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Deferred Nörlund statistical convergence in probability, mean and distribution for sequences of random variables

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Abstract

We introduce and study deferred Nörlund statistical convergence in probability, mean of order r, distribution and study the interrelation among them. Based upon the proposed method to illustrate the findings, we present new Korovkin-type theorems for the sequence of random variables via deferred Nörlund statistically convergence and present compelling examples to demonstrate the effectiveness of the results. ©2017 All rights reserved.

Keywords: Probability convergence, Deferred Nörlund, Mean convergence, Distribution convergence, Statistical convergence.

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1. Introduction and Preliminaries

Fast [6] and also Schoenberg [22] studied the concept of statistical convergence and continued by Rath–Tripathy [21] and Gadjiev–Orhan [8].

Suppose that (x_m) and (y_m) are the sequences of non-negative integers fulfilling

$$x_m < y_m, \ \forall\, m \in \mathbb{N} \qquad \text{and} \qquad \lim_{x\to\infty} y_m = \infty. \tag{1.1}$$

Further, let (e_m) and (g_m) be two sequences of non-negative real numbers such that

$$\mathcal{E}_m = \sum_{n=x_m+1}^{y_m} e_n \ \text{ and } \ \mathcal{F}_m = \sum_{n=x_m+1}^{y_m} g_n. \tag{1.2}$$

The convolution of (1.2) is defined as

$$\mathcal{R}_m = \sum_{v=x_m+1}^{y_m} e_v g_{y_m - v}.$$

As introduced by Srivastava et al. in [23], the deferred Nörlund (DN) mean is defined as

$$t_m = \frac{1}{\mathcal{R}_m} \sum_{n=x_m+1}^{y_m} e_{y_m-n} g_n y_n.$$

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Suppose that (x_m) and (y_m) are the sequences fulfilling conditions (1.1) and $(e_m),(g_m)$ are sequences satisfying (1.2). A sequence (Y_m) is called as deferred Nörlund statistically convergent to Y if $\forall\ \varepsilon > 0$, the set

$$\{n : n \leqslant \mathcal{R}_m, \text{ and } e_{y_m-n}g_n|Y_m - Y| \geqslant \varepsilon\}$$

has zero deferred Nörlund density, i.e. if

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m}\left|\left\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_n|Y_m - Y| \geqslant \varepsilon\right\}\right| = 0.$$

We write it as

$$St_{DN} \lim Y_m = Y.$$

Suppose that (x_m) and (y_m) are the sequences fulfilling conditions (1.1) and $(e_m),(g_m)$ are sequences satisfying (1.2). A sequence (Y_m) is called as deferred Nörlund statistically probability (or $St_{DNP}-$) convergent to a random variable Y, if $\forall\ \varepsilon > 0$ and $\delta > 0$, the set

$$\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - Y| \geqslant \varepsilon) \geqslant \delta\}$$

has $DN-$density zero, i.e.,

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m}\left|\left\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - Y| \geqslant \varepsilon) \geqslant \delta\right\}\right| = 0$$

or

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m}\left|\left\{n : n \leqslant \mathcal{R}_m \text{ and } 1 - e_{y_m-n}g_nP(|Y_m - Y| \leqslant \varepsilon) \geqslant \delta\right\}\right| = 0,$$

and it is denoted as

$$St_{DNP} \lim_{m\to\infty} e_{y_m-n}g_nP(|Y_m - Y| \geqslant \varepsilon) = 0$$

or

$$St_{DNP} \lim_{m\to\infty} e_{y_m-n}g_nP(|Y_m - Y| \leqslant \varepsilon) = 1.$$

2. Deferred Nörlund statistically probability convergence

In this section, we study deferred Nörlund statistically probability convergence, for a historical review and basic concept we refer [3, 4, 11, 15, 20, 12, 18, 9, 5, 13, 26].

Theorem 2.1. *Suppose that (Y_m) and (Z_m) are sequences of random variables and consider two random variables Y and Z. Then the following assertions are satisfied*

1. $St_{DNP}Y_m \to Y$ *and* $St_{DNP}Y_m \to Z \Rightarrow P(Y = Z) = 1$,
2. $St_{DNP}Y_m \to y \Rightarrow St_{DNP}Y_m^2 = y^2$,
3. $St_{DNP}Y_m \to y$ *and* $St_{DNP}Z_m \to z \Rightarrow St_{DNP}Y_mZ_m \to yz$,
4. $St_{DNP}Y_m \to y$ *and* $St_{DNP}Z_m \to z \Rightarrow St_{DNP}\frac{Y_m}{Z_m} \to \frac{y}{z}$, $z \neq 0$,

5. $St_{DNP}Y_m \to Y$ *and* $St_{DNP}Z_m \to Z \Rightarrow St_{DNP}Y_mZ_m \to YZ$,

6. *if* $St_{DNP}Y_m \to Y\ \forall\ \varepsilon, \delta > 0$, *then* $\exists\ a \in \mathbb{N}$ *s.t.*

$$d(\{n : n \leqslant \mathcal{R}_m \ \ and \ \ e_{y_m-n}g_nP(|Y_m - Y_a| \geqslant \varepsilon) \geqslant \delta\}) = 0.$$

Proof. Let ϵ and δ be positively small real numbers. Also consider (x_m) and (y_m) are the sequences fulfilling conditions (1.1) and $(e_m), (g_m)$ are sequences satisfying (1.2).

1. Suppose that $a \in \Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\Big(|Y_m - Y| \geqslant \frac{\varepsilon}{2}\Big) < \frac{\delta}{2}\Big\} \cap \Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\Big(|Y_m - Z| \geqslant \frac{\varepsilon}{2}\Big) < \frac{\delta}{2}\Big\}$ (as the limit density of both the sets is 1). Then,
   $e_{y_m-n}g_nP\Big(|Y - Z| \geqslant \varepsilon\Big) \leqslant e_{y_m-n}g_nP\Big(|Y_a - Y| \geqslant \frac{\varepsilon}{2}\Big) + e_{y_m-n}g_nP\Big(|Y_a - Z| \geqslant \frac{\varepsilon}{2}\Big) < \delta.$
   It means $P\{Y = Z\} = 1$.
2. If $St_{DNP}Y_m \to 0$, then $St_{DNP}Y_m^2 \to 0$. Here, we see that $a \in \{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - 0| \geqslant \varepsilon) > \delta\} = a \in \{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m^2 - 0| \geqslant \varepsilon > \delta\}$. Now, take $Y_m^2 = (Y_m - y)^2 + 2y(Y_m - y) + y^2$. Thus, $St_{DNP}Y_m^2 \to y^2$.

3. Suppose that $St_{DNP}Y_m \to y$ and $St_{DNP}Z_m \to z$. As $St_{DNP}Y_mZ_m = St_{DNP}\frac{1}{4}\{(Y_m + Z_m)^2 - (Y_m - Z_m)^2\} = \frac{1}{4}\{(y_m + z_m)^2 - (y_m - z_m)^2\} = yz$.

4. Suppose that R and S be two events correspond $|Z_m - z| < |z|$ and $\left|\frac{1}{Z_m} - \frac{1}{z}\right| \geqslant \varepsilon$. We have
$$\left|\frac{1}{Z_m} - \frac{1}{z}\right| = \frac{|Z_m - z|}{|zZ_m|} = \frac{|Z_m - z|}{|z| \cdot |z + (Z_m - z)|} \leqslant \frac{|Z_m - z|}{|z| \cdot |(|z| - |Z_m - z|)|}.$$
   If the events R and S occurs at same time, then
$$|Z_m - z| \geqslant \frac{\varepsilon|z^2|}{1 + \varepsilon|z|}.$$
   Further, let $\varepsilon_0 = \varepsilon|z|^2/(1 + \varepsilon|z|)$ and A be the event such that $|Z_m - z| \geqslant \varepsilon_0$. Thus,
$$RS \subseteq A \Rightarrow P(S) \leqslant P(A) + P(R^c).$$
   Thus,
$$\Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\Big(|\tfrac{1}{Z_m} - \tfrac{1}{z}| \geqslant \varepsilon\Big) \geqslant \delta\Big\} \subseteq \Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\big(|Z_m - z| \geqslant \varepsilon_0\big) \geqslant \frac{\delta}{2}\Big\} \cup \Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_n P\big(|Z_m - z| \geqslant |z|\big) \geqslant \frac{\delta}{2}\Big\}.$$
   Therefore, $St_{DNP}\frac{1}{Z_m} \to \frac{1}{z}$. Hence, we write $St_{DNP}\frac{Y_m}{Z_m} \to \frac{y}{z}$, $z \neq 0$.

5. Suppose that $St_{DNP}Y_m \to Y$ and X be a random variable such that $Y_mX \to YX$. Since X is a random variable such that $\forall \varepsilon > 0$, $\exists\ \delta > 0$ and $e_{y_m-n}g_nP(|X| > \delta) \leqslant \frac{\varepsilon}{2}$. Next, $\forall\ \varepsilon' > 0$,
$$\begin{aligned} e_{y_m-n}g_nP\big(|Y_mX - YX| \geqslant \varepsilon'\big) &= e_{y_m-n}g_nP\big(|Y_m - Y||X| \geqslant \varepsilon', |Z| > \delta\big) \\ &\quad + e_{y_m-n}g_nP\big(|Y_m - Y||X| \geqslant \varepsilon', |Z| \leqslant \delta\big) \leqslant \frac{\varepsilon}{2} \\ &\quad + e_{y_m-n}g_nP\big(|Y_m - Y| \geqslant \frac{\varepsilon'}{\delta}\big). \end{aligned}$$

Which implies, $\big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\big(|Y_mX - YX| \geqslant \varepsilon'\big)\big\} \subseteq \big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\big(|Y_m - y| \geqslant \frac{\varepsilon'}{\delta}\big) \geqslant \frac{\varepsilon}{2}\big\}$. Therefore,

$$St_{DNP}(Y_m - Y)(Z_m - Z) \to 0.$$

Thus,

$$St_{DNP}Y_mZ_m \to YZ.$$

6. Suppose that (x_m) and (y_m) be two non-negative sequences such that

$$e_{y_m-n}g_nP\Big(|Y_m - Y| \geqslant \frac{\varepsilon}{2}\Big) < \frac{\delta}{2}$$

and

$$\Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\Big(|Y_m - Y| \geqslant \frac{\varepsilon}{2}\Big) < \frac{\delta}{2}\Big\} = 1.$$

Now,
$\big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\big(|Y_m - Y| \geqslant \varepsilon\big) \geqslant \delta\big\} \subseteq \big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\big(|Y_m - Y| \geqslant \frac{\varepsilon}{2}\big) < \frac{\delta}{2}\big\} = 1$. Which implies that

$$d(\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - Y| \geqslant \varepsilon) \geqslant \delta\}) = 0.$$

□

Theorem 2.2. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is uniform continuous on* $\mathbb{R}$ *and* $St_{DNP}Y_m \to Y$. *Then* $St_{DNP}f(Y_m) \to f(Y)$.

Proof. Let us consider a random variable Y such that for each $\delta > 0$, $\exists\ \beta \in \mathbb{R}$ such that $P(Y > \beta) \leqslant \delta/2$. Since, f is uniformly continuous on $[\beta, \beta]$ $\forall \varepsilon > 0$, $\exists \delta_0$ such that

$$|f(y_m) - f(y)| < \varepsilon \text{ whenever } |y_m - y| < \delta_0.$$

Thus,

$$\begin{aligned} P(|f(Y_m) - f(Y)| \geqslant \varepsilon) &\leqslant P(|Y_m - Y| \geqslant \delta_0) + P(|Y > \beta|) \\ &\leqslant P(|Y_m - Y| \geqslant \delta_0) + \delta/2. \end{aligned}$$

However, from the definition of St_{DNP}–convergence, we have

$$\Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\Big(|f(Y_m) - f(Y)| \geqslant \varepsilon\Big) \geqslant \delta\Big\}$$

$$\subseteq \Big\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\Big(|Y_m - Y| \geqslant \delta_0\Big) < \frac{\delta}{2}\Big\}.$$

□

3. Deferred Nörlund statistical mean convergence

Definition 3.1. Suppose that $r \geqslant 1$ be a fixed number. A sequence (Y_m) is r^{th} mean convergent to Y, if

$$\lim_{m\to\infty} E(|Y_m - Y|^r) = 0.$$

Definition 3.2. A sequence (Y_m) is statistically r^{th} mean convergent (MC) to a random variable Y, where $Y : S \to \mathbb{R}$ if,

$$\lim_{m\to\infty} \frac{1}{m}\Big|n : n \leqslant m \ \text{ and } \ E(|Y_m - Y|^r \geqslant \varepsilon)\Big| = 0$$

for any $\varepsilon > 0$. We write it as

$$St_{MC} \lim_{m\to\infty} E(|Y_m - Y|^r) = 0.$$

Definition 3.3. Suppose that (x_m) and (y_m) are the sequences fulfilling conditions (1.1) and $(e_m), (g_m)$ are sequences satisfying (1.2). A sequence (Y_m) is said to be deferred Nörlund statistically r^{th} $(r \geqslant 1)$ mean convergent to Y $(Y : S \to \mathbb{R})$, if for $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m}\left|\left\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m - n} g_n E\big(|Y_m - Y|^r \geqslant \varepsilon\big)\right\}\right| = 0.$$

It is denoted as

$$St_{DNM} \lim_{m\to\infty} E(|Y_m - Y|^r) = 0.$$

Theorem 3.4. *Let* $St_{DNM} \lim_{m\to\infty} E(|Y_m - Y|^r) = 0$ *for* $r \geqslant 1$, *then* $St_{DNM} \lim_{m\to\infty} P(|Y_m - Y| \geqslant \varepsilon) = 0$.

Proof. For every $\varepsilon > 0$, we have from Markov's inequality

$$\begin{aligned} St_{DNM} \lim_{m\to\infty} P(|Y_m - Y| \geqslant \varepsilon) &= St_{DNM} \lim_{m\to\infty} P(|Y_m - Y|^r \geqslant \varepsilon^r) \ \ (r \geqslant 1) \\ &\leqslant St_{DNM} \lim_{m\to\infty} \frac{E(|Y_m - Y|^r)}{\varepsilon^r} = 0. \end{aligned}$$

From definition of statistically deferred Nörlund mean convergence

$$St_{DNM} \lim_{m\to\infty} E(|Y_m - Y|^r) = 0,$$

it implies that

$$St_{DNP} \lim_{m\to\infty} P(|Y_m - Y| \geqslant \varepsilon) = 0.$$

□

We now present an example to show that a sequence of random variables is statistically probability convergent but not statistically r^{th}−mean convergent.

Example 1: Suppose that $x_m = 2m - 1, y_m = 4m - 1$. Also, suppose that $e_{y_m - m} = 2m$ and $g_m = 1$. Further, consider a sequence (z_m) of random variables such that

$$Y_m = \begin{cases} m, \text{ with probability } \frac{1}{\sqrt{m}} \\ 0, \text{ with probability } 1 - \frac{1}{\sqrt{m}}. \end{cases}$$

Then the statistically deferred Nörlund convergence of Y_m is given as

$$\begin{aligned} \lim_{m\to\infty} \frac{1}{2m}\left|\left\{n : n \leqslant \mathcal{R}_m \text{ and } 2mP(|Y_m - 0| \geqslant \varepsilon)\right\}\right| &= \lim_{m\to\infty} P(Y_m = m) \\ &= \lim_{m\to\infty} \frac{1}{\sqrt{m}} \\ &= 0. \end{aligned}$$

However, statistically deferred Nörlund mean convergence, for $r \geqslant 1$, is

$$\begin{aligned} \lim_{m\to\infty} \frac{1}{2m}\left|\left\{n : n \leqslant \mathcal{R}_m \text{ and } 2mE(|Y_m - 0|^r)\right\}\right| &= \lim_{m\to\infty} \left(m^r\left(\frac{1}{\sqrt{m}}\right) + 0\left(1 - \frac{1}{\sqrt{m}}\right)\right) \\ &= \lim_{m\to\infty} m^{r-1/2} \\ &= \infty. \end{aligned}$$

This implies that the sequence (Y_m) is St_{DNP}−convergent but not St_{DNM}−convergent.

4. Statistical distribution convergence via Deferred Nörlund

Definition 4.1. The sequence of random variables (Y_m) is said to be distribution convergent (or convergent in distribution) to Y, if

$$\lim_{m\to\infty} F_{Y_m}(y) = F_Y(y)$$

for all $y \in \mathbb{R}$ at which $F_Y(y)$ is continuous.

Thorughout the paper $(F_{Y_m}(y))$ is the sequence of distribution functions of (Y_m) and $F_Y(y)$ is the distribution function of Y.

Definition 4.2. The sequence $(F_{Y_m}(y))$ is called as statistically distribution convergent (or St_{DC}), if there exists $F_Y(y)$ of random variable Y such that for each $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m}\left|\left\{n : n \leqslant m \text{ and } |F_{Y_m}(y) - F_Y(y)| \geqslant \varepsilon\right\}\right| = 0.$$

We may write this as

$$St_{DC} \lim_{m\to\infty} F_{Y_m}(y) = F_Y(y).$$

Definition 4.3. The sequence $(F_{Y_m}(y))$ of distribution functions is called as deferred Nörlund statistically distribution convergent (or St_{DNDC}), if there exists $F_Y(y)$ of Y such that for each $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m}\left|\left\{n : n \leqslant \mathcal{R}_m \text{ and } e_{y_m - n} g_n |F_{Y_m}(y) - F_Y(y)| \geqslant \varepsilon\right\}\right| = 0.$$

In this case, we say

$$St_{DNDC} \lim_{m\to\infty} F_{Y_m}(y) = F_Y(y).$$

Theorem 4.4. *Suppose that* $St_{DNP} \lim_{m\to\infty} P(|Y_m - Y| \geqslant \varepsilon) = 0$, *then*

$$St_{DNDC} \lim_{m\to\infty} F_{Y_m}(y) = F_Y(y).$$

Proof. Suppose that $(F_{Y_m}(y))$ is distribution functions of (Y_m), and $F_Y(y)$ be the distribution function of Y. For $i, j \in \mathbb{R}$ such that $i < j$, we have

$$(Y \leqslant i) = (Y_m \leqslant j, Y \leqslant i) + (Y_m \geqslant j, Y \leqslant i).$$

Further,

$$(Y_m \leqslant j, Y \leqslant i) \subseteq (Y_m \leqslant j),$$

which implies that

$$(Y \leqslant i) \subseteq (Y_m \leqslant j) + (Y_m \geqslant j, Y \leqslant i). \tag{4.1}$$

Let us take the probability to left hand side and right hand side of equation (4.1)

$$\begin{aligned} P(Y \leqslant i) &\leqslant P\{(Y_m \leqslant j) + (Y_m \geqslant j, Y \leqslant i)\} \\ &\leqslant P(Y_m \leqslant j) + P(Y_m \geqslant j, Y \leqslant i). \end{aligned}$$

It means that

$$F_{Y_m}(j) \geqslant F_Y(i) - P(Y_m \geqslant j, Y \leqslant i). \tag{4.2}$$

If $Y_m \geqslant j, Y \leqslant i$, then $Y_m \geqslant j, -Y \geqslant -i$, so that $Y_m - Y > j - i$, that is,

$$(Y_m \geqslant j, Y \leqslant i) \subseteq (Y_m - Y > j - i) \subseteq (|Y_m - Y| > j - i).$$

This means

$$P(Y_m \geqslant j, Y \leqslant i) \leqslant P(|Y_m - Y| > j - i).$$

As we know that $i < j$ and $St_{DNP}Y_m \to Y$, we obtain

$$St_{DNP} \lim_{m\to\infty} P(Y_m \geqslant j, Y \leqslant i) = 0.$$

From (4.2) we get

$$St_{DNDC} \lim_{m\to\infty} F_{Y_m}(j) \geqslant F_Y(i).$$

Similarly, if $j < a$ for any real constant a, then

$$(Y \leqslant j) = (Y \leqslant a, Y_m \leqslant j) + (Y > a, Y_m \leqslant j).$$

Consequently,

$$F_{Y_m}(j) \leqslant F_Y(a) + P(Y > a, Y_m \leqslant j)$$

and

$$St_{DNDC} \lim_{m\to\infty} P(Y > a, Y_m \leqslant j) = 0.$$

Therefore, we get

$$St_{DNDC} \lim_{m\to\infty} F_{Y_m}(j) \leqslant F_Y(a).$$

Thus, with $i < j < a$, we have

$$St_{DNDC} \lim_{m\to\infty} F_{Y_m}(j) = F_Y(i).$$

□

Example 2: Consider the random variables $((Y_m), Y)$ of two dimensions as $\{(0,0), (0,1), (1,0), (1,1)\}$ such that

$$(Y_m, Y) = \begin{cases} 0, & [P(Y_m = 0, Y = 0) = 0 = P(Y_m = 1, Y = 1)] \\ \frac{1}{2}, & [P(Y_m = 1, Y = 0) = 0 = P(Y_m = 0, Y = 1)]. \end{cases}$$

The distribution function of Y_m is given by $Y_m = (\lambda_1 = 0, 1)$, with probability mass function

$$(p_{y_m,\lambda_1}) = P(Y_m = \lambda_1), \text{ where } p_{y_m,0} = \frac{1}{2} = p_{y_m,1}$$

and for $Y = \lambda_2(\lambda_2 = 0, 1)$, with probability mass function

$$(p_{y_m,\lambda_2}) = P(Y_m = \lambda_2), \text{ where } p_{y,0} = \frac{1}{2} = p_{y,1}.$$

If $(F_{Y_m}(y))$ is distribution functions of (Y_m) and $F_Y(y)$ is the distribution function of Y, then

$$F_Y(y) = \lim_{m\to\infty} F_{Y_m}(y) = \begin{cases} 0, & (y < 0) \\ \frac{1}{2}, & (0 \leqslant y < 1). \\ 1, & (z > 1). \end{cases}$$

Thus, we get

$$St_{DNDC} \lim_{m\to\infty} F_{Y_m}(y) = F_Y(y), \text{ where } x_m = 2m-1, y_m = 4m-1, e_{y_m - m} = 2m \text{ and } g_m = 1.$$

But, it is not St_{PC} for the sequence of random variables, i.e.

$$St_{DNPC} \lim_{m\to\infty} P(|Y_m - Y| \geqslant \varepsilon) \neq 0, \text{ where } x_m = 2m-1, y_m = 4m-1, e_{y_m - m} = 2m \text{ and } g_m = 1.$$

5. Applications

The hypothesis of the Korovkin-type theorems have been studied by several researchers in various field in different ways such as in, summability theory, functional analysis and probability theory. Korovkin-type approximation theorems have been investigated by many mathematicians under various background, involving function spaces, Banach spaces, and so on. Recently, Mohiuddine and Alamri studied Korovkin and Voronovskaya type approximation theorems in [14]. Further, Hazarika et al. [10] studied Korovkin approximation theorem for Bernstein operator of rough statistical convergence of triple sequences. For detailed study on Korovkin approximation theorem one may refer [2], [16], [17], [19], [27].
By $\mathcal{C}(Y)$, we denote the space of all continuous probability functions defined on a compact subset $Z \subset \mathbb{R}$. The space $\mathcal{C}(Y)$ is a Banach space with respect to the norm

$$||f||_\infty = \sup_{z\in Y}\{|f(z)|\}, \qquad f \in \mathcal{C}(Y).$$

We say that $\mathcal{Y}$ is a positive linear operator of sequence of random variables if

$$\mathcal{Y}(f,z) \geqslant 0 \text{ whenever } f \geqslant 0.$$

Throughout, $\mathcal{Y}_n : \mathcal{C}(Y) \to \mathcal{C}(Y)$ be a sequence of random variables of positive linear operators.

Theorem 5.1 ([25])**.** *Let* $\mathcal{Y}_n : \mathcal{C}(Y) \to \mathcal{C}(Y)$*. Then for all* $f \in \mathcal{C}(Y)$*, we have*

$$\mathrm{St}_{DNP} \lim_{n\to\infty} ||\mathcal{Y}_n(f,z) - f(z)||_\infty = 0$$

iff

$$\mathrm{St}_{DNP} \lim_{n\to\infty} ||\mathcal{Y}_n(1,z) - 1||_\infty = 0, \tag{5.1}$$

$$\mathrm{St}_{DNP} \lim_{n\to\infty} ||\mathcal{Y}_n(z,z) - z||_\infty = 0, \tag{5.2}$$

$$\mathrm{St}_{DNP} \lim_{n\to\infty} ||\mathcal{Y}_n(z^2,z) - z^2||_\infty = 0. \tag{5.3}$$

Theorem 5.2. *Let* $\mathcal{Y}_n : \mathcal{C}(Y) \to \mathcal{C}(Y)$*. Then for all* $f \in \mathcal{C}(Y)$*, we have*

$$\mathrm{St}_{DNM} \lim_{n\to\infty} ||\mathcal{Y}_n(f,z) - f(z)||_\infty = 0$$

iff

$$\mathrm{St}_{DNM} \lim_{n\to\infty} ||\mathcal{Y}_n(1,z) - 1||_\infty = 0,$$
$$\mathrm{St}_{DNM} \lim_{n\to\infty} ||\mathcal{Y}_n(z,z) - z||_\infty = 0,$$
$$\mathrm{St}_{DNM} \lim_{n\to\infty} ||\mathcal{Y}_n(z^2,z) - z^2||_\infty = 0.$$

Theorem 5.3. *Let* $\mathcal{Y}_n : \mathcal{C}(Y) \to \mathcal{C}(Y)$*. Then for all* $f \in \mathcal{C}(Y)$*, we have*

$$\mathrm{St}_{DNDC} \lim_{n\to\infty} ||\mathcal{Y}_n(f,z) - f(z)||_\infty = 0$$

iff

$$\mathrm{St}_{DNDC} \lim_{n\to\infty} ||\mathcal{Y}_n(1,z) - 1||_\infty = 0,$$
$$\mathrm{St}_{DNDC} \lim_{n\to\infty} ||\mathcal{Y}_n(z,z) - z||_\infty = 0,$$
$$\mathrm{St}_{DNDC} \lim_{n\to\infty} ||\mathcal{Y}_n(z^2,z) - z^2||_\infty = 0.$$

Example 3: Let $\mathcal{M}_m(f, y)$ be a Meyer-König and Zeller operators on $\mathcal{C}[0,1]$ and $Z = [0,1]$ as defined in [1] as

$$\mathcal{M}_m(f, y) = (1-y)^{m+1} \sum_{t=0}^{\infty} f\Big(\frac{t}{t+m+1}\Big) \binom{m+t}{t} y^t.$$

Further, let us consider a sequence of operators $\mathcal{Y}_n : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ and (Y_n) as defined in example 2.4 such that

$$\mathcal{Y}_n(f, y) = [1 + F_{Y_m}(y)]\mathcal{M}_n(f), \quad (f \in \mathcal{C}[0,1]), \tag{5.4}$$

where $(F_{Y_m}(y))$ is defined in Example 2. Now we observe that

$$\mathcal{Y}_n(1, y) = [1 + F_{Y_m}(y)] \cdot 1 = [1 + F_{Y_m}(y)],$$
$$\mathcal{Y}_n(u, z) = [1 + F_{Y_m}(y)] \cdot y = [1 + F_{Y_m}(y)] \cdot y$$

and

$$\mathcal{Y}_n(u^2, z) = [1 + F_{Y_m}(y)] \cdot \Big\{y^2 \Big(\frac{m+2}{m+1}\Big) + \frac{y}{m+1}\Big\}.$$

Therefore, we have

$$St_{DNDC} \lim_{n\to\infty} \|\mathcal{Y}_n(1, y) - 1\|_\infty = 0,$$

$$St_{DNDC} \lim_{n\to\infty} \|\mathcal{Y}_n(y, y) - y\|_\infty = 0,$$

$$St_{DNDC} \lim_{n\to\infty} \|\mathcal{Y}_n(y^2, y) - y^2\|_\infty = 0.$$

Hence, $\mathcal{Y}_n(f, y)$ fulfills (5.1), (5.2) and (5.3). Thus, from Theorem 5.3

$$St_{DNDC} \lim_{n\to\infty} \|\mathcal{Y}_n(f, y) - f\|_\infty = 0.$$

Hence, it is $(DNDC)-$convergent. However, (Y_m) is neither $(DN)-$statistical convergent nor $(DN)-$ convergent. Thus, we can exhibit that the work in [23] does not hold for our operators described in (5.4). Hence, our Theorem 5.3 is stronger than the theorem proved in [23].

6. Conclusion

Upon prior analysis, our interest is to modify the studies of Srivastava et al. [24] and introduce various aspects of statistical convergence for the sequences of random variables and sequences of real numbers via deferred Norlund summability mean. We first study various results presenting the connection by using fundamental limit concepts of sequences of random variables. As an applications of our findings, we present new Krorvkin-type approximation results and also demonstrated the effectiveness of the findings. As a future work one can obtain the corresponding results of the present paper using deferred Euler summability mean.

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