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Stability and boundedness analysis for a system of two nonlinear delay differential equations

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Abstract

In this paper, the stability and boundedness analysis of a certain system of two nonlinear delay differential equations with variable delay $\rho(t)$ is carried out. By using the Lyapunov's second method and Lyapunov-Krasovskii's functional derived from the differential equations describing the system which yielded a better stability and boundedness estimate to establish sufficient conditions for the uniform asymptotic stability of the trivial solution and uniform ultimate boundedness of solution. These new results improve and generalize some results that can be found in the literature.

Keywords: Stability, boundedness, delay differential equation, Lyapunov's method.

2020 MSC: 34D20, 34D23, 34K40.

1. Introduction

The system of nonlinear delay differential equation to be considered is given by

$$x' = ay + q_1(x(t-\rho(t))), \qquad y' = by + q_2(x(t-\rho(t))) + p(t,x,y), \tag{1.1}$$

where the constants $a, b > 0$ and functions q_1, q_2, p are continuous in their respective arguments. Also, $0 \leqslant \rho(t) \leqslant \eta$, $\rho'(t) \leqslant \varepsilon$, $0 \leqslant \varepsilon \leqslant 1$, and $\eta, \varepsilon > 0$ are some constants, η will be determined later. The functions q_1, q_2 also satisfy a Lipschitz condition in x.

Many authors have investigated the qualitative properties of solutions of various and more general form of scalar delay differential equations of second and higher orders. For instance, see [1, 2, 12–14, 16, 26]. In particular, some authors have contributed to the study of qualitative properties of solutions of system of differential equations of second order without delay or delay being zero, for example, [8, 9, 22, 23], where they obtained results for stability, boundedness or both for the system considered while few others for instance, [24, 25] obtained results for the nonlinear terms depending on either constant, variable or multiple delay for stability and boundedness of solutions of system of differential equations

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which are quite different from the system (1.1). However, in the relevant literature, there is no work on the qualitative properties of solutions of system of two nonlinear delay differential equations of the form (1.1). So, it is worthwhile to investigate the stability and boundedness of solutions of (1.1). Special cases of (1.1) (where the variable delay is absent) and $p = 0$ arising from the so-called Aizermann problem have been investigated by [6, 10, 11] who extended the result of the former to a more general form to obtain the stability of the zero solution. In almost all the papers mentioned, Lyapunov's second method has been used with the aid of suitable continuously differentiable scalar functions to establish their results. For a more satisfactory result, one needs to construct a suitable complete Lyapunov-Krasovskii's functional. Unfortunately, the construction of this function remains a hard task (see [4]). Stability and boundedness of solutions perform significant roles in defining the behaviour of solutions of fairly complicated nonlinear physical systems arising from applied sciences such as after effect, equation with variable delay, time-delay system and nonlinear oscillations, see [5, 19, 20]. More importantly, the study of nonlinear delay system (1.1) will present some new beneficence to the qualitative theory of delay differential equations and also some recent areas of mathematical ecology. See [3, 7, 15, 18].

The motivation for this work comes from the paper of [10]. The aim is to improve the result proved in [10] for the uniform asymptotic stability of the trivial solution (when $p = 0$) and uniform ultimate boundedness (when $p \neq 0$) of solutions of system (1.1) using a suitable Lyapunov-Krasovskii's functional derived from the differential equations describing the system. Results obtained are not only new but will improve and generalize the results of [6, 10, 11]. An example is given to illustrate the effectiveness and significance of the results obtained as well as provide geometric arguments to support our findings on the behaviour of solutions of the system.

Now, we write the equation (1.1) as the following equivalent system:

$$x' = ay + q_1(x) + \int_{t-\rho(t)}^{t} q_1'(x(s))y(s)ds, \quad y' = by + q_2(x) + \int_{t-\rho(t)}^{t} q_2'(x(s))y(s)ds + p(t,x,y), \tag{1.2}$$

where q_1' and q_2' are continuous for all x with $q_1(0) = q_2(0) = 0$.

2. Statement of result

The following is the theorem on uniform asymptotic stability of solutions of (1.1) when $p = 0$.

Theorem 2.1. *Apart from the earlier declaration of* q_1 *and* q_2*, we also assume that there are constants* $\delta, \nu, L, M > 0$ *and* $a\nu - b\delta > 0$ *such that the following conditions hold:*

(a_1) $\frac{q_1(x)}{x} \geqslant \delta$, $\frac{q_2(x)}{x} \geqslant \nu$, $x \neq 0$;
(a_2) $|q_1'(x)| \leqslant L$, $|q_2'(x)| \leqslant M$, *for every* $x \in \mathbb{R}$;

provided

$$\eta < \min\left\{\frac{2(b+\delta)(a\nu - b\delta)}{k_1}; \frac{2ab(1-\varepsilon)(b+\delta)(a\nu-b\delta)}{\delta\nu(k_2+k_3+k_4)}\right\}$$

with

$$\begin{aligned} k_1 &= [b(b+\delta) + a(b\nu^2 - \nu)]L + ab(1+\delta\nu)M > 0, \\ k_2 &= a(1-\varepsilon)\Big\{\left[(a+b\delta^2) + b(a\nu - b\delta)\right]M + b(1+\delta\nu)L\Big\} > 0, \\ k_3 &= 2[(b+\delta) + a(b\nu^2-\nu) + ab(1-\varepsilon)(1+\delta\nu)]L > 0, \\ k_4 &= 2a[(a+b\delta^2) + b(a\nu - b\delta) + b(1-\varepsilon)(1+\delta\nu)]M > 0. \end{aligned}$$

Then, the trivial solution of (1.1) *is uniformly asymptotically stable.*

2.1. Preliminary result

The tool in proving the result is the scalar functional $V(t) = V(x_t, y_t)$ defined by

$$\begin{aligned} 2V(t) &= \frac{1}{ab}\Big[\int_0^x (aq_2(\sigma) - bq_1(\sigma))d\sigma + b^2x^2 - 2abxy + a^2y^2\Big] \\ &\quad + \frac{1}{\delta\nu}\Big[q_2^2(x) - 2\delta y q_2(x) + \delta^2y^2 + (a\nu - b\delta)y^2\Big] \\ &\quad + 2n_1\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds + 2n_2\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds, \end{aligned} \tag{2.1}$$

where $n_1, n_2 > 0$ are constants that will be determined later.

The proof of Theorem 2.1 depends on the following lemma.

Lemma 2.2. *Assume that the hypotheses* (a_1)-(a_2) *of Theorem* 2.1 *hold, then there exists the constants* $D_1, D_2 > 0$ *such that*

$$D_1(x^2 + y^2) \leqslant 2V(t) \leqslant D_2(x^2 + y^2). \tag{2.2}$$

Proof. Clearly, (2.1) vanishes for $x = y = 0$ and can be re-arranged as follows

$$\begin{aligned} 2V(t) = \frac{1}{ab}\Big[\int_0^x (a\nu - \frac{bq_1(\sigma)}{\sigma})\sigma d\sigma + \int_0^x (\frac{aq_2(\sigma)}{\sigma} - a\nu)\sigma d\sigma + (bx - ay)^2\Big] \\ + \frac{1}{\delta\nu}\Big[(q_2(x) - \delta y)^2 + (a\nu - b\delta)y^2\Big] + 2n_1\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds + 2n_2\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds. \end{aligned}$$

By the hypotheses (a_1)-(a_2) of Theorem 2.1, we have that the terms

$$\int_0^x (a\nu - \frac{bq_1(\sigma)}{\sigma})\sigma d\sigma \geqslant (a\nu - b\delta)x^2 \quad \text{and} \quad \int_0^x (\frac{aq_2(\sigma)}{\sigma} - a\nu)\sigma d\sigma \geqslant 0.$$

It follows that

$$\begin{aligned} 2V(t) &\geqslant \frac{1}{ab}\Big[(a\nu - b\delta)x^2 + (bx - ay)^2\Big] + \frac{1}{\delta\nu}\Big[(q_2(x) - \delta y)^2 + (a\nu - b\delta)y^2\Big] \\ &\quad + 2n_1\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds + 2n_2\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds. \end{aligned}$$

Since by Theorem 2.1, $a\nu - b\delta > 0$ and the integrals $2n_1\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds$ and $2n_2\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds$ being non-negative, thus, it is evident from the terms contained in the above inequality that there exists a constant $D_1 > 0$ small enough such that

$$2V(t) \geqslant D_1(x^2 + y^2).$$

In order to prove the right side of inequality (2.2), we consider the hypotheses (a_1)-(a_2) of Theorem 2.1 and the fact that $2|x||y| \leqslant x^2 + y^2$, yields for $V(t)$ in (2.1) term by term,

$$\begin{aligned} 2ab|x||y| &\leqslant ab(x^2 + y^2), & 2\delta|y||q_2(x)| &\leqslant \delta\nu(x^2 + y^2), \\ |q_2^2(x)| &\leqslant \nu^2x^2, & \int_0^x (aq_2(\sigma) - bq_1(\sigma))\sigma d\sigma &\leqslant (a\nu - b\delta)x^2, \\ 2n_1\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds &\leqslant n_1\rho^2(t)sy^2, & 2n_2\int_{-\rho(t)}^0\int_{t+s}^t y^2(\sigma)d\sigma ds &\leqslant n_2\rho^2(t)sy^2. \end{aligned}$$

Thus,

$$
\begin{aligned}
2V(t) &\leqslant \left(\frac{a^2-b\delta-2ab}{ab}+\nu(\frac{1}{\delta}+\frac{1}{b})\right)x^2+\left(-1+a\frac{(b+\delta)}{b\delta}+\frac{\delta-b}{\nu}+\frac{\nu}{\delta}+\rho^2(t)s(n_1+n_2)\right)y^2\\
&\leqslant D_2^*(x^2+y^2),
\end{aligned}
$$

where $D_2^*=\max\left\{\frac{a^2-b\delta-2ab}{ab}+\nu(\frac{1}{\delta}+\frac{1}{b});-1+a\frac{(b+\delta)}{b\delta}+\frac{\delta-b}{\nu}+\frac{\nu}{\delta}+\rho^2(t)s(n_1+n_2)\right\}$. We choose a constant $D_2>0$, so that

$$2V(t)\leqslant D_2(x^2+y^2).$$

□

Thus, (2.2) of Lemma 2.2 is established, where D_1, D_2 are finite constants. Next, we present the Proof of Theorem 2.1.

Proof. Now, differentiating (2.1) with respect to t along the system (1.2) and after simplification we get

$$
\begin{aligned}
\frac{dV(t)}{dt}=&-\frac{1}{ab}\left[a\frac{q_1(x)}{x}\frac{q_2(x)}{x}x^2+ab\frac{q_2(x)}{x}x^2-b\frac{q_1^2}{x^2}x^2-b^2\frac{q_1(x)}{x}x^2\right]\\
&-\frac{1}{\delta\nu}\left[a\delta q_2'(x)y^2+ab\nu y^2-b\delta^2y^2-b^2\delta y^2\right]\\
&+\left[\frac{\delta}{a}|x|+\frac{b}{a}|x|+(\nu^2-\frac{\nu}{b})|x|+(1+\delta\nu)|y|\right]n_1\int_{t-\rho(t)}^{t}q_1'(x(s))y(s)ds\\
&+\left[\frac{a}{b}|y|+\delta^2|y|+(a\nu-b\delta)|y|+(1+\delta\nu)|x|\right]n_2\int_{t-\rho(t)}^{t}q_2'(x(s))y(s)ds\\
&+(n_1+n_2)\rho(t)y^2+(n_1+n_2)(1-\rho'(t))\int_{t-\rho(t)}^{t}y^2(s)ds.
\end{aligned}
$$

By the hypotheses (a_1)-(a_2) of Theorem 2.1, we have

$$
\begin{aligned}
\frac{dV(t)}{dt}\leqslant&-\frac{1}{ab}\Big[(b+\delta)(a\nu-b\delta)\Big]x^2-\frac{1}{\delta\nu}\Big[(b+\delta)(a\nu-b\delta)\Big]y^2\\
&+\left[\frac{\delta}{a}|x|+\frac{b}{a}|x|+(\nu^2-\frac{\nu}{b})|x|+(1+\delta\nu)|y|\right]\times\int_{t-\rho(t)}^{t}q_1'(x(s))y(s)ds\\
&+\left[\frac{a}{b}|y|+\delta^2|y|+(a\nu-b\delta)|y|+(1+\delta\nu)|x|\right]\times\int_{t-\rho(t)}^{t}q_2'(x(s))y(s)ds\\
&+(n_1+n_2)\rho(t)y^2+(n_1+n_2)(1-\rho'(t))\int_{t-\rho(t)}^{t}y^2(s)ds.
\end{aligned}
\tag{2.3}
$$

Using the hypothesis (a_2) of Theorem 2.1 and the fact that $2uv\leqslant u^2+v^2$, the term in (2.3) yields

$$
\begin{aligned}
&\left[\frac{\delta}{a}|x|+\frac{b}{a}|x|+(\nu^2-\frac{\nu}{b})|x|-(1+\delta\nu)|y|\right]\times\int_{t-\rho(t)}^{t}q_1'(x(s))y(s)ds\\
&\leqslant\left[(\frac{b+\delta}{a}+\frac{b\nu^2-\nu}{b})\rho(t)\frac{L}{2}x^2+(\frac{b+\delta}{a}+\frac{b\nu^2-\nu}{b})\frac{L}{2}\int_{t-\rho(t)}^{t}y(s)ds\right.\\
&\quad+(1+\delta\nu)\rho(t)\frac{L}{2}y^2+(1+\delta\nu)\rho(t)\frac{L}{2}\int_{t-\rho(t)}^{t}y(s)ds
\end{aligned}
$$

and

$$\left[\frac{a}{b}|y|+\delta^2|y|+(a\nu-b\delta)|y|-(1+\delta\nu)|x|\right]\times\int_{t-\rho(t)}^{t} q_2'(x(s))y(s)ds$$
$$\leqslant\left[\left(\frac{a+b\delta^2}{b}+(a\nu-b\delta)\right)\rho(t)\frac{M}{2}y^2+\left(\frac{a+b\delta^2}{b}+(a\nu-b\delta)\right)\frac{M}{2}\int_{t-\rho(t)}^{t} y(s)ds\right.$$
$$+(1+\delta\nu)\rho(t)\frac{M}{2}x^2+(1+\delta\nu)\rho(t)\frac{M}{2}\int_{t-\rho(t)}^{t} y(s)ds,$$

and (2.3) becomes

$$\begin{aligned}\frac{dV(t)}{dt}\leqslant&-\frac{1}{ab}\Big[(b+\delta)(a\nu-b\delta)\Big]x^2-\frac{1}{\delta\nu}\Big[(b+\delta)(a\nu-b\delta)\Big]y^2\\
&+\frac{1}{2}\Big[\big(\frac{b+\delta}{a}+\frac{b\nu^2-\nu}{b}\big)L+(1+\delta\nu)M\Big]\rho(t)x^2\\
&+\frac{1}{2}\Big[\big(\frac{a}{b}+\frac{\delta^2}{b}+(a\nu-b\delta)\big)M+(1+\delta\nu)L+2n_1+2n_2\Big]\rho(t)y^2\\
&+\frac{1}{2}\Big[\big(\frac{b+\delta}{a}+\frac{b\nu^2-\nu}{b}\big)L+(1+\delta\nu)L-2n_1(1-\rho'(t))\Big]\int_{t-\rho(t)}^{t} y^2(s)ds\\
&+\frac{1}{2}\Big[\big(\frac{a+b\delta^2}{b}+(a\nu-b\delta)\big)M+(1+\delta\nu)M-2n_2(1-\rho'(t))\Big]\int_{t-\rho(t)}^{t} y^2(s)ds.\end{aligned}$$

By the assumption on $\rho(t), \rho'(t)$, and choosing

$$n_1=\frac{[b(b+\delta)+a(b\nu^2-\nu)+ab(1-\varepsilon)(1+\delta\nu)]L}{2ab(1-\varepsilon)}>0$$

and

$$n_2=\frac{[(a+b\delta^2+b(a\nu-b\delta))+b(1-\varepsilon)(1+\delta\nu)]M}{2b(1-\varepsilon)}>0,$$

thus,

$$\begin{aligned}\frac{dV(t)}{dt}\leqslant&-\frac{1}{2ab}\left\{2(b+\delta)(a\nu-b\delta)-\Big[(b(b+\delta)+a(b\nu^2-\nu))L+ab(1+\delta\nu)]M\Big]\eta\right\}x^2\\
&-\frac{1}{2\delta\nu}\left\{2(b+\delta)(a\nu-b\delta)-\delta\nu\left[\frac{[a+b\delta^2+b(a\nu-b\delta)]M+b(1+\delta\nu)L}{b}\right.\right.\\
&+2\frac{[b(b+\delta)+a(b\nu^2-\nu)+ab(1-\varepsilon)(1+\delta\nu)]L}{ab(1-\varepsilon)}\\
&\left.\left.+2\frac{[(a+b\delta^2+b(a\nu-b\delta))+b(1-\varepsilon)(1+\delta\nu)]M}{b(1-\varepsilon)}\right]\eta\right\}y^2.\end{aligned}$$

Choosing

$$\eta<\min\left\{\frac{2(b+\delta)(a\nu-b\delta)}{k_1};\frac{2ab(1-\varepsilon)(b+\delta)(a\nu-b\delta)}{\delta\nu(k_2+k_3+k_4)}\right\},$$

where

$$k_1=[b(b+\delta)+a(b\nu^2-\nu)]L+ab(1+\delta\nu)M>0,$$
$$k_2=a(1-\varepsilon)\Big\{[(a+b\delta^2)+b(a\nu-b\delta)]M+b(1+\delta\nu)L\Big\}>0,$$

$$k_3 = 2[(b+\delta) + a(b\nu^2 - \nu) + ab(1-\varepsilon)(1+\delta\nu)]L > 0,$$
$$k_4 = 2a[(a+b\delta^2) + b(a\nu - b\delta) + b(1-\varepsilon)(1+\delta\nu)]M > 0,$$

we get

$$\frac{dV(t)}{dt} \leqslant -D_3(x^2+y^2), \tag{2.4}$$

for some $D_3 > 0$. □

The conclusion of this proof follows by the same reasoning in [2, 21] using (2.2) and (2.4). This shows that the trivial solution of (1.1) is asymptotically stable. Hence, the proof of Theorem 2.1 is complete.

3. Statement of result

The following is the theorem on uniform ultimate boundedness of solutions of (1.1), when $p \neq 0$.

Theorem 3.1. *Assume that all the hypotheses of Theorem 2.1 hold and there exists a constant $\Delta_0 > 0$ such that*

$$|p(t,x,y)| \leqslant \Delta_0, \tag{3.1}$$

then the solution of (1.1) is uniformly ultimately bounded.

Proof. In view of $V'_{(1.2)}$, when $p = 0$ in (2.4), we have for $p \neq 0$ in (1.1), along any solution (x_t, y_t) of (1.2) to get for

$$\frac{dV(t)}{dt} \leqslant -D_3(x^2+y^2) + [(1+\delta\nu)|x| + (\frac{a}{b} + \delta^2 + (a\nu - b\delta))|y|]|p(t,x,y)|.$$

By (3.1) of Theorem 3.1, we have

$$\frac{dV(t)}{dt} \leqslant -D_3(x^2+y^2) + D_4(|x|+|y|)\Delta_0(x^2+y^2),$$

where $D_4 = \max\{(1+\delta\nu), \frac{a}{b} + \delta^2 + (a\nu - b\delta)\}$,

$$\frac{dV(t)}{dt} \leqslant -D_3(x^2+y^2) + D_4(|x|+|y|)\Delta_0, \quad \frac{dV(t)}{dt} \leqslant -D_3(x^2+y^2) + \sqrt{2}D_4\Delta_0(x^2+y^2)^{\frac{1}{2}}.$$

Moreover,

$$\frac{dV(t)}{dt} \leqslant -D_3(x^2+y^2) + D_5(x^2+y^2)^{\frac{1}{2}}, \tag{3.2}$$

where $D_5 = \sqrt{2}D_4\Delta_0$. Choose

$$(x^2+y^2)^{\frac{1}{2}} \geqslant D_6 = D_5 D_3^{-1}.$$

Inequality (3.2) implies that

$$\frac{dV(t)}{dt} \leqslant -D_3(x^2+y^2).$$

So, we have a D_7 such that

$$\frac{dV(t)}{dt} \leqslant -D_7,$$

provided $x^2+y^2 \geqslant D_7 D_3^{-1}$. □

Thus, (3.2) satisfies the conditions of [2, Lemma 2.2] and [21, Lemmas 2 and 3]. The conclusion of the proof of Theorem 3.1 may now be obtained by the use of inequalities (2.2) and (2.4) and an obvious adaptation of the reasoning in [17]. Hence, we omit this part of the proof.

4. Numerical example

Consider (1.1) in the form

$$\begin{aligned} x' &= 12y + 0.2x(t-\rho(t)) + \frac{0.01x(t-\rho(t))}{1+x^2}, \\ y' &= 10y + 0.3x(t-\rho(t)) + \frac{0.01x(t-\rho(t))}{1+x^2} + \frac{1}{1+t^2+x^2+y^2}, \end{aligned} \tag{4.1}$$

with equivalent system of (4.1) as

$$\begin{aligned} x' &= 12y + \left(0.2x + \frac{0.01x}{1+x^2}\right) + \int_{t-(\rho(t))}^{t} \left(0.2 + \frac{0.01}{1+x^2(s)}\right) ds, \\ y' &= 10y + \left(0.3x + \frac{0.01x}{1+x^2}\right) + \int_{t-(\rho(t))}^{t} \left(0.3 + \frac{0.01}{1+x^2(s)}\right) ds + \frac{1}{1+t^2+x^2+y^2}. \end{aligned} \tag{4.2}$$

Comparing (1.2) with (4.2), it is easy to see that $a = 12$ and $b = 10$, and

$$q_1(x) = \left(0.2x + \frac{0.01x}{1+x^2}\right) \quad \text{and} \quad q_2(x) = \left(0.3x + \frac{0.01x}{1+x^2}\right).$$

It is obvious from the equations that

$$\frac{q_1(x)}{x} \geqslant 0.2 = \delta > 0, \quad x \neq 0, \qquad \frac{q_2(x)}{x} \geqslant 0.3 = \nu > 0, \quad x \neq 0,$$

and

$$(a\nu - b\delta) = 1.6 > 0.$$

Also,

$$|q_1'(x)| \leqslant 0.21 = L \quad \text{and} \quad |q_2'(x)| \leqslant 0.31 = M.$$

Since $0 \leqslant \varepsilon \leqslant 1$, choosing $\varepsilon = 0.5$, we have

$$\eta < \min\{0.54, 93.01\}.$$

If we choose $\eta = 0.5$, then $\rho(t) \leqslant 0.5$, if the delay is extended beyond this interval, the behavior of solution of (4.1) may also be useful. Finally,

$$p(t,x,y) = \frac{1}{1+t^2+x^2+y^2} \leqslant \frac{1}{1+t^2} \leqslant 1.$$

Thus, all the conditions of Theorems 2.1 and 3.1 are satisfied. The trivial solution of (4.1) is asymptotically stable and the solution of the same equation is ultimately bounded.

Figures 1 and 2 show that the solution (x_t, y_t) of the delay system (4.1) is asymptotically stable as $t \to \infty$ and in Figure 3, the parametric plot of $x(t)$ versus $y(t)$ shows the trajectory of solution of system (4.1) remains stable as t increases and the delay function causes the trajectory to be bounded as $t \to \infty$. Figure 4 shows the direction field associated with the system (4.1) showing that the solution is asymptotically stable for $\rho(t) = 0.5$, that is, the origin is a stable spiral. Figures 5 and 6 show that the solution (x_t, y_t) of the delay system (4.1) is ultimately bounded by a single constant.

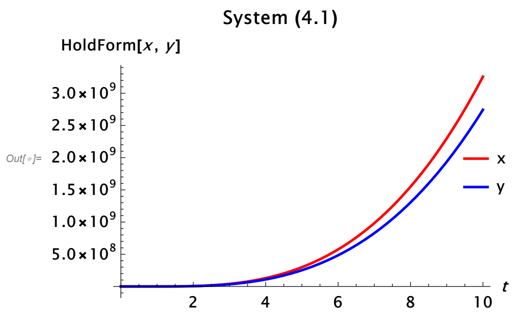

Figure 1: The plot of $x(t)$ (in blue) and $y(t)$ (red) satisfying the conditions of Theorem 2.1 if $\rho(t) = 0.5$ and $p = 0$ for $0 \leqslant t \leqslant 10$.

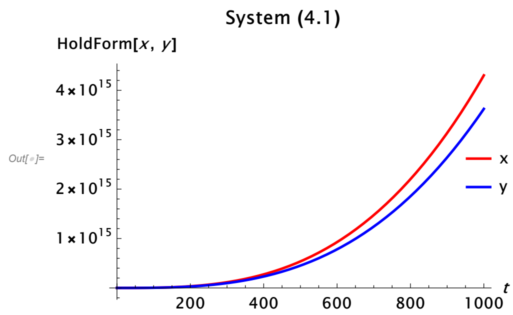

Figure 2: The plot of $x(t)$ (in blue) and $y(t)$ (red) satisfying the conditions of Theorem 2.1 if $\rho(t) = 0.5$ and $p = 0$ for $0 \leqslant t \leqslant 1000$.

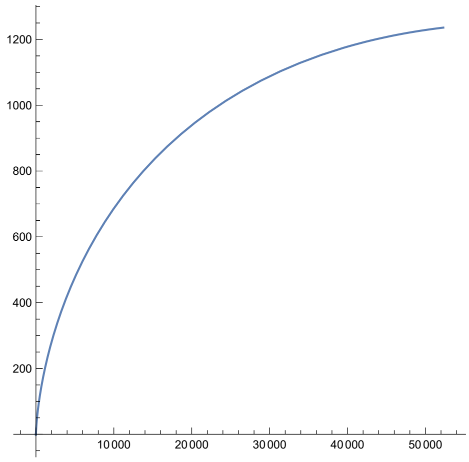

Figure 3: The parametric plot of $x(t)$ versus $y(t)$ satisfying the conditions of Theorem 2.1 if $\rho(t) = 0.5$ and $p = 0$ for $0 \leqslant t \leqslant 10$ cycle and $0 \leqslant t \leqslant 1000$ cycle.

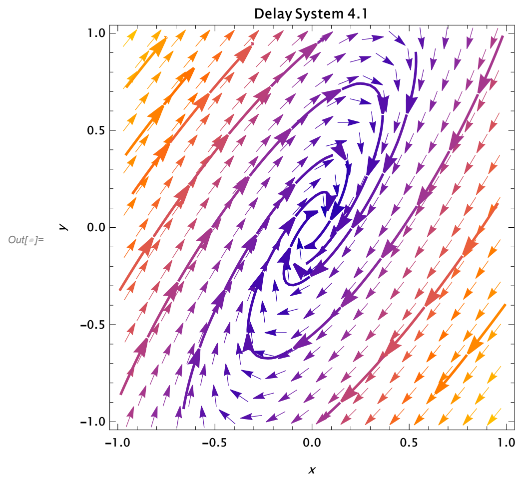

Figure 4: The direction field associated with the delay system (4.1) satisfying the conditions of Theorem 2.1 and $\rho(t) = 0.5$ together with several solutions of the system.

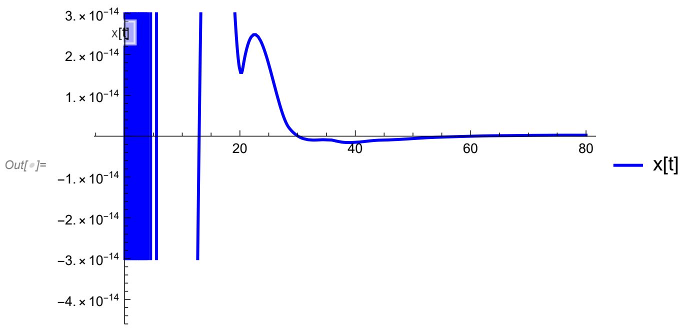

Figure 5: The boundedness of $x(t)$ of system (4.1) satisfying the conditions of Theorems 2.1 and 3.1 with $\rho(t) = 0.5$ and $p \neq 0$ as $t \to \infty$.

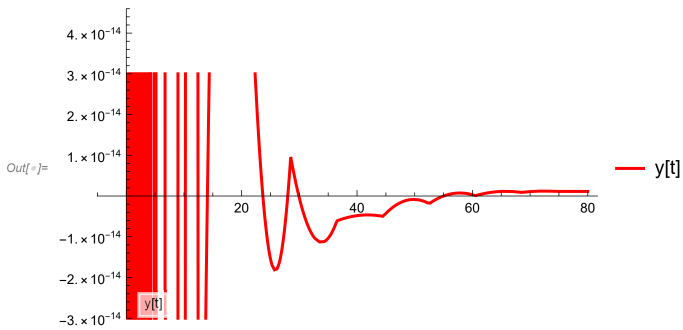

Figure 6: The boundedness of $y(t)$ of system (4.1) satisfying the conditions of Theorems 2.1 and 3.1 with $\rho(t) = 0.5$ and $p \neq 0$ as $t \to \infty$.

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References

[1] D. O. Adams, A. L. Olutimo, *Some results on the boundedness of solutions of a certain third order non-autonomous differential equations with delay*, Adv. Stud. Contemp. Math., **29** (2019), 237–250. 1

[2] A. U. Afuwape, M. O. Omeike, *Stability and boundedness of solutions of a kind of third order delay differential equations*, Comput. Appl. Math., **29** (2010), 329–342. 1, 2.1, 3

[3] H. Arifah, K. P. Krisnawan, *A model of predator-prey differential equation with time delay*, J. Phys.: Conf. Ser., **1320** (2019), 1–6. 1

[4] P. S. M. Chin, *Stability results for the solutions of certain fourth order autonomous differential equations*, Internat. J. Control, **49** (1989), 1163–1173. 1

[5] J. Cronin-Scanlon, *Some mathematics of biological oscillations*, SIAM Rev., **19** (1997), 100–138. 1

[6] N. P. Erugin, *On certain questions of stability of motion and the qualitative theory of differential equations in the large (Russian)*, Akad. Nauk SSSR. Prikl. Mat. Meh., **14** (1950), 459–512. 1

[7] N. H. Gazi, M. Bandyopadhyay, *Effect of time delay on a harvest predator-prey model*, J. Appl. Math. Comput., **26** (2008), 263–280. 1

[8] J. R. Graef, L. Hatvani, J. Karsai, P. W. Spikes, *Boundedness and asymptotic behavior of solutions of second order nonlinear differential equations*, Publ. Maths. Debrecen, **36** (1989), 85–99. 1

[9] J. R. Graef, P. W. Spikes, *Asymptotic behavior of solutions of a second order nonlinear differential*, J. Differential Equations, **17** (1975), 461–476. 1

[10] N. N. Krasovskii, *Theorems on stability of motions determined by a system of two equations (Russian)*, Akad. Nauk SSSR. Prikl. Mat. Meh., **16** (1952), 547–554. 1

[11] I. G. Malkin, *On a problem of the theory of stability of systems of automatic regulation (Russian)*, Akad. Nauk SSSR. Prikl. Mat. Meh., **16** (1952), 365–368. 1

[12] A. L. Olutimo, *On the stability and ultimate boundedness of solutions for certain third order non-autonomous delay differential equations*: In Proceedings of the 14th International Conference: Dynamical Systems Theory and Applications, Lodz, Poland. Vibration, Control and stability of Dynamical Systems, (Department of Automation, Bio-mechanics and Mechatronics), (2018), 389–400. 1

[13] A. L. Olutimo, D. O. Adams, *On the stability and boundedness of solutions of certain non-autonomous delay differential equations of third order*, Appl. Math., **7** (2016), 457–467.

[14] A. L. Olutimo, A. O. Abosede, I. D. Omoko, *On the existence of periodic or almost periodic solutions of a kind of third order nonlinear delay differential equations*, J. Nig. Assoc. Math. Physics., **57** (2020), 45–54. 1

[15] A. L. Olutimo, D. O. Adams, A. A. Abdurasid, *Stability and boundedness analysis of a prey-predator system with predator cannibalism*, J. Nigerian Math. Soc., **41** (2022), 275–286. 1

[16] A. L. Olutimo, I. D. Omoko, *The problem of convergence of solutions of certain third order nonlinear delay differential equations*, Differ. Uravn. Protsessy Upr., **2020** (2020), 12–29. 1

[17] M. O. Omeike, *New result in the ultimate boundedness of solutions of a third-order nonlinear ordinary differential equation*, J. Inequal. Pure Appl. Math., **9** (2008), 8 pages. 3

[18] B. Rahman, M. A. Yau, Y. N. Kyrychko, K. B. Blyss, *Dynamics of a predator model with discrete and distributed delay*, Int. J. Dyn. Syst. Differ. Equ., **10** (2020), 475–449. 1

[19] L. L. Rauch, *Oscillations of a third-order nonlinear autonomous system in contributions to the theory of nonlinear oscillations*, In: Contributions to the Theory of Nonlinear Oscillations, Princeton University Press, (1950), 39–88. 1

[20] J.-P. Richard, *Time-delay system: an overview of some recent advances and open problems*, Automatica, **39** (2003), 1667–1694. 1

[21] A. I. Sadek, *Stability and boundedness of a kind of third order delay differential system*, Appl. Math. Lett., **16** (2003), 657–662. 2.1, 3

[22] C. Tunç, *A note of boundedness of solutions to a class of non-autonomous differential equations of second order*, Appl. Anal. Discrete Math., **4** (2010), 361–372. 1

[23] C. Tunç, *Boundedness analysis for certain two-dimensional differential systems via Lyapunov approach*, Bull. Math. Soc. Sci. Math. Roumanie, **53(101)** (2010), 61–68. 1

[24] C. Tunç, *On the stability and boundedness of solutions of a class of Liénard equations with multiple deviating arguments*, Vietnam J. Math., **39** (2011), 177–190. 1

[25] C. Tunç, O. Tunç, Y. Wang, J.-C. Yao, *Qualitative analyses of differential systems with time-varying delays via Lyapunov–Krasovskiĭ approach*, Mathematics, **9** (2021), 1–20. 1

[26] H. Yao, W. Meng, *On the stability of solutions of certain non-linear third order delay differential equations*, Int. J. Nonlinear Sci., **6** (2008), 230–237. 1