vartheta}_{\lambda}(qz^{j})}{e_{\lambda}(qz)-1}=\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}. $$

*Proof.* By (2.2), we get

$$ \begin{aligned} \sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}&=\mathrm{Ei}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)\frac{q}{z}\frac{ze^{\sigma+1}_{\lambda}(qz)\,e^{\vartheta}_{\lambda}(qz^{j})}{e_{\lambda}(qz)-1}\\ &=\frac{q}{z}\sum_{\psi=1}^{\infty}\left(\sum_{\rho=\psi}^{\infty}q^{\rho}S_{1,\lambda}(\rho,\psi)\frac{z^{\rho}}{\rho!}\right)\left(\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(j)}_{\rho,\lambda;q}(\sigma+1,\vartheta)\frac{z^{\rho}}{\rho!}\right)\frac{q^{-\psi}}{\psi^{\upsilon-1}}\\ &=\frac{1}{z}\left(\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(j)}_{\rho,\lambda;q}(1+\sigma,\vartheta)\frac{z^{\rho}}{\rho!}\right)\left(\sum_{\rho=\psi}^{\infty}\sum_{\psi=1}^{\rho}q^{\rho+1-\psi}S_{1,\lambda}(\rho,\psi)\frac{z^{\rho}}{\psi^{\upsilon-1}\rho!}\right)\\ &=\sum_{\rho=0}^{\infty}\left[\sum_{\gamma=0}^{\rho}\sum_{\psi=1}^{\gamma}{}_{GH}\beta^{(j)}_{\rho,\lambda;q}(\sigma+1,\vartheta)S_{1,\lambda}(\gamma,\psi)\binom{\rho}{\gamma}\frac{q^{\gamma-1-\psi}}{\psi^{\upsilon-1}}\right]\frac{z^{\rho-1}}{\rho!}, \end{aligned} $$

which gives the desired result. □

**Theorem 2.13.** *The following relation holds*

$$ \sum_{\psi=1}^{\rho}S^{(j)}_{1,\lambda;q}(\rho,\psi;\sigma+1,\vartheta)\frac{q^{-\psi+1}}{\psi^{\upsilon-1}}={}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(1+\sigma,\vartheta)-{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta). $$

*Proof.* Multiplying $(e_{\lambda}(qz)-1)$ on both sides of (2.2), we observe that

$$ \begin{aligned} \sum_{\rho=0}^{\infty}\left({}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma+1,\vartheta)-{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\right)\frac{z^{\rho}}{\rho!}&=e^{\sigma+1}_{\lambda}(qz)\,\mathrm{Ei}_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)e^{\vartheta}_{\lambda}(qz^{j})\,q\\ &=\sum_{\psi=1}^{\infty}e^{\sigma+1}_{\lambda}(qz)\,e^{\vartheta}_{\lambda}(qz^{j})\left(\frac{(\log_{\lambda}(1+qz))^{\psi}}{\psi!}\right)\frac{q^{-\psi+1}}{\psi^{\upsilon-1}}\\ &=\sum_{\psi=1}^{\infty}\left(\sum_{\rho=\psi}^{\infty}S^{(j)}_{1,\lambda;q}(\rho,\psi;\sigma+1,\vartheta)\frac{z^{\rho}}{\rho!}\right)\frac{q^{-\psi+1}}{\psi^{\upsilon-1}}\\ &=\sum_{\rho=\psi}^{\infty}\left(\sum_{\psi=1}^{\rho}S^{(j)}_{1,\lambda;q}(\rho,\psi;\sigma+1,\vartheta)\frac{q^{-\psi+1}}{\psi^{\upsilon-1}}\right)\frac{z^{\rho}}{\rho!}. \end{aligned} $$

□

**Theorem 2.14.** *We have*

$$ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)=\sum_{\gamma=0}^{\psi-1}\sum_{s=0}^{\rho+1}\sum_{\psi=1}^{s}\frac{{}_{GH}\beta^{(j)}_{\rho+1-s,\lambda/\psi;q\psi}\left(\frac{\sigma+\gamma-1}{\psi}-1,\frac{\vartheta}{\psi}\right)}{\psi^{\upsilon-1}}q^{s-\psi+1}\binom{\rho+1}{s}\frac{S_{1,\lambda}(s,\psi)}{\rho+1}. $$

*Proof.* By (2.2), we acquire

$$\begin{aligned}
&\sum_{\rho=0}^{\infty} {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}\\
&\quad=\frac{e^{\psi}_{\lambda}(qz)-1^{\psi}}{e_{\lambda}(qz)-1}e^{\sigma+1}_{\lambda}(qz)\frac{qEi_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)}{e^{\psi}_{\lambda}(qz)-1}e^{\vartheta}_{\lambda}(qz^{j})\\
&\quad=qEi_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)\frac{\sum_{\gamma=0}^{\psi-1}e^{\sigma+\gamma+1-\psi}_{\lambda}(qz)\,e^{\vartheta}_{\lambda}(qz^{j})}{1-e^{-\psi}_{\lambda}(qz)}\\
&\quad=\sum_{\gamma=0}^{\psi-1}qEi_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)\frac{e^{\sigma+\gamma+1+\psi}_{\lambda}(qz)\,e^{\vartheta}_{\lambda}(qz^{j})}{1-e^{-\psi}_{\lambda}(qz)}\\
&\quad=\left(\sum_{\gamma=0}^{\psi-1}\left(\sum_{\rho=\psi}^{\infty}\sum_{\psi=1}^{\rho}q^{-\psi+1+\rho}\frac{S_{1,\lambda}(\rho,\psi)}{\psi^{\upsilon-1}}\right)\frac{z^{\rho-1}}{\rho!}\right)\left(\sum_{\rho=0}^{\infty}z^{\rho}\frac{{}_{GH}\beta^{(j)}_{\rho,\lambda/\psi;q\psi}\left(\frac{\sigma+\gamma-1}{\psi}-1,\frac{\vartheta}{\psi}\right)}{\rho!}\right)\\
&\quad=\sum_{\gamma=0}^{\psi-1}\sum_{\rho=0}^{\infty}\sum_{s=0}^{\rho}\sum_{\psi=1}^{s}\frac{q^{s-\psi+1}}{\psi^{\upsilon-1}}\binom{\rho}{s}S_{1,\lambda}(s,\psi)\;{}_{GH}\beta^{(j)}_{\rho-s,\lambda/\psi;q\psi}\left(\frac{\sigma+\gamma-1}{\psi}-1,\frac{\vartheta}{\psi}\right)\frac{z^{\rho-1}}{\rho!}.
\end{aligned}$$

□

**Theorem 2.15.** *We have*

$$ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma_1+\sigma_2,\vartheta_1+\vartheta_2)=\sum_{\gamma=0}^{\rho}\binom{\rho}{\gamma}\,{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma_1,\vartheta_1)\,GH^{(j)}_{\rho-\gamma,\lambda;q}(\sigma_2,\vartheta_2).$$

*Proof.* From (2.2), we attain

$$\begin{aligned}
\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma_1+\sigma_2,\vartheta_1+\vartheta_2)\frac{z^{\rho}}{\rho!}&=\frac{qEi_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)}{1-e^{-1}_{\lambda}(qz)}e^{\sigma_1+\sigma_2}_{\lambda}(qz)\,e^{\vartheta_1+\vartheta_2}_{\lambda}(qz^{j})\\
&=\left(\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma_1,\vartheta_1)\frac{z^{\rho}}{\rho!}\right)\left(\sum_{\rho=0}^{\infty}GH^{(j)}_{\rho,\lambda;q}(\sigma_2,\vartheta_2)\frac{z^{\rho}}{\rho!}\right)\\
&=\sum_{\rho=0}^{\infty}\left(\sum_{\gamma=0}^{\rho}\binom{\rho}{\gamma}\,{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma_1,\vartheta_1)\,GH^{(j)}_{\rho-\gamma,\lambda;q}(\sigma_2,\vartheta_2)\right)\frac{z^{\rho}}{\rho!},
\end{aligned}$$

which means the claimed result. □

**Theorem 2.16.** *We have*

$$ {}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)=\sum_{s=0}^{\lfloor\rho/j\rfloor}\sum_{\psi=0}^{\rho-js}\binom{\rho-js}{\psi}\beta^{(\upsilon)}_{\rho-js-\psi,\lambda;q}\left(\frac{\sigma}{\lambda}\right)_{\psi}\left(\frac{\vartheta}{\lambda}\right)_{s}\lambda^{\psi+s}q^{\psi+s}\frac{\rho!}{s!(\rho-js)!}.$$

*Proof.* By (2.2), we get

$$\begin{aligned}
&\sum_{\rho=0}^{\infty}{}_{GH}\beta^{(\upsilon,j)}_{\rho,\lambda;q}(\sigma,\vartheta)\frac{z^{\rho}}{\rho!}\\
&\quad=\frac{qEi_{\upsilon}\left(\frac{\log_{\lambda}(1+qz)}{q}\right)}{1-e^{-1}_{\lambda}(qz)}e^{\sigma}_{\lambda}(qz)\,e^{\vartheta}_{\lambda}(qz^{j})
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{\rho=0}^{\infty}\binom{\sigma}{\lambda}_{\rho}\lambda^{\rho}q^{\rho}\frac{z^{\rho}}{\rho!}\right)\left(\sum_{\rho=0}^{\infty}\binom{\vartheta}{\lambda}_{\rho}\lambda^{\rho}q^{\rho}\frac{z^{j\rho}}{\rho!}\right)\left(\sum_{\rho=0}^{\infty}{}_{GH}\beta_{\rho,\lambda;q}^{(\upsilon)}\frac{z^{\rho}}{\rho!}\right)\\
&= \left(\sum_{\rho=0}^{\infty}\lambda^{\rho}\binom{\vartheta}{\lambda}_{\rho}q^{\rho}\frac{z^{j\rho}}{\rho!}\right)\left(\sum_{\rho=0}^{\infty}\sum_{\psi=0}^{\rho}\binom{\sigma}{\lambda}_{\psi}\lambda^{\psi}\binom{\rho}{\psi}{}_{GH}\beta_{\rho-\psi,q}^{(\upsilon)}(\sigma,\vartheta;\lambda)\,q^{\psi}\frac{z^{\rho}}{\rho!}\right)\\
&= \sum_{\rho=0}^{\infty}\sum_{s=0}^{\lfloor\frac{\rho}{j}\rfloor}\sum_{\psi=0}^{\rho-js}\binom{\rho-js}{\psi}{}_{GH}\beta_{\rho-js-\psi,q}^{(\upsilon,j)}(\sigma,\vartheta;\lambda)\binom{\sigma}{\lambda}_{\psi}\binom{\vartheta}{\lambda}_{s}\lambda^{\psi+s}q^{\psi+s}\frac{\rho!}{s!\,(\rho-js)!}\frac{z^{\rho}}{\rho!}. \qquad \square
\end{aligned}
$$

The following rules hold ([15]):

$$
\sum_{\rho=0}^{\infty}\frac{(\sigma+\vartheta)^{\rho}}{\rho!}f(\rho)=\sum_{\rho,\psi=0}^{\infty}\frac{\sigma^{\rho}}{\rho!}\frac{\vartheta^{\psi}}{\psi!}f(\rho+\psi) \tag{2.4}
$$

and

$$
\sum_{\upsilon,l=0}^{\infty}A(l,\upsilon)=\sum_{\upsilon=0}^{\infty}\sum_{l=0}^{\upsilon}A(l,\upsilon-l). \tag{2.5}
$$

**Theorem 2.17** (Implicit summation formula)**.** *We have*

$$
{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\tau,\vartheta)=\sum_{\rho,\psi=0}^{s,l}\binom{s}{\rho}\binom{l}{\psi}{}_{GH}\beta_{s+l-\rho-\psi,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\,(\tau-\sigma)_{\rho+\psi,\lambda}. \tag{2.6}
$$

*Proof.* Substituting $z$ by $\gamma+z$ in (2.2), then, we derive

$$
\frac{q\mathrm{Ei}_{\upsilon}\left(\frac{\log_{\lambda}(1+q(z+\gamma))}{q}\right)}{1-e_{\lambda}^{-1}(q(z+\gamma))}e_{\lambda}^{\vartheta}\left(q(z+\gamma)^{j}\right)=e_{\lambda}^{-\sigma}(q(z+\gamma))\sum_{s,l=0}^{\infty}{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\frac{z^{s}}{s!}\frac{\gamma^{l}}{l!}.
$$

Also, subtituting $\tau$ by $\sigma$ in the last equation, and in view of (2.4), it is observed that

$$
e_{\lambda}^{-\tau}(q(z+\gamma))\sum_{s,l=0}^{\infty}{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\tau,\vartheta)\frac{z^{s}}{s!}\frac{\gamma^{l}}{l!}=\frac{qe_{\lambda}^{\vartheta}\left(q(z+\gamma)^{j}\right)}{1-e_{\lambda}^{-1}(q(z+\gamma))}\mathrm{Ei}_{\upsilon}\left(\frac{\log_{\lambda}(1+q(z+\gamma))}{q}\right).
$$

By the last two equations, we obtain

$$
\sum_{s,l=0}^{\infty}{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\tau,\vartheta)\frac{z^{s}}{s!}\frac{\gamma^{l}}{l!}=e_{\lambda}^{\tau-\sigma}(q(z+\gamma))\sum_{s,l=0}^{\infty}{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\frac{z^{s}}{s!}\frac{\gamma^{l}}{l!},
$$

which yields

$$
\sum_{s,l=0}^{\infty}{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\tau,\vartheta)\frac{z^{s}}{s!}\frac{\gamma^{l}}{l!}=\sum_{\rho,\psi=0}^{\infty}(\tau-\sigma)_{\rho+\psi,\lambda}\frac{z^{\rho}}{\rho!}\frac{\gamma^{\psi}}{\psi!}e_{\lambda}^{\tau-\sigma}(q(z+\gamma))\sum_{s,l=0}^{\infty}{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\frac{z^{s}}{s!}\frac{\gamma^{l}}{l!}.
$$

Utilizing (2.5), we acquire

$$
\sum_{s,l=0}^{\infty}{}_{GH}\beta_{s+l,\lambda;q}^{(\upsilon,j)}(\tau,\vartheta)\frac{z^{s}}{s!}\frac{\gamma^{l}}{l!}=\sum_{s,l=0}^{\infty}\sum_{\rho,\psi=0}^{s,l}\frac{{}_{GH}\beta_{s+l-\rho-\psi,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)\,(\tau-\sigma)_{\rho+\psi,\lambda}}{(s-l)!\rho!\psi!\,(l-\psi)!}z^{s}\gamma^{l},
$$

which completes the proof. $\square$

**Corollary 2.18.** *Upon setting* $s=0$ *in* (2.6) *gives*

$$ {}_{GH}\beta_{l,\lambda;q}^{(\upsilon,j)}(\tau,\vartheta)=\sum_{\psi=0}^{l}\binom{l}{\psi}\,{}_{GH}\beta_{l-\psi,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)(\tau-\sigma)_{\psi,\lambda}. $$

**Corollary 2.19.** *Choosing* $s=0$ *and substituting* $\tau$ *by* $\tau+\sigma$ *in* (2.6), *it is attained that*

$$ {}_{GH}\beta_{l,\lambda;q}^{(\upsilon,j)}(\tau+\sigma,\vartheta)=\sum_{\psi=0}^{l}\binom{l}{\psi}\,{}_{GH}\beta_{l-\psi,\lambda;q}^{(\upsilon,j)}(\sigma,\vartheta)(\tau)_{\psi,\lambda}. $$

**Theorem 2.20** (Symmetric property)**.** *We have*

$$ \begin{aligned} &\sum_{\psi=0}^{\rho}\sum_{s=0}^{\psi}\sum_{\gamma=0}^{\rho-\psi}\binom{\rho}{\psi}\binom{\psi}{s}\binom{\rho-\psi}{\gamma}\left(a^j\vartheta\right)_{\gamma,a^j\lambda}\left(b^j\vartheta\right)_{s,b^j\lambda}\,{}_{GH}\beta_{\rho-\psi-\gamma,a\lambda;q}^{(\upsilon,j)}(a\sigma)\,{}_{GH}\beta_{\psi-s,b\lambda;q}^{(\upsilon,j)}(b\sigma)\,a^{\psi}b^{\rho-\psi}\\ &=\sum_{\psi=0}^{\rho}\sum_{s=0}^{\psi}\sum_{\gamma=0}^{\rho-\psi}\binom{\rho}{\psi}\binom{\psi}{s}\binom{\rho-\psi}{\gamma}\left(b^j\vartheta\right)_{\gamma,b^j\lambda}\left(a^j\vartheta\right)_{s,a^j\lambda}\,{}_{GH}\beta_{\rho-\psi-\gamma,b\lambda;q}^{(\upsilon,j)}(b\sigma)\,{}_{GH}\beta_{\psi-s,a\lambda;q}^{(\upsilon,j)}(a\sigma)\,b^{\psi}a^{\rho-\psi} \end{aligned} \tag{2.7} $$

*holds for* $\alpha\in\mathbb{N}$, $a,b\in\mathbb{R}$ *and* $\rho\geqslant 0$.

*Proof.* Let

$$ \Upsilon=\frac{q^2\mathrm{Ei}_{\upsilon}\left(\frac{\log_{\lambda}(1+qaz)}{q}\right)\mathrm{Ei}_{\upsilon}\left(\frac{\log_{\lambda}(1+qbz)}{q}\right)}{\left(1-e_{\lambda}^{-1}(qaz)\right)\left(1-e_{\lambda}^{-1}(qbz)\right)}e_{\lambda}^{2\sigma}(qabz)\,e_{\lambda}^{2\vartheta}\left(q\,(abz)^j\right). $$

It is seen that

$$ \begin{aligned} \Upsilon&=\sum_{\rho=0}^{\infty}{}_{GH}\beta_{\rho,b\lambda;q}^{(\upsilon,j)}(b\sigma)\frac{(az)^{\rho}}{\rho!}\sum_{\rho=0}^{\infty}\left(b^j\vartheta\right)_{\rho,b^j\lambda}\frac{(az)^{\rho}}{\rho!}\sum_{\rho=0}^{\infty}{}_{GH}\beta_{\rho,a\lambda;q}^{(\upsilon,j)}(a\sigma)\frac{(bz)^{\rho}}{\rho!}\sum_{\rho=0}^{\infty}\left(a^j\vartheta\right)_{\rho,a^j\lambda}\frac{(bz)^{\rho}}{\rho!}\\ &=\sum_{\rho=0}^{\infty}\sum_{s=0}^{\rho}\binom{\rho}{s}\left(b^j\vartheta\right)_{s,b^j\lambda}\,{}_{GH}\beta_{\rho-s,b\lambda;q}^{(\upsilon,j)}(b\sigma)\frac{(az)^{\rho}}{\rho!}\sum_{\rho=0}^{\infty}\sum_{s=0}^{\rho}\binom{\rho}{s}\left(a^j\vartheta\right)_{s,a^j\lambda}\,{}_{GH}\beta_{\rho-s,a\lambda;q}^{(\upsilon,j)}(a\sigma)\frac{(bz)^{\rho}}{\rho!}\\ &=\sum_{\rho=0}^{\infty}\sum_{\psi=0}^{\rho}\binom{\rho}{\psi}\left(\sum_{s=0}^{\psi}\binom{\psi}{s}\left(b^j\vartheta\right)_{s,b^j\lambda}\,{}_{GH}\beta_{\psi-s,b\lambda;q}^{(\upsilon,j)}(b\sigma)\right)\\ &\quad\times\left(\sum_{s=0}^{\rho-\psi}\binom{\rho-\psi}{s}\left(a^j\vartheta\right)_{s,a^j\lambda}\,{}_{GH}\beta_{\rho-\psi-s,a\lambda;q}^{(\upsilon,j)}(a\sigma)\right)a^{\psi}b^{\rho-\psi}\frac{z^{\rho}}{\rho!} \end{aligned} $$

and similarly

$$ \begin{aligned} \Upsilon=&\sum_{\rho=0}^{\infty}\sum_{\psi=0}^{\rho}\binom{\rho}{\psi}\left(\sum_{s=0}^{\psi}\binom{\psi}{s}\left(a^j\vartheta\right)_{s,a^j\lambda}\,{}_{GH}\beta_{\psi-s,a\lambda;q}^{(\upsilon,j)}(a\sigma)\right)\\ &\times\left(\sum_{s=0}^{\rho-\psi}\binom{\rho-\psi}{s}\left(b^j\vartheta\right)_{s,b^j\lambda}\,{}_{GH}\beta_{\rho-\psi-s,b\lambda;q}^{(\upsilon,j)}(b\sigma)\right)a^{\rho-\psi}b^{\psi}\frac{z^{\rho}}{\rho!}, \end{aligned} $$

which provides the asserted identity (2.7). □

## 3. Conclusions

In this paper, the Gould-Hopper based fully degenerate type2 poly-Stirling polynomials of the first kind with a q parameter have been considered and some of their diverse identities and properties have been investigated. Then, the Gould-Hopper based fully degenerate type2 poly-Bernoulli polynomials with a q parameter have been introduced and some of their properties have been analyzed and derived. Furthermore, several formulas and relations covering implicit summation formulas, recurrence relations and symmetric property have been attained.

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