J. Nonlinear Sci. Appl., 16 (2023), 99–110

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Fuzzy implications based on quasi-copula and fuzzy negations

Georgios Souliotis[a], Michael Th. Rassias[b,c], Basil Papadopoulos[a,*]

[a]*Department of Civil Engineering, Section of Mathematics and Informatics, Democritus University of Thrace, 67100 Kimeria, Greece.*
[b]*Hellenic Military Academy, Greece.*
[c]*Institute for Advanced Study, Princeton, USA.*

Abstract

In this particular paper the connection of fuzzy implications to the basic concepts of probability theory such as copula, quasi-copula and semi-copula is being studied. This study showed that fuzzy implications produced through copula, quasi-copula or semi-copula, apart from having as a common characteristic the Lipschitz condition with constant 1, this characteristic is also the cornerstone for grouping fuzzy implications according to the original generator which is no other than a copula, quasi-copula or semi-copula.

Keywords: Fuzzy implications, fuzzy negations, copula, quasi-copula, semi-copula, aggregations functions.

2020 MSC: 03B52, 03B50, 94D05, 03E72.

1. Introduction

Probabilistic S- Implications and fuzzy implications based on semicopulas introduced by Przemyslaw Grzegorzewski and Michal Baczynski, Przemyslaw Grzegorzewski, Radko Mesiar, Piotr Helbin and Wanda Niemyska, respectively, have attracted the research interest of many scientists specializing in fuzzy implications. The above papers connect the concept of fuzzy implication with probability theory and specifically with the quasi-copula, copula and semicopula conjuctors. All the above effort is being made in the context of a reasoning in problems where logic and randomness interact, consequently Aristotle logic is not enough. This particular work becomes the bridge between fuzzy implications and a special category of aggregations functions, of conjunctive aggregations functions, trying as much as possible to provide the appropriate tools for the production of fuzzy implications that interact with probability theory. Also through this bridge connection it is proved that the set of all probabilistic S-Implications is a subset of fuzzy implications $I(x, y) = 1 - Q(x, 1 - y)$ for some quasi-copula Q. The structure of the paper follows the following pattern. Initially, the basic definitions and propositions that will be needed are provided. The new results are then given with the proofs and examples where necessary, along with grouping of the fuzzy implications. Specifically, two classes of implications are defined, the first being: "the family of fuzzy implications of copula" and the second: "the family of fuzzy implications of quasi-copula" and in fact the first is a subset of the second.

*Corresponding author
Email addresses: gsouliot@civil.duth.gr (Georgios Souliotis), michail.rassias@math.uzh.ch (Michael Th. Rassias), papadob@civil.duth.gr (Basil Papadopoulos)

doi: 10.22436/jnsa.016.02.03

Received: 2022-10-12 *Revised:* 2023-01-19 *Accepted:* 2023-03-03

2. Basic notions

In this paragraph we recall the basic theory that we will be needed to understand the key points of this paper.

2.1. Aggregation functions

Aggregation functions combine is the process of combining several numerical values into a single representative value, and an aggregation function performs this operation. Many well-known binary aggregation operators, such as the arithmetic mean, the product, Min and Max operators, as well as weighted means aggregation operators (for more details see, e.g., [11]). Apart from the theoretical extensions of these functions that penetrate several concepts of mathematics such as copulas, quasi-copulas, stochastic processes and other, these also have diverse applications, including approximate reasoning, control systems, image processing, etc.

Definition 2.1 ([4, 13])**.** A mapping $A : [0,1]^2 \to [0,1]$ is called a binary aggregation function whenever

(A1) it is non-decreasing in both variables, i.e., for all $x_1, x_2, y_1, y_2 \epsilon\, [0,1]$ with $x_1 \leqslant x_2, y_1 \leqslant y_2$ implies $A(x_1, y_1) \leqslant A(x_2, y_2)$;
(A2) $A\,(0,0) = 0,\ A\,(1,1) = 1$.

The class of all binary aggregation functions will be denoted by $\mathfrak{A}$.

Definition 2.2 ([11])**.** An aggregation function A is called conjunctive aggregation function whenever $0 \leqslant A \leqslant \mathrm{Min}$.

Obviously, the smallest aggregation function $A_\perp(x,y) = \begin{cases} 1, & \text{if } (x,y) = (1,1), \\ 0, & \text{otherwise,} \end{cases}$ is also the smallest conjunctive aggregation function, while Min is the greatest conjunctive aggregation. Though more explanation will be given in subsequent sections, we list here some distinguished classes of conjunctive aggregation functions.

Definition 2.3 ([11])**.** Let $A : [0,1]^2 \to [0,1]$ be an aggregation function.

(bwt-norm) A is a boundary weak triangular norm (bwt-norm for short) if it is associative, i.e., $A(x, A(z,y)) = A(A(x,y), z)$ and symmetric, i.e., $A\,(x,y) = A\,(y,x)$ conjunctive aggregation function.

(SC) If A has a neutral element $e = 1$, i.e., $A\,(x,1) = A\,(1,x) = x$, then A is called semi-copula.

(t-norm) A is a triangular norm (t-norm in short) if it is bwt-norm and a semi-copula.

(QC) A is a quasi-copula if it is a conjunctive aggregation function and satisfies the Lipschitz condition with constant 1, i.e., such that for all $x, y, u, v \epsilon [0,1]$ we have $|A\,(x,y) - A\,(u,v)| \leqslant |x-u| + |y-v|$.

(C) A is a copula if it is a semi copula and fulfills the moderate growth property, i.e., such that for all $x, y, u, v \epsilon [0,1]$ with $x \leqslant u, y \leqslant v$, we have $A\,(x,y) + A\,(u,v) \geqslant A\,(x,v) + A\,(u,y)$.

We will then look at the definitions of copula, semi-copula and quasi-copula from a different perspective. Since definitions are equivalent, we just resorted to the definition of the above concepts through aggregation functions in order to be able to connect aggregation functions with fuzzy implications.

Proposition 2.4 ([13] or [16])**.** *For any* $A \epsilon \mathfrak{A}$ *the mapping* $\bar{A} : [0,1]^2 \to R$ *by*

$$\bar{A}(x,y) = x + y - A(x,y) \tag{2.1}$$

is an aggregation function only if A *is 1-Lipschitz aggregation function.*

Remark 2.5 ([13]). In the class $A^{(1)}$ of all 1-Lipschitz aggregation functions, the construction method $\bar{\ }{:}A^{(1)} \to A^{(1)}$ defined by equation (2.1) is an involutive mapping. Another involutive construction method on $A^{(1)}$ as in [19] is the reverse $\widehat{\ }{:}A^{(1)} \to A^{(1)}$ given by

$$\widehat{A}(x,y) = x + y - 1 - A(1-x, 1-y).$$

Definition 2.6 ([13]). For any aggregation function $A \epsilon \mathfrak{A}$ we can define two mappings $A^*, A_* : [0,1]^2 \to R$ given by

$$A^*(x,y) = y + A(1,0) - A(1-x,y), \qquad A_*(x,y) = x + A(0,1) - A(x,1-y).$$

Proposition 2.7 ([13, Theorem 1]). *Let $A \epsilon \mathfrak{A}$. Then $A^* \epsilon \mathfrak{A} (A_* \epsilon \mathfrak{A})$ if and only if*

(i) *A is 1-Lipschitz in the second variable (first variable respectively); and*
(ii) $A(0,1) = A(1,0)$.

2.2. *Copula, quasi copula, and semi copula*

Copulas play a key role in the probability theory and statistics. The basic result in this context is Sklar's Theorem [18]. The family of functions that generalize copulas are quasi copulas, while the family of quasi copulas belongs to a wider family, the family of semi copula and the semi copula belongs to the family of conjunctive aggregation functions see Definition 2.3. A very good picture of the above families is given in Figure 1.

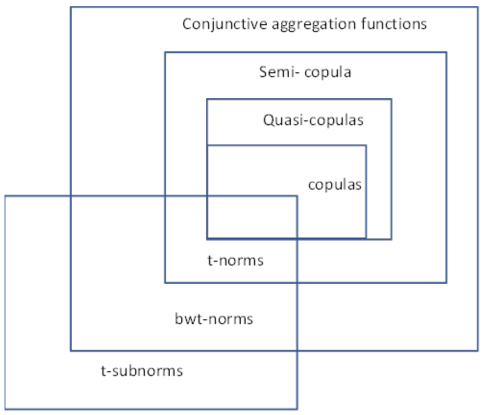

Figure 1: Relations between various binary aggregation functions.

Definition 2.8 ([18]). A copula (specifically, a 2-copula) is a function $C : [0,1]^2 \to [0,1]$ which satisfies the following condition

(C1) $C(x,0) = C(0,y) = 0$ for all $x, y \epsilon [0,1]$;
(C2) $C(x,1) = x$ for all $x \epsilon [0,1]$;
(C3) $C(1,y) = y$ for all $y \epsilon [0,1]$;
(C4) $C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \geqslant 0$ for all $x_1, x_2, y_1, y_2 \epsilon [0,1]$ such that $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

The family of all copulas will be denoted by $\mathcal{C}$.

Remark 2.9 ([18]). The following is very useful for the sequel: for each copula C, the following functions $C_*, C^*, \hat{C} : [0,1]^2 \to [0,1]$ are copulas too:

$$C_*(x,y) = x - C(x, 1-y), \tag{2.2}$$
$$C^*(x,y) = y - C(1-x, y), \tag{2.3}$$
$$\hat{C}(x,y) = x + y - 1 - C(1-x, 1-y). \tag{2.4}$$

Several interesting families of copulas, like Fréchet's family, Farlie-Gumbel-Morgenstern's family, Marshall-Olkin's family, etc, are considered in the literature. It can be shown that every copula is bounded by the so-called Fréchet-Hoeffding bounds, i.e., for any copula C and for all $x, y \epsilon [0,1]$ applies $W(u,v) \leqslant C(u,v) \leqslant M(u,v)$, where, $W(u,v) = \max\{u+v-1, 0\}$ and $M(u,v) = \min\{u,v\}$ are also copulas. For more information on copulas, we refer the reader to the famous monograph by Nelsen [18]. The importance of the copulas is clarified by the Sklar theorem showing that copulas link joint distribution functions to their one-dimensional margins. Many scientists have focused on the theory of copulas, providing numerous and interesting families of copulas, like Fréchet's family, Marshall-Olkin's family etc. due to the fact that in general Copulas are neither symmetric (i.e., C(x,y)=C(y,x)) nor associative (i.e., if $C(x, C(y,z)) = C(C(x,y), z)$ hold, then C is associative, different will be called non associative). In general, we will provide the following example as a symmetric but non associative case and a copula case where it is symmetric and associative.

Example 2.10.

(i) The Farlie-Gumbel-Morgenstern family of copulas symmetric but not associative. For $\lambda \neq 0$ is given by $C_\lambda^{FGM} = (x,y) = xy + \lambda xy(1-x)(1-y)$ for $\lambda \in [-1,1]$.

(ii) A linear combination $C = p\Pi(x,y) + (1-p)\min\{x,y\}$ is another case of a general non associative symmetric copula where $p\epsilon(0,1)$ and $\Pi(x,y) = xy$.

(iii) We must note that the copulas $\Pi(x,y) = xy$ and $\min\{x,y\}$ are symmetric and associative.

In order to generalize the notion of copulas, quasi-copulas of were introduced by Alsina et al. [1] as special functions.

Definition 2.11 ([10, 11])**.** A function $Q : [0,1]^2 \to [0,1]$ which satisfies conditions **(C1)-(C3)** and

(C4') $Q(x_2,y_2) - Q(x_2,y_1) - Q(x_1,y_2) + Q(x_1,y_1) \geqslant 0$ for all $x_1, x_2, y_1, y_2 \epsilon [0,1]$ such that $x_1 \leqslant x_2, y_1 \leqslant y_2$, where at least one of all x_1, x_2, y_1, y_2 belongs to $\{0,1\}$,

is called a quasi-copulas. The set of all quasi-copulas will be denoted by $\mathcal{QC}$.

Remark 2.12 ([2, 11]).

(a) The condition **(C4')** is equivalent to the following two conditions.
  (i) Implies **(A1)**, i.e., the functions Q are non-decreasing in each variable; and
  (ii) Q satisfy the 1-Lipschitz property, i.e., such that for all $x, y, u, v \epsilon [0,1]$ we have $|Q(x,y) - Q(u,v)| \leqslant |x-u| + |y-v|$.

(b) It becomes apparent that Definition 2.11 and definition 2.3 with respect to quasi-copula are equivalent definitions, i.e., a quasi-copula is a conjunctive aggregation function which satisfies the 1-Lipschitz property and vice versa, is a conjunctive aggregation function which satisfies the 1-Lipschitz property is a quasi-copula.

(c) It is clear that every copula is a quasi-copula. Quasi-copulas that are not copulas are called proper quasi-copulas.

(d) An interesting example of a symmetric quasi-copula which is not a copula is given by $Q = \mathrm{Med}(a, T_L, \Pi)$, $a \in (0,1)$, where $T_L = \max\{x+y-1, 0\}$ (see [11]).

e. It is evident that each associative quasi-copula is also an associative copula (see [11]).

Remark 2.13 ([5, 13]). As in Remark 2.9 we have, for each quasi-copula Q, the functions Q^*, Q_*, $\hat{Q} : [0,1]^2 \to [0,1]$ are quasi-copula too:

$$Q_*(x,y) = x - Q(x, 1-y), \tag{2.5}$$
$$Q^*(x,y) = y - Q(1-x, y), \tag{2.6}$$
$$\hat{Q}(x,y) = x + y - 1 + Q(1-x, 1-y). \tag{2.7}$$

Another set aggregation function which is worth noting is a semi-copula [19]. Semi-copulas are known under several different names, e.g., conjunctors [11], weak t-norms [9], etc.

Definition 2.14 ([6, 7, 19])**.** A function $B : [0,1]^2 \to [0,1]$ is called semi-copula if it satisfies conditions **(C2)**, **(C3)**, and **(A1)**, i.e.,

(C2) $B(x,1) = x$ for all $x \epsilon [0,1]$;
(C3) $B(1,y) = y$ for all $y \epsilon [0,1]$;
(A1) it is non-decreasing in both variables, i.e., for all $x_1, x_2, y_1, y_2 \epsilon [0,1]$ with $x_1 \leqslant x_2, y_1 \leqslant y_2$ implies $B(x_1,y_1) \leqslant B(x_2,y_2)$.

The family of all semi-copulas will be denoted by $(\mathcal{SC})$.

Lemma 2.15 ([19])**.** *Each semi-copula satisfies condition* **(C1)**.

Proof. By Definition 2.14 we have $0 \leqslant B(x,0) \leqslant B(1,0) = 0, 0 \leqslant B(0,y) \leqslant B(0,1) = 0$, therefore $B(x,0) = B(0,y) = 0$, i.e., satisfies condition **(C1)**. □

From the above definitions and propositions, the connections and overlaps that exist in the concepts of aggregation functions, copula, quasi-copula and semi-copula become apparent. The following proposition attempts to put in order these concepts as well as their overlaps.

Proposition 2.16 ([19])**.**

(i) *A semi-copula* C *which satisfies condition* **(C4)**, *is a copula.*
(ii) *A semi-copula* Q *which satisfies the 1-Lipschitz property is a quasi-copula.*

Proposition 2.17.

1. *A semi-copula* C *is an aggregation function which has a neutral element* $e = 1$ *(see Definition 2.3 or Remark 2.12).*
2. *For each semi-copula* B *which is 1-Lipschitz in the first variable, the following function,* $B_* : [0,1]^2 \to [0,1]$, *is semi-copula too (see [13])*
$$B_*(x,y) = x - B(x, 1-y). \tag{2.8}$$
3. *For each semi-copula B which is 1-Lipschitz in the second variable, the following function,* $B^* : [0,1]^2 \to [0,1]$ *is semi-copula too (see Proposition 2.7)*
$$B^*(x,y) = y - B(1-x,y). \tag{2.9}$$

Proposition 2.17 will be needed to link some of the results of Probabilistic S-Implications (see [12]) and Fuzzy implications papers, based on semicopulas (see [2]).

2.3. *Fuzzy implications and fuzzy negations*

In the international literature, we can find several definitions of fuzzy implication based on the characteristic properties of classical implication. In this paper we will use the equivalent definition proposed by Kitainik [14] (see also Foodor and Roubens [8]), below we quote the definition of fuzzy implication with its properties.

Definition 2.18. A function $I : [0,1]^2 \to [0,1]$ will be considered to be a fuzzy implication if it satisfies for each $x, x_1, x_2, y, y_1, y_2 \in [0,1]$ the following:

(I1) $x_1 \leqslant x_2$, then $I(x_1,y) \geqslant I(x_2,y)$ decreasing with respect to the 1st variable;
(I2) $y_1 \leqslant y_2$, then $I(x,y_1) \leqslant I(x,y_2)$ increasing to the 2nd variable (elsewhere the term non-decreasing is used);
(I3) $I(0,0) = 1$;

(I4) $I(1,1) = 1$;
(I5) $I(1,0) = 0$.

We will denote the class of all fuzzy implications by $\mathcal{I}$. Directly from Definition 2.18 we can deduce that each fuzzy implication $\mathcal{I}$ satisfies the following properties:

(LB) left boundary condition $I(0,y) = 1,\ y\epsilon[0,1]$;
(RB) Right boundary condition $I(x,1) = 1,\ x\epsilon[0,1]$.

We list the most important additional properties of implications. Most are generalizations of bivalent logic.

(NP) Neutrality property $I(1,y) = y,\ y\epsilon[0,1]$;
(EP) Exchange principle $I(x, I(y,z)) = I(y, I(x,z)),\ y,x,z \in [0,1]$;
(IP) Identidy principle $I(x,x) = 1,\ x \in [0,1]$;
(OP) Ordering property $I(x,y) = 1 \Longleftrightarrow x \leqslant y,\ x,y \in [0,1]$.

Example 2.19. The following functions from $[0,1]^2 \to [0,1] \longrightarrow [0,1]$ are common examples of fuzzy implications:

(IŁ) the Łukasiewicz fuzzy implication $I_L(x,y) = \min\{1, 1-x+y\}$;
(IR) the Reichenbach fuzzy implication $I_R(x,y) = 1-x+xy$;
(IKD) the Kleene-Dienes fuzzy implication $I_{KD}(x,y) = \max\{1-x, y\}$;
(IGG) the Goguen fuzzy implication $I_{GG}(x,y) = \begin{cases} 1, & \text{if } x \leqslant y, \\ \frac{y}{x}, & \text{otherwise.} \end{cases}$

In the theory of fuzzy logic as well as in classical logic, negation plays a decisive role, and the following definition is given.

Definition 2.20 ([15])**.** A non-increasing function $N : [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1,\ N(1) = 0$. Moreover, a fuzzy negation N is called strict if it is strictly decreasing and continuous; strong if it is involution, i.e., $N(N(x) = x$ for all $x\epsilon\,[0,1]$.

Example 2.21. The following functions $N : [0,1] \to [0,1]$ are common examples of fuzzy strong negations.

Sugeno class $N^{\lambda}(x) = \frac{1-x}{1+\lambda x},\ \lambda \in (-1, +\infty)$;

Yager class $N^{w}(x) = (1-x^{w})^{\frac{1}{w}},\ w \in (0, +\infty)$;

Classical negation $N_C(x) = 1-x$, which is the one and only 1-Lipschitz fuzzy negation, i.e., $|f(x_1) - f(x_2)| \leqslant |x_1 - x_2|$, for all $x_1, x_2 \epsilon [0,1]$ (see [2]) and

Conical negation $N_{CN}(x) = \sqrt{(m^2-1)x^2+1} + mx,\ \ x \in [0,1],\ m \leqslant 0$ (see [20]).

One of the most important tautologies in the classical logic is the law of contraposition:

1. $p \to q \equiv \neg q \to \neg p$ and its variants;
2. $\neg p \to q \equiv \neg q \to p$;
3. $p \to \neg q \equiv q \to \neg p$.

Consequently, we can consider different laws of contraposition in fuzzy logic.

Definition 2.22 ([3])**.** Let $I \in \mathcal{I}$ and N be a fuzzy negation. I is said to satisfy

- low of contraposition (or, the contrapositive symmetry) with respect to N, if $I(x,y) = I(N(y), N(x))$, $x,y\epsilon[0,1]$ **(CP)**;
- low of left contraposition with respect to N, if $I(N(x), y) = I(N(y), x)$, $x, y\epsilon\,[0,1]$ **(L-CP)**;
- low of right contraposition with respect to N, if $I(x, N(y)) = I(y, N(x))$, $x, y\epsilon\,[0,1]$ **(R-CP)**.

2.4. Constructing family of fuzzy implications from semi-copula

In Grzegorzewski's paper entitled probabilistic implications (see [12]) he introduced probabilistic S-implications creating a bridge of fuzzy implications with copulas, specifically: a function $I_C : [0,1]^2 \to [0,1]$ given by

$$I_C(u,v) = C(u,v) - u + 1, \tag{2.10}$$

where C which is any copula, is a fuzzy implication which is called a probabilistic S-implication. Some of the properties that probabilistic S-implications can satisfy are as following.

Proposition 2.23.

(i) *A probabilistic S-implication* I_C *based on a copula satisfies the left neutrality principle* **(NP)**.
(ii) *A probabilistic S-implication* I_C *based on a copula satisfies the identity principle if and only if* I_C *is the Łukasiewicz implication, i.e.,* $I_C(x,y) = I_{LK}(x,y) = \min\{1, 1-x+y\}$.
(iii) *A probabilistic S-implication* I_C *based on a copula satisfies the ordering principle if and only if* I_C *is the Łukasiewicz implication.*

From another point of view the authors of [2] studied among other fuzzy implication,

$$I(x,y) = 1 - Q(x, 1-y) \tag{2.11}$$

for some quasi-copula Q. In fact, they proved that the only fuzzy implications (based on Definition 2.18) that satisfies 1-Lipschitz property, are given by equation (2.11) and vice versa. Later Mesiar and Kolesárová in [17] completed the above effort with the following notes.

Proposition 2.24 ([17])**.** *For every semi-copula* $B \in \mathcal{SC}$ *and* $I \in \mathcal{I}$ *we have*

(i) *the function* $I_B : [0,1]^2 \to [0,1]$ *given by*

$$I_B(x,y) = 1 - B(x, 1-y) \tag{2.12}$$

is a fuzzy implication;
(ii) *the function* $B_I : [0,1]^2 \to [0,1]$ *given by*

$$B_I(x,y) = 1 - I(x, 1-y) \tag{2.13}$$

is a semi-copula, i.e., $B \in \mathcal{SC}$*;*
(iii) $B_{I_B} = B$ *and* $I_{B_I} = I$.

They also led to additional fuzzy implications, which are given in the following proposition.

Proposition 2.25 ([17])**.** *For every quasi-copula* $Q \in \mathcal{QC}$ *the following functions* $I : [0,1]^2 \to [0,1]$ *are fuzzy implications:*

$$I(x,y) = 1 - x + Q(x,y), \tag{2.14}$$
$$I(x,y) = y + Q(1-x, 1-y), \tag{2.15}$$
$$I(x,y) = y + 1 - x - Q(1-x, y). \tag{2.16}$$

As we see, the connection of fuzzy implications and semi-copula and consequently of quasi-copula and copula has been studied a lot. However in the next paragraph we will show that the fuzzy implications studied in the aforementioned papers besides being made by quasi-copulas, have another common feature. They constitute different manifestations of the same implication. That is, let assume that we have the implication $I_1(x,y) = 1 - x + Q_1(x,y)$ produced by the quasi-copula $Q_1(x,y)$, then there is a unique $Q_2(x,y) \in \mathcal{QC}$ that produces the implication $I_1(x,y) = 1 - x + Q_1(x,y)$ produced by the quasi-copula $Q_1(x,y)$, then there is a unique $Q_2(x,y) \in \mathcal{QC}$ that produces the implication $I_2(x,y) = 1 - Q_2(x, 1-y)$ so that $I_1(x,y) = I_2(x,y)$ for each $x, y \epsilon [0,1]$.

3. Results

Next in this paper, we will formulate the corresponding result of Proposition 2.25, but for copulas which as we know satisfies condition 1-Lipschitz, thus defining the family of fuzzy implications of copula.

3.1. The family of fuzzy implications of copula

Proposition 3.1. *For every copula $C \in \mathcal{C}$ the following functions, $I : [0,1]^2 \to [0,1]$ are fuzzy implications:*

$$I(x,y) = 1 - C(x, 1-y), \tag{3.1}$$

$$I(x,y) = y + C(1-x, 1-y), \tag{3.2}$$

$$I(x,y) = y + 1 - x - C(1-x, y), \tag{3.3}$$

and

$$I_C(x,y) = 1 - x + C(x,y), \tag{3.4}$$

where is a probabilistic S-implication (see equation (2.10)).

Proof. We can easily see from Remark 2.12, which tells us, every copula is a quasi-copula, therefore: from $I(x,y) = 1 - Q(x,1-y)$ (equation (2.11)) we have $I(x,y) = 1 - C(x,1-y)$. Similarly, equations (3.2) and (3.3) are the result of Proposition 2.25, but simply in place of quasi-copula Q we put a copula C. While for the proof of equation (3.4) see ([12]). □

The question is the following: are implications mentioned in Proposition 3.1 different from each other? The answer to this question is given in the following proposition.

Proposition 3.2.

(i) *For each copula $C \in \mathcal{C}$ that produces fuzzy implications $I_C(x,y) = 1 - x + C(x,y)$, there is a unique C_* that produces fuzzy implication $I_{C_*}(x,y) = 1 - C_*(x, 1-y)$, such that, $I_C(x,y) = I_{C_*}(x,y)$.*

(ii) *For each copula $C \in \mathcal{C}$ and fuzzy implication $I_C(x,y) = 1 - x + C(x,y)$, there is a unique C^* that produces fuzzy implication $I_{C^*}(x,y) = y + 1 - x - C^*(1-x, y)$, such that, $I_C(x,y) = I_{C^*}(x,y)$.*

(iii) *For each copula $C \in \mathcal{C}$ and fuzzy implication $I_C(x,y) = 1 - x + C(x,y)$, there is a unique $\hat{C}$ that produces fuzzy implication $I_{\hat{C}}(x,y) = y + \hat{C}(1-x, 1-y)$, such that, $I_C(x,y) = I_{\hat{C}}(x,y)$.*

Proof. From Remark 2.9 we have the following, for each copula C, the following functions $C_*, C^*, \hat{C}$ with equations (2.2), (2.3), and (2.4) are copulas too.

(i) We have from equation (2.2) that there is copula C_*, such as $C(x,y) = x - C_*(x, 1-y)$, by substitution in equation (2.7) for each $x, y \epsilon [0,1]$ we have:

$$I_C(x,y) = 1 - x + C(x,y) = 1 - x + x - C_*(x, 1-y) = 1 - C_*(x, 1-y) = I_{C_*}(x,y).$$

Similarly for cases (ii) and (iii), so we finally have:

$$I_C(x,y) = I_{C_*}(x,y) = I_{C^*}(x,y) = I_{\hat{C}}(x,y), \text{ for each } x, y \in [0,1].$$

□

Therefore we have four "different" implications that produce exactly the same space of fuzzy implications and as a representative of them we keep the following equation

$$I(x,y) = 1 - C(x, 1-y).$$

Conclusion 3.3. The above implication with formula $I(x,y) = 1 - C(x, 1-y)$ of equations (2.10), (3.2), and (3.3) expresses the total of fuzzy implications and only those that are produced by all copula $C \in \mathcal{C}$. The family of all fuzzy implications produced via copulas will be denoted by $\mathcal{I}^{\mathcal{C}}$.

An example is given below.

Example 3.4. Let us consider the upper Fréchet-Hoeffding bound $M(x,y) = \min\{x,y\}$. The fuzzy implication based on M is

$$I_M(x,y) = \min\{x,y\} - x + 1 = \min\{x - x + 1, y - x + 1\} = \min\{1, y - x + 1\},$$

which is nothing else than the Łukasiewicz fuzzy implication ($\mathbf{I_L}$), $I_M(x,y) = I_L(x,y)$. From the forms (2.2), (2.3), and (2.4), and $C(x,y) = M(x,y)$ we have respectively:

(i) $M_*(x,y) = x - M(x, 1-y) = x - \min\{x, 1-y\} = \max\{0, x+y-1\} = W(x,y)$, then, $I_{M_*}(x,y) = 1 - M_*(x, 1-y) = 1 - \max\{0, x+1-y-1\} = 1 + \min\{0, y-x\} = \min\{1, 1-x+y\} = I_L(x,y)$.

(ii) Similarly $M^*(x,y) = y - M(1-x, y) = y - \min\{1-x, y\} = \max\{y+x-1, 0\} = W(x,y)$. Therefore, $I_{M^*}(x,y) = y + 1 - x - M^*(1-x, y) = y + 1 - x - \max\{y + 1 - x - 1, 0\} = y + 1 - x + \min\{x - y, 0\}$ $= \min\{1, y - x + 1\} = I_L(x,y)$.

(iii) $\hat{M}(x,y) = x + y - 1 + M(1-x, 1-y) = x + y - 1 + \min\{1-x, 1-y\} = \min\{x + y - 1 + 1 - x, x + y - 1 + 1 - y\} = \min\{y, x\}$. Therefore, $I_{\hat{M}}(x,y) = y + \hat{M}(1-x, 1-y) = y + \min\{1-x, 1-y\} = \min\{y - x + 1, 1\} = I_L(x,y)$.

The following are some of the properties that are satisfied in Subsection 2.3.

Lemma 3.5. *Each $I \in \mathcal{I}^{\mathcal{C}}$ satisfies the neutrality principle* (NP)*.*

Proof. Actually, $I(1,y) = 1 - C(1, 1-y) = 1 - 1 + y = y$ for every $y \in [0,1]$, which means that each $I \in \mathcal{I}^{\mathcal{C}}$ satisfies (NP). □

Lemma 3.6. *The only fuzzy implication $I \in \mathcal{I}^{\mathcal{C}}$ which satisfies the identity principle (IP) is the Łukasiewicz implication.*

Proof. From the equation (3.1) we have

$$I(x,x) = 1 \Longleftrightarrow 1 - C(x, 1-x) = 1 \Leftrightarrow C(x, 1-x) = 0 \tag{3.5}$$

for every $x \in [0,1]$, according to equation (2.2) there exists copula C_* such that

$$C_*(x,x) = x - C(x, 1-x). \tag{3.6}$$

From (3.5) and (3.6) we have $C_*(x,x) = x$ for every $x \in [0,1]$, therefore, the copula C_* is idempotent (see Definition 2.35 and Proposition 3.1. in [11]). However, the only idempotent copula is $M(x,y) = \min(x,y)$, therefore $I(x,y) = 1 - x + \min\{x,y\} = \min\{1, 1-x+y\}$, i.e., the only fuzzy implication $I \in \mathcal{I}^{\mathcal{C}}$ which satisfies the identity principle (IP) is the Łukasiewicz implication. □

Lemma 3.7. *The only fuzzy implication $I \in \mathcal{I}^{\mathcal{C}}$ which satisfies the ordering principle (OP) is the Łukasiewicz implication.*

Proof. If $I = I_L$, then (OP) holds. To see the opposite, i.e., (OP) satisfied then apply the (IP), from Lemma 3.6 the only fuzzy implication $I \in \mathcal{I}^{\mathcal{C}}$ which satisfies the identity principle (IP) is the Łukasiewicz implication. □

Lemma 3.8. *If $I \in \mathcal{I}^{\mathcal{C}}$, based on a copula C where C is symmetric and associative, then I satisfies exchange principle* (EP)*, i.e., $I(x, I(y,z)) = I(y, I(x,z))$, $x, y, z \epsilon [0,1]$.*

Proof. Let $I \in \mathcal{I}^{\mathcal{C}}$, then there exists a copula C such as $I(x,y) = 1 - C(x, 1-y)$. Also C is symmetric and associative, i.e., $C(x,y) = C(y,x)$ symmetric and $C(x, C(y,z)) = C(C(x,y), z)$ associative. We will have that the following property also applies

$$C(x, C(y,z)) \underset{\text{associative}}{=} C(C(x,y), z) \underset{\text{symmetric}}{=} C(C(y,x), z) \underset{\text{associative}}{=} C(y, C(x,z)).$$

In short

$$C(x, C(y,z)) = C(y, C(x,z)). \tag{3.7}$$

Therefore $I(x, I(y,z)) = 1 - C(x, 1 - I(y,z)) = 1 - C(x, 1 - (1 - C(y, 1-z)) = 1 - C(x, C(y, 1-z)) \underset{(3.3)}{=} 1 - C(y, C(x, 1-z)) = 1 - C(y, 1 - (1 - C(x, 1-z)) = 1 - C(y, 1 - I(x,z)) = I(y, I(x,z))$, i.e., $I(x, I(y,z)) = I(y, I(x,z))$, $y, x, z \epsilon [0,1]$ so I satisfies exchange principle (EP). □

An additional very important element that characterizes the $\mathcal{I}^{\mathcal{C}}$ family is the 1-Lipchitz property that you are satisfied with all the fuzzy implications of this family. Proof is given by the next lemma.

Lemma 3.9. *Let* I *be a fuzzy implication in* $\mathcal{I}^{\mathcal{C}}$. *Then* I *is1-Lipschitz.*

Proof. Since $\mathcal{I} \in \mathcal{I}^{\mathcal{C}}$, there exist copula C which, as we know is 1-Lip, given by the formula $I(x,y) = 1 - C(x, 1-y)$ (see (2.8)). Then, for $x_1 \leqslant x_2, y_1 \leqslant y_2$ we have $|I(x_1,y_1) - I(x_2,y)| = |1 - C(x_1, 1-y_1) - (1 - C(x_2, 1-y_2))| = |1 - C(x_1, 1-y_1) - 1 + C(x_2, 1-y_2)| = |C(x_1, 1-y_1) - C(x_2, 1-y_2)| \leqslant |x_1 - x_2| + |1 - y_1 - (1-y_2)| \leqslant |x_1 - x_2| + |y_1 - y_2|$, so I is 1-Lipschitz. □

3.2. *The family of fuzzy implications of quasi-copula*

Let's attempt to extend family $\mathcal{I}^{\mathcal{C}}$ and in place of copula let's place their extension, quasi-copula. This becomes feasible with transformation of the formulas (2.5), (2.6), (2.7), as

$$Q_*(x,y) = x - Q(x, 1-y), \quad Q^*(x,y) = y - Q(1-x, y), \quad \hat{Q}(x,y) = x + y - 1 + Q(1-x, 1-y).$$

Also the following is proved by the paper of Michał Baczyński et al., entitled "Fuzzy implications based on semicopulas" (see[2]) as follows.

Proposition 3.10 ([2, Lemma 2.13])**.** *Let* I *be a Fuzzy implication. Then,* I *is 1-Lipscitz if and only if* $I(x,y) = 1 - Q(x, 1-y)$ *for some quasi-copula Q.*

The result of the above is the total of fuzzy implications which satisfies 1-Lipschitz property, being produced solely by formation (2.11). Furthermore, in accordance with transformations (2.5), (2.6), and (2.7) and fuzzy implications in Proposition 2.25 we are led to (exactly as happened in Subsection 3.1 "The family of fuzzy implications of copula" and Proposition 3.2) the fact that all fuzzy implications produced by equations (2.14), (2.15), and (2.16), that were set in the paper [17] are nothing but different forms of equation $I(x,y) = 1 - Q(x, 1-y)$. The following proposition follows similar steps and proofs as in Proposition 3.2.

Proposition 3.11.

(i) *For each copula* $Q \in \mathcal{Q}$ *that produces the fuzzy implication* $I_Q(x,y) = 1 - x + Q(x,y)$, *there is a unique* Q_* *that produces the fuzzy implication* $I_{Q_*}(x,y) = 1 - Q_*(x, 1-y)$, *such that,* $I_Q(x,y) = I_{Q_*}(x,y)$.

(ii) *For each copula* $Q \in \mathcal{Q}$ *and fuzzy implication* $Q(x,y) = 1 - x + Q(x,y)$, *there is a unique* Q^* *that produces the fuzzy implication* $I_{Q^*}(x,y) = y + 1 - x - Q^*(1-x, y)$, *such that,* $I_Q(x,y) = I_{Q^*}(x,y)$.

(iii) *For each copula* $Q \in \mathcal{Q}$ *and fuzzy implication* $I_Q(x,y) = 1 - x + Q(x,y)$, *there is a unique* $\hat{Q}$ *that produces the fuzzy implication* $I_{\hat{Q}}(x,y) = y + \hat{Q}(1-x, 1-y)$, *such that,* $I_Q(x,y) = I_{\hat{Q}}(x,y)$.

In brief, the implication with formation $I(x, y) = 1 - Q(x, 1 - y)$ with its equivalent expressions (2.11), (2.12), and (2.13) expresses the total of fuzzy implications and of only those produced by quasi-copula $Q \in \mathcal{Q}$. The family of all fuzzy implications produced via quasi-copulas will be denoted by $\mathcal{I}^{\mathcal{Q}}$. The space of all fuzzy implications which is 1-Lipschitz (let's denote it by $\mathcal{I}^{1-\mathrm{Lip}}$) is identical with the space of all fuzzy implications produced via quasi-copulas, i.e., $\mathcal{I}^{\mathcal{Q}} = \mathcal{I}^{1-\mathrm{Lip}}$. Finally, we have $I^{C} \subset I^{Q} = I^{1-\mathrm{Lip}}$, a good picture of the above families is provided in Figure 2.

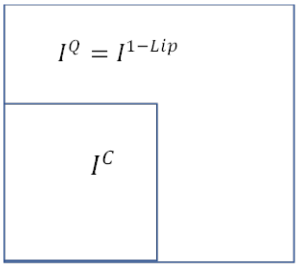

Figure 2: Relations between families $\mathcal{I}^{\mathcal{C}}$, $\mathcal{I}^{\mathcal{Q}}$, $\mathcal{I}^{1-\mathrm{Lip}}$.

Remark 3.12. Due to the affinity of copula and quasi-copula, the properties of Lemmas 3.5, 3.6, and 3.7 are valid under the same conditions in the family of fuzzy implications $\mathcal{I}^{\mathcal{Q}}$ as well, that is,

(i) each $I \in \mathcal{I}^{\mathcal{Q}}$ satisfies the neutrality principle (NP);

(ii) the only fuzzy implication $I \in \mathcal{I}^{\mathcal{Q}}$ which satisfies the identity principle (IP) is the Łukasiewicz implication;

(iii) the only fuzzy implication $I \in \mathcal{I}^{\mathcal{Q}}$ which satisfies the ordering principle (OP) is the Łukasiewicz implication;

(iv) if $I \in \mathcal{I}^{\mathcal{Q}}$, based on quasi-copula Q, where Q is symmetric and associative, then I satisfies exchange principle (EP), i.e., $I(x, I(y, z)) = I(y, I(x, z)),\ \ y, x, z \in [0, 1]$. Here we must stress that each associative quasi-copula Q is also an associative copula (see [11]), therefore Lemma 3.8 applies.

The next proposition declares that the fuzzy implications that belong in family $\mathcal{I}^{\mathcal{Q}}$ and consequently $\mathcal{I}^{\mathcal{C}}$ fully satisfy the laws of contrapositive with respect to classical negation $N_C(x) = 1 - x$ under one condition, the quasi-copulas that produce implication I are symmetric.

Proposition 3.13. *If $I \in \mathcal{I}^{\mathcal{Q}}$ and consequently $I \in \mathcal{I}^{\mathcal{C}}$ and Q and C that produce implication I are symmetric, then*

(i) *I satisfies (CP) with respect to $N_C(x) = 1 - x$;*

(ii) *I satisfies (L-CP) with respect to $N_C(x) = 1 - x$;*

(iii) *I satisfies (R-CP) with respect to $N_C(x) = 1 - x$.*

Proof.

(i) $I(N_C(y), N_C(x)) = 1 - Q(N_C(y), 1 - N_C(x)) = 1 - Q(1 - y, 1 - (1 - x)) = 1 - Q(1 - y, x) = 1 - Q(x, 1 - y) = I(x, y)$.

(ii) $I(N_C(x), y) = 1 - Q(1 - x, 1 - y) = 1 - Q(1 - y, 1 - x) = I(N_C(y), x)$.

(iii) $I(x, N_C(y)) = 1 - Q(x, 1 - (1 - y)) = 1 - Q(x, y) = 1 - Q(y, x) = 1 - Q(y, 1 - (1 - x)) = 1 - Q(y, 1 - N_C(x)) = I(y, N_C(x))$. □

4. Conclusion

The results of this paper give a very promising connection between the theory of fuzzy implications and the theory of probabilities. It may also lead to new avenues for approximate reasoning and decision theories. The connection of these two different worlds is achieved with the help of the links of aggregation functions contained in copula, semi-copula and quasi-copula, which for a strange reason also connect different mathematical theories. Through this connection we may come quite close to understanding the decision-making that dominates our daily lives in common experience. This work constitutes the beginning for the continuation of the attempt to connect the stochastic process with the fuzzy implication by studying these implications produced by aggregation functions under the influence of fuzzy negations other than classical negation $(N(x) = 1 - x)$.

References

[1] C. Alsina, R. B. Nelsen, B. Schweizer, *On the characterization of a class of binary operations on distribution functions*, Statist. Probab. Lett., **17** (1993), 85–89. 2.10

[2] M. Baczyński, P. Grzegorzewski, R. Mesiar, P. Helbin, W. Niemyska, *Fuzzy implications based on semicopulas*, Fuzzy Sets and Systems, **323** (2017), 138–151. 2.12, 2.2, 2.21, 2.4, 3.2, 3.10

[3] M. Baczyński, B. Jayaram, *Fuzzy implications*, Springer, Berlin, (2008). 2.22

[4] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, *Aggregation operators: properties, classes and construction methods*, In: Aggregation operators, Stud. Fuzziness Soft Comput., **97** (2002), 3–104. 2.1

[5] M. Dibala, S. Saminger-Platz, R. Mesiar, E. P. Klement, *Defects and transformations of quasi-copulas*, Kybernetika, **52** (2016), 848–865. 2.13

[6] F. Durante, J. Quesada-Molina, C. Sempi, *Semicopulas: characterizations and applicability*, Kybernetika, **42** (2006), 287–302. 2.14

[7] F. Durante, C. Sempi, *Semicopulæ*, Kybernetika, **41** (2005), 315–328. 2.14

[8] J. C. Fodor, M. Roubens, *Fuzzy preference modelling and multicriteria decision support*, Kluwer Academic Publishers, Dordrecht, (1994). 2.3

[9] F. S. García, P. G. Álvarez, *Two families of fuzzy integrals*, Fuzzy Sets and Systems, **18** (1986), 67–81. 2.2

[10] C. Genest, J. J. Q. Molina, J. A. R. Lallena, C. Sempi, *A characterization of quasi-copulas*, J. Multivariate Anal., **69** (1999), 193–205. 2.11

[11] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, Cambridge University Press, Cambridge, (2009). 2.1, 2.2, 2.3, 2.11, 2.12, d, 2.2, 3.6, iv

[12] P. Grzegorzewski, *Probabilistic implications*, Fuzzy Sets and Systems, **226** (2013), 53–66. 2.2, 2.4, 3.1

[13] J. Kalická, *On some construction methods for 1-Lipschitz aggregation functions*, Fuzzy sets and Systems, **160** (2009), 726–732. 2.1, 2.4, 2.5, 2.6, 2.7, 2.13, 2

[14] L. Kitainik, *Fuzzy decision procedures with binary relations*, Kluwer Academic Publishers, Boston, (1993). 2.3

[15] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms: Basic notions and properties*, In: Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms, Elsevier B. V., Amsterdam, (2005), 17–60. 2.20

[16] A. Kolesárová, *1-Lipschitz aggregation operators and quasi-copulas*, Kybernetika, **39** (2003), 615–629. 2.4

[17] R. Mesiar, A. Kolesárová, *Quasi-copulas, copulas and fuzzy implicators*, Int. J. Comput. Intell. Syst., **13** (2020), 681–689. 2.4, 2.24, 2.25, 3.2

[18] R. B. Nelsen, *An introduction to copulas*, Springer, New York, (2006). 2.2, 2.8, 2.9, 2.2

[19] M. Šabo, P. Strežo, *On reverses of some binary operators*, Kybernetika, **41** (2005), 435–450. 2.5, 2.2, 2.14, 2.15, 2.16

[20] G. Souliotis, B. Papadopoulos, *An algorithm for producing fuzzy negations via conical sections*, Algorithms, **12** (2019), 8 pages. 2.21