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Effect of an amplitude modulated force on vibrational resonance, chaos, and multistability in a modified Van der Pol-Duffing oscillator

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Abstract

This paper deals with the effects of an amplitude modulated (AM) excitation on the nonlinear dynamics of reactions between four molecules. The computation of the fixed points of the autonomous nonlinear chemical system has been made in detail using Cardan's method. Routes to chaos have been investigated through bifurcations structures, Lyapunov exponent and phase portraits. The effects of the control force on chaotic motions have been strongly analyzed and the control efficiency is found in the cases $g = 0$ (unmodulated case), $g \neq 0$ with $\Omega = n\omega$; n a natural number and $\frac{\Omega}{\omega} \neq \frac{p}{q}$; p and q are simple positive integers. Vibrational Resonance (VR), hysteresis and coexistence of several attractors have been studied in details based on the relationship between the frequencies of the AM force. Results of analytical investigations are validated and complemented by numerical simulations.

Keywords: Modified Van der Pol oscillator, Cardan's method, chaos, coexisting attractors, vibrational resonance.

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1. Introduction

Several natural phenomena observed in fields such as mathematics, chemistry, biology, physics, engineering are non-linear[28]. The dynamics of nonlinear systems can be very complicated and often counter-intuitive, and generally are controlled by fairly simple deterministic or stochastic laws. Nonlinearity, inherent in most systems, can appear in many forms, such as physical, structural, frictional or geometric forces and external forces in many contexts [24]. This nonlinearity is the basis of the complex situations or phenomena traversed by systems, thus making their operation more profitable or not. For example, the intense study of nonlinear dynamics in oscillating chemical reactions under the influence of external disturbances various results revealed that these chemical reactions are the seat of interesting phenomena such as roads to chaos, the coexistence of attractors, hysteresis, vibrational resonance, etc [1–5, 7, 8, 10, 13–15, 22, 24–26, 29]. Indeed, the oscillating chemical reactions in a continuous stirred tank

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reactor (CSTR) are one of the first biochemical oscillations discovered. Thus, the known chemical oscillators were either from biological origin, such as the glycolytic and oxidase-peroxidase systems; either discovered accidentally, such as the reactions of Bray and Belousov-Zhabotinsky (BZ); or variants of these reactions [1, 2, 4, 5, 8, 10, 15, 22, 24–26]. In most studies performed on chemical oscillations, one of the main challenges is to predict and control nonlinear dynamics for potential applications [3, 7, 13, 14, 29].

From the analysis of the research cited above, it follows that the study of these oscillations is more advantageous for systems with AME (amplitude modulation excitation) because some new dynamic phenomena, including controllable frequency, may also be exhibited [13]. For example, Blekhman and Landa [2] studied for a Duffing oscillator the resonances caused by a biharmonic external force with two different frequencies (called vibrational resonances). They showed that when the oscillator is weakly damped, these resonances are conjugate and occur when the amplitude of the high frequency excitation varies for appropriate low and high frequency conditions. Furthermore, Roy-Layinde et al. [24] have identified the origin of Vibrational Resonance (VR) in the plasma model under the influence of a biharmonic force taking into account the effective potential of the plasma and also contributions from the effective nonlinear dissipation. From the numerical simulations, they found several dynamical changes including symmetry-breaking bifurcations, attractor leaks and period-reversed bifurcations; also they found single and double resonances induced by symmetry-breaking bifurcations. Landa and McClintock [10] have investigated the effect of a high frequency force on the response of a bistable system to a low frequency signal for overdamped and weakly damped cases. They showed that the response can be optimized by an appropriate choice of vibration amplitude. Other equally important works have proven the conditions for obtaining VR as well as its applications [8, 19, 27].

Finally, another complex phenomenon encountered in nonlinear systems and which makes them difficult to contain is multistability. Indeed, for the same value of a parameter of the system for which multistability appears, the system is in several states or at several vibration amplitudes, thus making it difficult to control the system. On the other hand, megastability designates the coexistence of an infinite number of attractors while bistability translates the coexistence of two attractors for the same system [9, 11, 12, 16, 20, 21, 23, 30]. Therefore, due to the complexity of multistable systems many researchers work hard to predict and control multistability. This is one of the proofs of the many recent works studied during this decade on this rather interesting subject [9, 11, 12, 16, 20, 21, 23, 30]. In these different works, several techniques are used to research, analyze and control the coexistence of attractors and very conclusive results are obtained. Most of these works one can notice that a great flexibility in the performance of the system is made possible by the coexistence of various stable states without major changes of parameters.

In the present paper, it is question to seek hysteresis, vibrational resonance and chaos in the system of reactions between four molecules when it is subjected to an external amplitude modulation excitation. More precisely, after an in-depth analysis of the fixed points for the autonomous system, the effects of the modulated amplitude force on the dynamics of the chemical reaction considered have been studied in detail and the efficiency of the control force analyzed. By assuming that the model is influenced by an external sinusoidal excitation $(f + 2g \cos \Omega t) \sin \omega t$, Eq. (1.1) becomes a nonlinear single second order differential equation on the form

$$\ddot{x} + \mu \left(1 - x^2\right) \dot{x} + \alpha x + \gamma x^3 + \beta = (f + 2g \cos \Omega t) \sin \omega t. \tag{1.1}$$

Eq. (1.1) is a forced modified Van der Pol-Duffing oscillator equation. For the particular case where the constraint parameter term $\beta = 0$, Eq. (1.1) is reduced to the Van der Pol-Duffing equation treated extensively by many researchers. Also, Eq. (1.1) is used to model the nonlinear dynamics of chemical reactions without amplitude modulated excitation (see [15, 22]).

The paper is structured as follows. Section 2 analyses the fixed points and their stability. In Section 3, the vibrational resonance, bifurcation, route to chaos, bistability, coexistence of attractors and hysteresis are analyzed in a depth detail. The effect of AM excitation is studied when $\Omega = \omega$, $\Omega \gg \omega$ and $\frac{\Omega}{\omega} \neq \frac{p}{q}$, where p and q are simple positive integers. Finally the conclusion of the research is given in Section 4.

2. Equilibrium points and its stability

In this section, we determine the fixed points of the autonomous Van der Pol-Duffing oscillator and we analyze their stability. In this order, we write the equation of autonomous system as follow

$$\dot{x} = y, \quad \dot{y} = -\mu(1-x^2)y - \alpha x - \gamma x^3 - \beta.$$

All equilibrium points $E_*(x_*, y_*)$ of the autonomous system verify:

$$y_* = 0 \quad \text{and} \quad \gamma x_*^3 + \alpha x_* + \beta = 0. \tag{2.1}$$

Using the Cardan method [17, 22] to solve Eq. (2.1), we rewrite this equation in the form:

$$x_*^3 + a x_* + b = 0,$$

where $a = \frac{\alpha}{\gamma}$ and $b = \frac{\beta}{\gamma}$. The associate characteristic equation is

$$T^2 + bT - \frac{a^3}{27} = 0. \tag{2.2}$$

The equilibrium points of system depend on the parameters of the system and therefore the sign of η, with

$$\eta = 27\Delta = 4a^3 + 27b^2,$$

where Δ represents the determinant of Eq. (2.2). The values of α and γ seriously influence the sign of η because the parameter $\gamma > 0$ for the system in consideration.

Theorem 2.1. *The autonomous system has one fixed point $E_*(x_*, 0)$ with*

$$x_* = (-\frac{b}{2} - (\frac{b^2}{4} + \frac{a^3}{27})^{\frac{1}{2}})^{1/3} + (-\frac{b}{2} + (\frac{b^2}{4} + \frac{a^3}{27})^{\frac{1}{2}})^{1/3}.$$

This equilibrium point $E_*(x_*, 0)$ is semi-stable if $\beta > \alpha + \gamma$ while it is stable if $\beta < \alpha + \gamma$.

Proof. The real system modeled by the Van der Pol Duffing equation studied here is a chemical system and the parameters α and γ are strictly positive (see [15, 22]). Thus, in this case, the parameter $\eta > 0$ and the autonomous system has one equilibrium point $E_*(x_*, 0)$ with

$$x_* = (-\frac{b}{2} - (\frac{b^2}{4} + \frac{a^3}{27})^{\frac{1}{2}})^{1/3} + (-\frac{b}{2} + (\frac{b^2}{4} + \frac{a^3}{27})^{\frac{1}{2}})^{1/3}.$$

The eigenvalues of the corresponding Jacobian matrix at the equilibrium point $E_*(x_*, 0)$ are determined by solving

$$\lambda^2 + \lambda\sigma_1 + \sigma_2 = 0,$$

with

$$\sigma_1 = \mu - \mu x_*^2, \quad \sigma_2 = \alpha + 3\gamma x_*^2.$$

According to the Routh-Hurwitz criterion, $E_*(x_*, 0)$ is stable if and only if $\sigma_1 > 0$ and $\sigma_2 > 0$. However, for $\mu > 0, \alpha > 0$, and $\gamma > 0$, $\sigma_2 > 0$ and the stability of the fixed point depends on the sign of $\sigma_1 > 0$. Indeed, let's study the sign of σ_1. It is easy to show that if $\beta < \alpha + \gamma$, $\sigma_1 > 0$, and $E_*(x_*, 0)$ is stable and if $\beta > \alpha + \gamma$, $\sigma_1 < 0$, and $E_*(x_*, 0)$ is semi-stable. □

3. Effect of AM force on the system

3.1. Vibrational resonance and amplitude response

In a nonlinear dynamical system driven by a biharmonic force consisting of a low and high-frequencies ω and Ω with $\Omega \gg \omega$, when the amplitude g of the high frequency force is varied, the amplitude response at the low frequency ω exhibits a resonance so-called vibrational resonance [2, 5, 8, 10, 24]. In other words, vibrational resonance (VR) is a phenomenon wherein the response of a nonlinear oscillator driven by biharmonic forces with two different frequencies, and, such that $\Omega \gg \omega$, is enhanced by optimizing the parameters of high-frequency driving force. To determine the VR, we use the amplitude of the response at the frequency ω of the signal. Indeed, using the fourth order Runge-Kutta algorithm, with time step size $\frac{T}{1000}$, we numerically integrate the system (1.1) of the chemical reaction studied. Thus, the numerical solution $x(\tau)$ allows to calculate the amplitude response Q through the following formula

$$Q = \frac{\sqrt{Q_s^2 + Q_c^2}}{f},$$

where

$$Q_s = \frac{2}{n\pi}\int_0^{nT} x(\tau)\sin\omega\tau d\tau, \quad Q_c = \frac{2}{n\pi}\int_0^{nT} x(\tau)\cos\omega\tau d\tau,$$

with $T = \frac{2\pi}{\omega}$ the response period and $n = 500$. We compute Q first in the case of a low frequency force only, then in the case of a high frequency force only and finally in the case of the two forces. For $f = 0$, Q is determined as $Q = \sqrt{Q_s^2 + Q_c^2}$, Q_s and Q_c representing the Fourier coefficients of the output signal at the frequency $\frac{2\pi}{T}$ and Q the amplitude of the response to this same frequency. In general vibration resonance is observed by tuning the fast forcing g, so that the effective modified natural frequency, which should be a function of α, g, γ, β, resonates to the slow frequency ω. It is therefore for reason that the effects of each of these parameters are analyzed numerically. Thus, the results obtained represented on the Figures 1 and 2, respectively, in the planes (Q, g) and (Q, f), show the effects of the parameters of the systems on the vibrational resonance. From the analysis of these figures, it appears that the parameters β, γ as well as the parameters of the modulated amplitude force strongly influence the resonance amplitude and the values of the resonance parameters. We also note that the parameters β and ω influence the multiple resonance obtained.

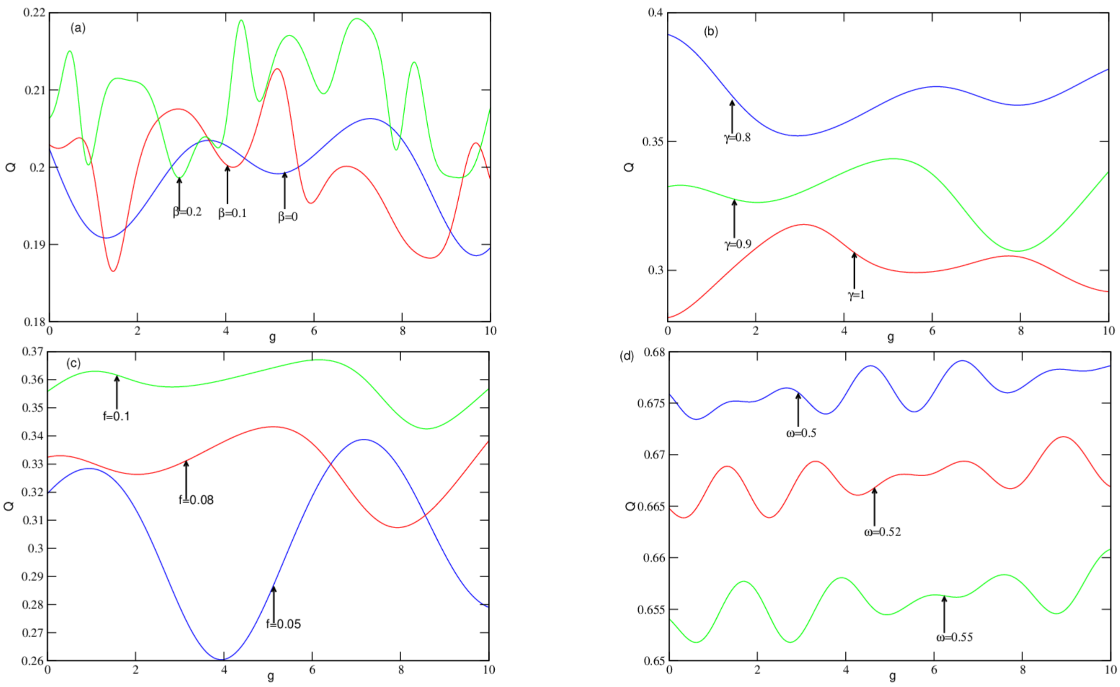

Figure 1: Vibrational resonance in (Q, g) for $\alpha = 1, \mu = 0.0001; \omega = 1.0; \Omega = 50 * w$. (a): effect of constraint parameter β with $\gamma = 1.8$ and $f = 0.08$; (b): effect of γ with $\beta = 0.1$ and $f = 0.08$; (c): effect of f with $\gamma = 0.9$ and $\beta = 0.1$; (d): effect of ω with $f = 0.08, \gamma = 0.9$, and $\beta = 0.1$.

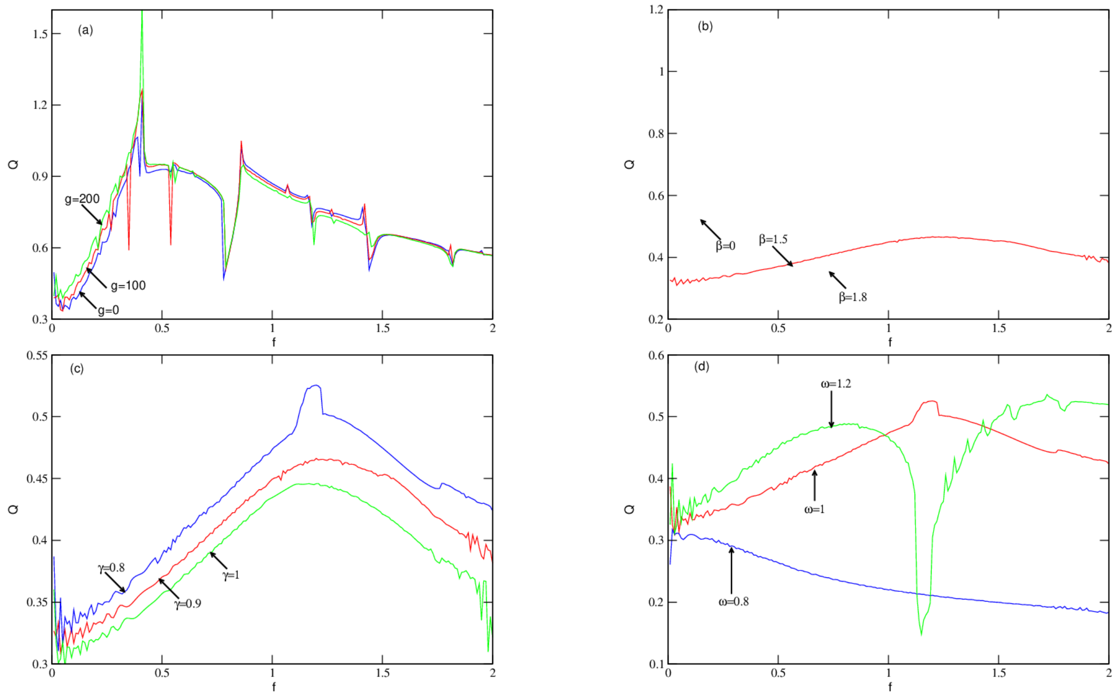

Figure 2: Vibrational resonance in (Q, f) for $\alpha = 1, \mu = 0.0001; \omega = 1.0; \Omega = 50 * w$. (a): effect of g with $\gamma = 0.9$ and $\beta = 0.2$; (b): effect of constraint parameter β with $\gamma = 0.9$ and $g = 5$; (c): effect of γ with $\beta = 1.5$ and $g = 5$ (d): effect of ω with $g = 5, \gamma = 0.8$, and $\beta = 1.5$.

3.2. *Hysteresis, coexistence of attractors, and multistability*

The modified Van der Pol-Duffing oscillator which modeled the a the nonlinear chemical reaction [15, 22] can exhibit the complex behaviors and that's why this section is dedicated to effects of an amplitude modulated excitation on the route to chaos and coexistence of attractors. Indeed, using the fourth order Runge-Kutta integration algorithm, we solve numerically Eq. (1.1) of a modified Van der Pol-Duffing oscillator and bifurcation diagrams, Lyapunov exponents and phase portraits are plotted. To search for the coexistence of attractors, we have studied the dynamics of the oscillator by varying the bifurcation parameter in the same interval in the increasing direction (blue curve) and in the decreasing direction (red curve) for fixed values of the others system parameters but with two different initial conditions. Thus, by superimposing the two curves (blue and red) in the same graph, if the system has the same dynamics with the same amplitude for a given value of the bifurcation parameter, then the oscillator does not present the phenomenon of coexistence of attractors. But if the dynamics of the oscillator have the same nature or have the same nature with different amplitudes, then there is coexistence of attractors. We say that there is coexistence of attractors of the same nature if the system has the same dynamics but with different amplitudes otherwise we say that attractors of different natures coexist. It is necessary to note that the Lyapunov exponent of stable periodic solution is less than zero while the Lyapunov exponent of chaos is greater than zero and Lyapunov exponent equal to zero means that bifurcation occurs, corresponding to a quasi-periodic dynamic [31].

Figure 3 represents the bifurcation diagram and its corresponding Lyapunov exponents versus the parameter f for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.6, g = 0, \omega = 1$. This figure shows the dynamics and the coexistence of the attractors of the modified Van der Pol oscillator studied in the absence of modulation of the external excitation. It is noted that this oscillator presents a varied dynamics and of coexistence of attractors of varied natures. Indeed, for $0 \leqslant f \leqslant 3.5$, the studied oscillator can have a mono-periodic, double-periodic, multi-periodic, and chaotic dynamics. We observe precisely that for $0.602317 \leqslant f < 0.723126$, attractors of period 1T and 3T coexist; for $1.53427 \leqslant f < 1.7802$, attractors of period 2T coexist; for $3.03143 \leqslant f < 3.29462$, chaotic attractors coexist, and when $3.29462 \leqslant f \leqslant 3.5$, chaotic attractors and period attractors 1T coexist. These different results are confirmed by the phase portraits of Figure 4. Now we are looking for the effect of amplitude modulation on the different dynamics and the coexistence of attractors. Indeed, for the degree of modulation $g = 5$ and $\Omega = 80\omega$ the bifurcation diagram and its corresponding Lyapunov exponents (Figure 5) reveal that the domains of the coexistence attractors, of period 1T and 3T, of period 2T have become domains of coexistence of attractors, of period 2T and 4T; the domains of coexistence of chaotic attractors and attractors of period 1T are preserved. Figure 6 represents for $\gamma = 3.6, \mu = 0.1\alpha = 1, \beta = 0.6, f = 3.4, \omega = 1, \Omega = 80$, the domain of the degree of modulation g, where attractors coexist and shows that for $0 \leqslant g \leqslant 10$, chaotic attractors and attractors of period 1T coexist. To verify the predictions of Figure 6, the phase portraits of appropriate values of g are plotted in Figure 7. This confirmes the coexistence of chaotic attractors and attractors of period 1T. For $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.2$, we study the effect of amplitude modulated excitation on the dynamics and the coexistence of the attractors of the modified Van der Pol oscillator. Indeed, Figure 8 shows the influence of excitation frequencies on these complex phenomena when $\Omega = n\omega$ with n a natural integer. Thus, for $f = 3.4$, we note that the domain of coexistence of chaotic attractors with periodic or multi-periodic attractors is greater for $n > 1$, than when $n = 1$, i.e., for $\Omega > \omega$ only for $\Omega = \omega$. Figure 9 also shows the effect of excitation frequencies on these complex phenomena when $\frac{\Omega}{\omega} \neq \frac{p}{q}$; p and q are simple positive integers. We observe through this figure that the domain of the coexistence of chaotic and periodic attractors are considerably reduced and we also note the coexistence of quasiperiodic and chaotic attractors. Finally, Figure 10 represents the amplitude effect f of the periodic excitation for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.2, \omega = \Omega = 1$. It is noted that the chaotic dynamic is considerably eliminated and the multiperiod dynamic is more abundant. Thus, the domain of the coexistence of quasi-periodic attractors is larger than the domain of the coexistence of other attractors.

From all these important results obtained, it can be concluded that the amplitude modulated excitation greatly influences the dynamics of the modified Van der Pol oscillator as well as the coexistence of attractors and their nature.

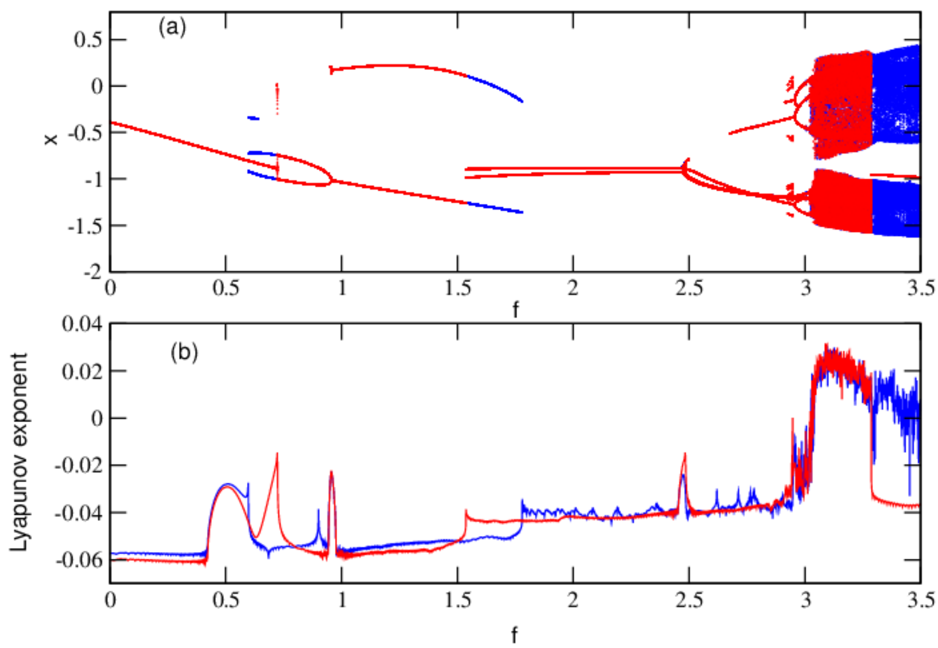

Figure 3: Bifurcation diagram and its corresponding Lyapunov exponents versus the parameter f for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.6, g = 0, \omega = 1$. Bifurcation diagrams are obtained by scanning the parameter f upwards (blue) and downwards (red).

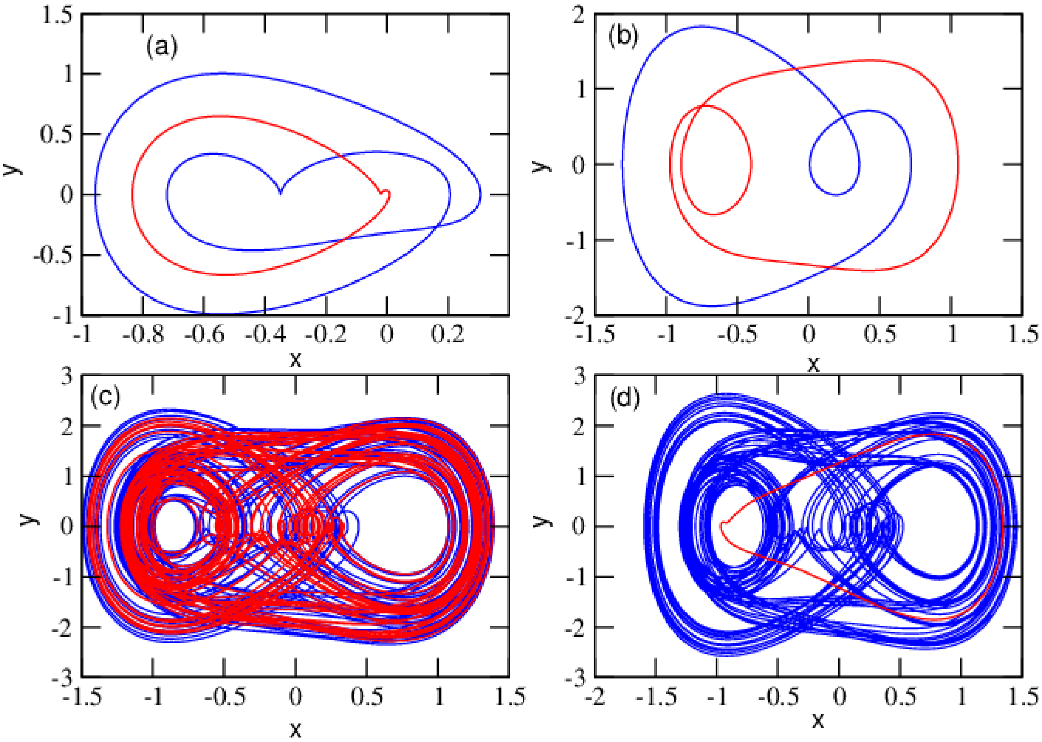

Figure 4: Phase portrait of coexistence of attractors with (a) $f = 0.64$; (b) $f = 1.65$; (c) $f = 3.1$; (d) $f = 3.4$; and other parameters of Figure 3. Blues curves correspond to initial condition $(x_0 = 1, y_0 = 0.8)$ and red curves correspond to initial condition $(x_0 = -0.2, y_0 = 0.5)$

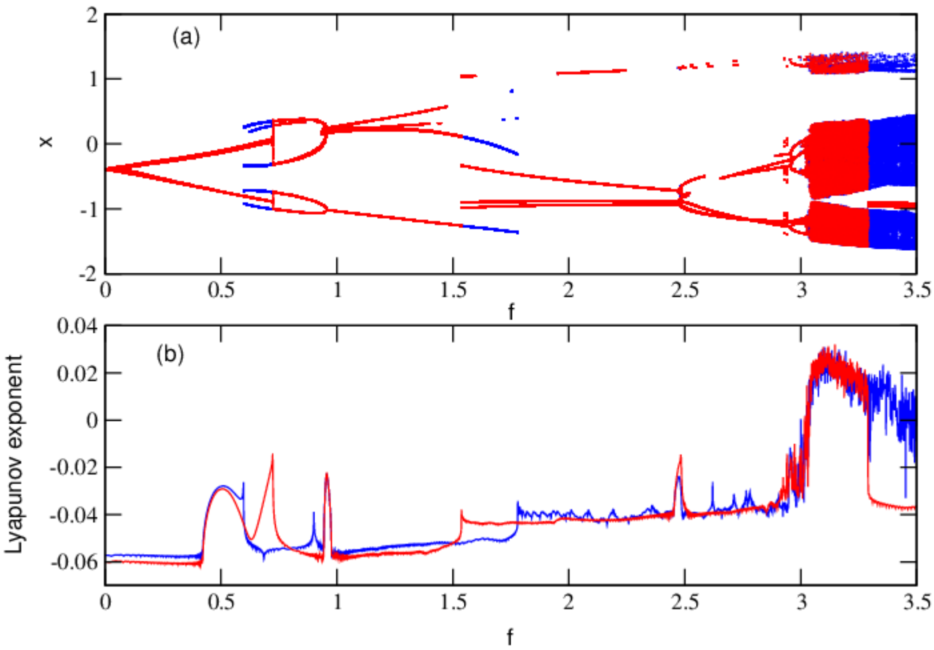

Figure 5: Bifurcation diagram and its corresponding Lyapunov exponents versus the parameter f for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.6, g = 5, \omega = 1, \Omega = 80$. Bifurcation diagrams are obtained by scanning the parameter f upwards (blue) and downwards (red).

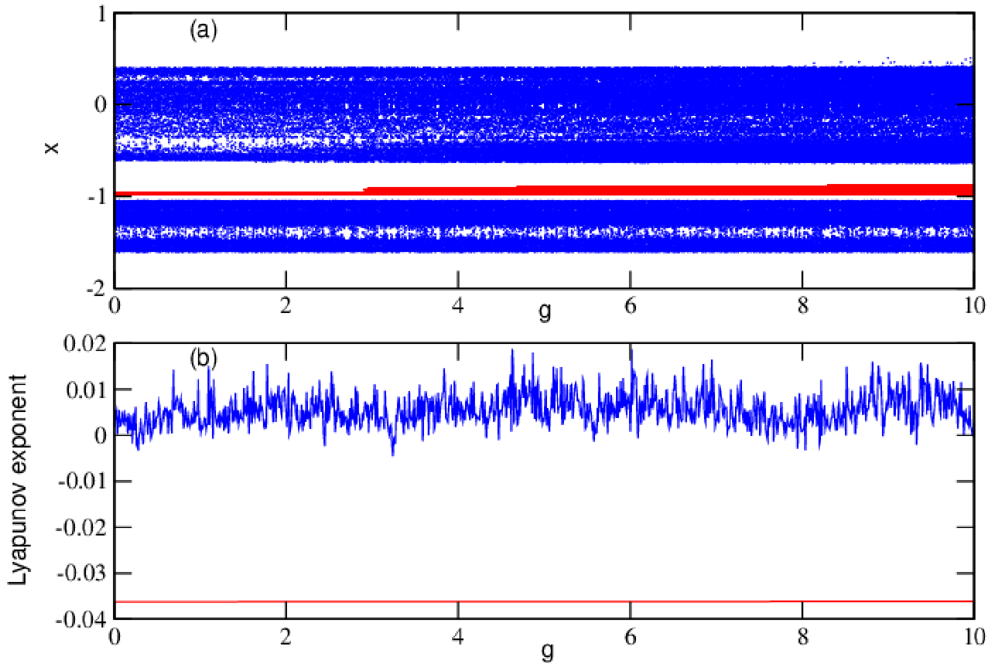

Figure 6: Bifurcation diagram and its corresponding Lyapunov exponents versus the parameter g for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.6, f = 3.4, \omega = 1, \Omega = 80$. Bifurcation diagrams are obtained by scanning the parameter g upwards (blue) and downwards (red).

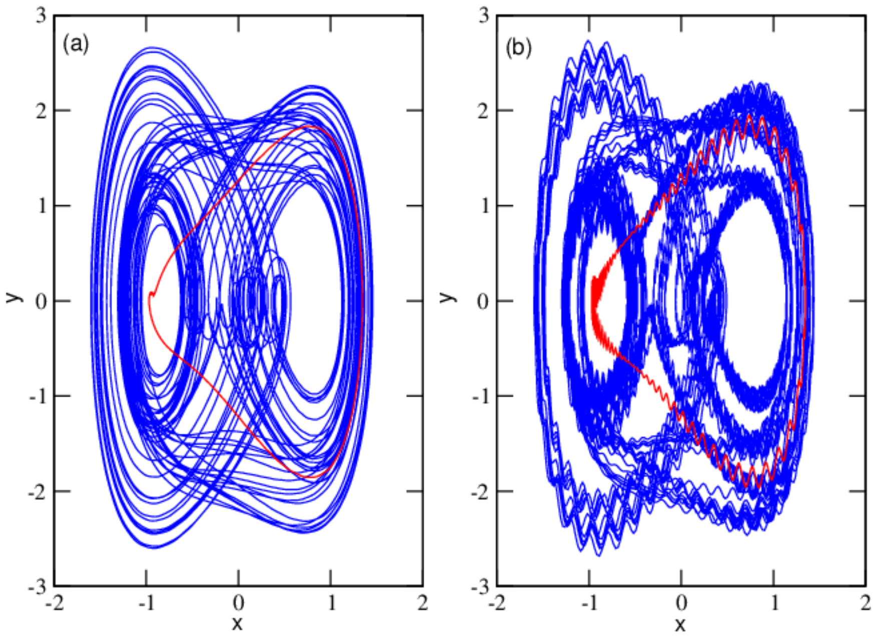

Figure 7: Phase portrait of coexistence of attractors with (a) $g = 0.1$; (b) $g = 5$, and other parameters of Figure 6. Blues curves correspond to initial condition $(x_0 = 1, y_0 = 0.8)$ and red curves correspond to initial condition $(x_0 = -0.2, y_0 = 0.5)$.

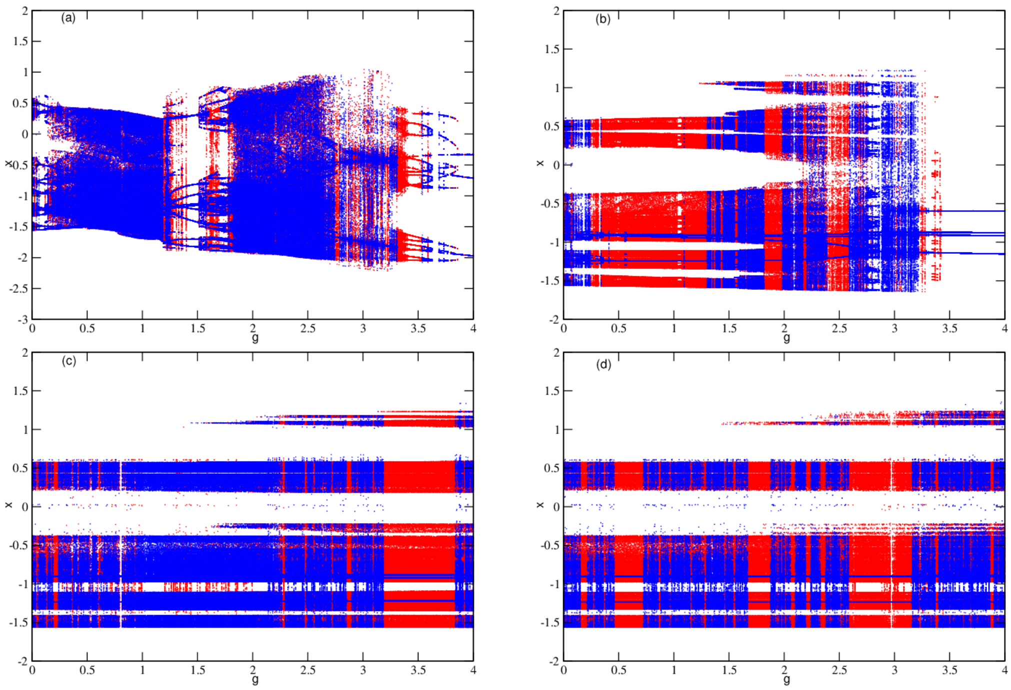

Figure 8: Effect of amplitude modulated excitation on dynamics of the system for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.2, f = 3.4, \omega = 1$. (a) $\Omega = \omega$; (b) $\Omega = 10\omega$; (c) $\Omega = 20\omega$; and (d) $\Omega = 80\omega$. Bifurcation diagrams are obtained by scanning the parameter g upwards (blue) and downwards (red).

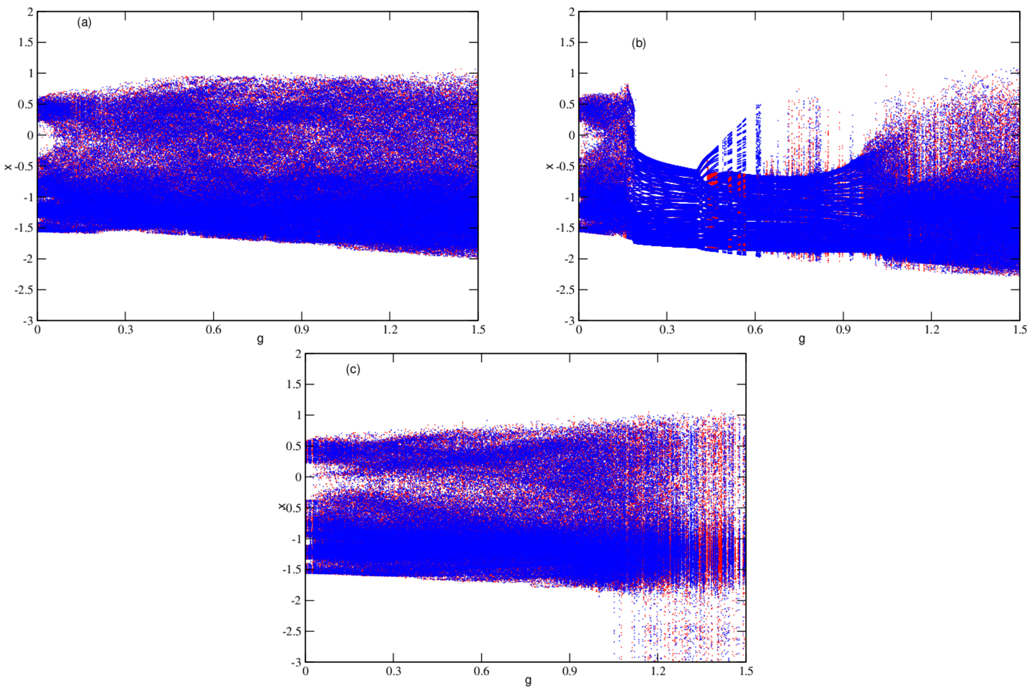

Figure 9: Effect of amplitude modulated excitation on dynamics of the system for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.2, f = 3.4, \omega = 1$. (a) $\Omega/\omega = \sqrt{2}/2$; (b) $\Omega/\omega = \sqrt{2}$; and (c) $\Omega/\omega = \sqrt{5}$. Bifurcation diagrams are obtained by scanning the parameter g upwards (blue) and downwards (red).

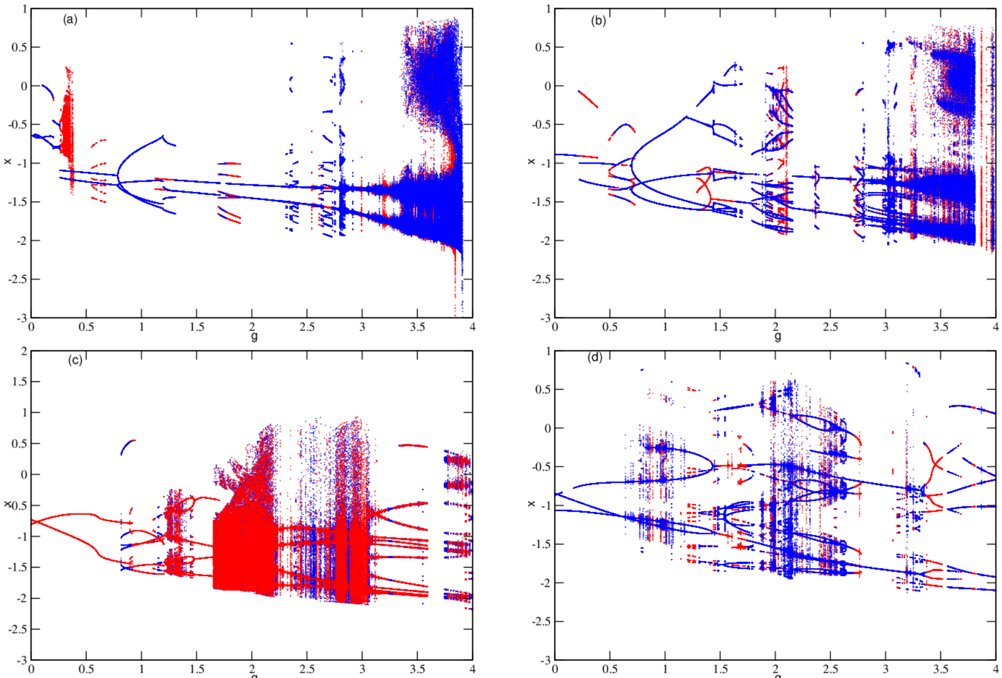

Figure 10: Effect of amplitude modulated excitation on dynamics of the system for $\gamma = 3.6, \mu = 0.1, \alpha = 1, \beta = 0.2, \omega = \Omega = 1$. (a) $f = 0.4$; (b) $f = 1$; (c) $f = 2$; and (d) $f = 3$. Bifurcation diagrams are obtained by scanning the parameter g upwards (blue) and downwards (red).

4. Conclusions

In this work, we studied the influence of an amplitude modulated excitation on the complex dynamics of a modified Van der Pol oscillator which models the complex oscillations of certain chemical reactions. Emphasis was placed on Hopf bifurcation, vibrational resonance, chaos, and the coexistence of attractors. After having sought the fixed points of the autonomous system by the method of Cardan, the stability of these points is analyzed by the use of the criterion of Routh-Hurwitz. Then, the vibrational resonance, which is also a very complex phenomenon for dynamical systems is searched numerically using the Runge-Kutta algorithm of order 4. It is obtained the amplitudes and the resonance frequencies showing the contribution of the force to modulated amplitude when the frequency of the modulated part is very high compared to the periodic excitation frequency. It appears from these results that the parameters β, γ as well as the parameters of the modulated amplitude force strongly affect the resonance amplitude and the values of the resonance parameters. We have also noticed that the parameters β and ω influence the multiple resonance obtained. To study in this work the effects of the amplitude modulated force on the routes to chaos and the coexistence of attractors, we used the Runge-Kutta algorithm of order 4 to numerically solve the equation of the modified oscillator by Van der Pol-Duffing. It follows that through the bifurcation diagrams, the Lyapunov exponents and the phase portraits obtained that the amplitude modulated excitation strongly influences the dynamics and the coexistence of the attractors. Indeed, it is obtained that the presence of the modulated amplitude force promotes chaotic dynamics and the coexistence of attractors of the same nature or of different nature for well-defined values of the excitation frequencies, the degree of amplitude modulation and the amplitude of the periodic excitation. The different results prove the importance of this study because it situates the researchers on the domains of values of each parameter of the system that must be considered to have a given behavior or a precise phenomenon. For example, in chemistry or biochemistry where the nonlinear dynamics of chemical reactions can be modeled by the modified Van der Pol-Duffing oscillator studied, a good choice of the values of the parameters of the chemical reaction and of the external excitation could make it possible to control the concentrations of the chemical species in reaction and to predict the yield of the reaction.

Precisely, in a chemical or biochemical reaction for which one wishes to know the yield, it will be necessary to choose precisely the values of the parameters in a domain where the dynamics are not chaotic and where attractors do not coexist. It will also be necessary to choose the values of the parameters at resonance if the aim is to have a high concentration of chemical species in reaction.

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References

[1] H. Binous, A. Bellagi, *Introducing nonlinear dynamics to chemical and biochemical engineering graduate students using MATHEMATICA©*, Comput. Appl. Eng. Educ., **27** (2018), 1–19. 1
[2] I. I. Blekhman, P. S. Landa, *Conjugate resonances and bifurcations in nonlinear systems under biharmonical excitation*, Int. J. Non-linear Mech., **39** (2004), 421–426. 1, 3.1
[3] H. G. Enjieu Kadji, B. R. Nana Nbendjo *Passive aerodynamics control of plasma instabilities*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 1779–1794. 1
[4] M. Gruebelle, P. G. Wolynes, *Vibrational Energy Flow and Chemical Reactions*, Acc. Chem. Res., **37** (2004), 261–267. 1
[5] S. Guruparan, D. N. B. Ravindran, V. Ravichandran, V. Chinnathambi, S. Rajasekar, *Hysteresis, Vibrational Resonance and Chaos in Brusselator Chemical System under the Excitation of Amplitude Modulated Force*, Chem. sci. rev. lett., **4** (2015), 870–879. 1, 3.1
[6] C. Hayashi, *Nonlinear Oscillations in Physical Systems*, McGraw-Hill Book Co., New York-Toronto-London, (1964).
[7] R. Imbihl, G. Ertl, *Oscillatory kinetics in heterogeneous catalysis*, Chem. Rev., **95** (1995), 697–733. 1
[8] C. Jeevarathinam, S. Rajasekar, M. A. F. Sanjúan, *Vibrational Resonance in the Duffing Oscillator with Distributed Time-Delayed Feedback*, J. Appl. Nonlinear Dyn., **4** (2015), 1–20. 1, 3.1
[9] I. G. Kevrekidis, L. D. Shmidt, R. Aris, *Some common features of periodically forced reacting systems*, Chem. Eng. Sci., **41** (1986), 1263–1276. 1
[10] P. S. Landa, P. V. E. McClintock, *Vibrational resonance*, J. Phys. A, **33** (2000), L433–L438. 1, 3.1
[11] G. D. Leutcho, S. Jafari, I. I. Hamarash, J. Kengne, Z. Tabekoueng Njitacke, I. Hussain, *A new megastable nonlinear oscillator with infinite attractors*, Chaos Solitons Fractals, **134** (2020), 7 pages. 1
[12] G. D. Leutcho, A. J. M. Khalaf, Z. Tabekoueng Njitacke, T. Fonzin Fozin, J. Kengne, S. Jafari, I. Hussain, *A new oscillator with mega-stability and its Hamilton energy: infinite coexisting hidden and self-excited attractors*, Chaos, **30** (2020), 8 pages. 1
[13] C. H. Miwadinou, A. V. Monwanou, L. A. Hinvi, J. B. Chabi Orou, *Effect of amplitude modulated signal on chaotic motions in a mixed Rayleigh-Liénard oscillator*, Chaos Solitons Fractals, **113** (2018), 89–101. 1
[14] C. H. Miwadinou, A. V. Monwanou, A. A. Koukpemedji, Y. J. F. Kpomahou, J. B. Chabi Orou, *Chaotic Motions in Forced Mixed Rayleigh-Liénard Oscillator with External and Parametric Periodic-Excitations*, Int. J. Bifurc. Chaos, **28** (2018), 1–16. 1
[15] C. H. Miwadinou, A. V. Monwanou, J. Yovogan, L. A. Hinvi, P. R. Nwagoum Tuwa, J. B. Chabi Orou, *Modeling nonlinear dissipative chemical dynamics by a forced modified Van der Pol-Duffing oscillator with asymmetric potential: Chaotic behaviors predictions*, Chin. J. Phys., **56** (2018), 1089–1104. 1, 1, 2, 3.2
[16] A. V. Monwanou, A. A. Koukpémèdji, C. Ainamon, P. R. Nwagoum Tuwa, C. H. Miwadinou, J. B. Chabi Orou, *Nonlinear Dynamics in a Chemical Reaction under an Amplitude-Modulated Excitation: Hysteresis, Vibrational Resonance, Multistability and Chaos*, Complexity, **2020** (2020), 1–16. 1
[17] T. R. Mukundan, *Solution of cubic equations: An alternative method*, Resonance, **15** (2010) 347–350. 2
[18] A. H. Nayfeh, *Introduction to perturbation techniques*, Wiley-Interscience [John Wiley & Sons], New York, (1981).
[19] G. Nicolis, I. Prigogine, *Self-organization in nonequilibrium systems* Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, (1977). 1
[20] Z. T. Njitacke, S. D. Isaac, T. Nestor, J. Kengne, *Window of multistability and its control in a simple 3D Hopfield neural network: application to biomedical image encryption*, Neural. Comput. Appl., **33** (2021), 6733–6752. 1
[21] Z. T. Njitacke, R. L. T. Mogue, G. D. Leutcho, T. Fonzin Fozin, J. Kengne, *Heterogeneous multistability in a novel system with purely nonlinear terms*, Int. J. Electron., **108** (2021), 1166–1182. 1
[22] D. L. Olabodé, C. H. Miwadinou, V. A. Monwanou, J. B. Chabi Orou, *Effects of passive hydrodynamics force on harmonic and chaotic oscillations in nonlinear chemical dynamics*, Phys. D, **386/387** (2019), 49–59. 1, 1, 2, 2, 3.2
[23] K. Rajagopal, S. T. Kingni, G. H. Kom, V.-T. Pham, A. Karthikeyan, S. Jafari, *Self-Excited and Hidden Attractors in a Simple Chaotic Jerk System and in Its Time-Delayed Form: Analysis, Electronic Implementation, and Synchronization*, J. Korean Phys. Soc., **77** (2020), 145–152. 1
[24] T. O. Roy-Layinde, J. A. Laoye, O. O. Popoola, U. E. Vincent, *Analysis of vibrational resonance in bi-harmonically driven plasma*, Chaos, **26** (2016), 1–9. 1, 3.1

[25] P. Sarkar, D. S. Ray, *Vibrational antiresonance in nonlinear coupled systems*, Phys. Rev. E, **99** (2019), 1–7.

[26] A. Shabunin, V. Astakhov, V. Demidov, A.Provata, F. Baras, G. Nicolis, V. Anishchenko, *Modeling chemical reactions by forced limit-cycle oscillator: synchronization phenomena and transition to chaos*, Chaos Solit. Fract., **15** (2003), 395–405. 1

[27] A. V. Shabunin, F. Baras, A. Provata, *Oscillatory reactive dynamics on surfaces: a lattice limit cycle model*, Phys. Rev. E, **66** (2002), 1–11. 1

[28] S. Strogatz, *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering, 2nd ed.*, CRC Press, Boca Raton, (2018). 1

[29] C. G. Takoudis, L. D. Schmidt, R. Aris, *Isothermal sustained oscillations in very simple surface reaction*, Surf. Sci, **105** (1981), 325–333. 1

[30] M. A. Taylor, I. G. Kevrekidis, *Some common dynamics features of coupled reacting systems*, Phys. D, **51** (1991), 274–292. 1

[31] C. Zhang, Q. Bi, X. Han, Z. Zhang, *On two-parameter bifurcation analysis of switched system composed of Duffing and van der Pol oscillators*, Commun. Nonlinear Sci. Numer. Simul., **19** (2014), 750–757. 3.2

[32] C. Zhang, X. Ma, Q. Bi, *Complex mixed-mode oscillations based on a modified Rayleigh-Duffing oscillator driven by low-frequency excitations*, Chaos Solitons Fractals, **160** (2022), 9 pages.

[33] C. Zhang, X. Ma, Q. Tang, Q. Bi, *Complex bifurcation structures and bursting oscillations of an extended Duffing-van der Pol oscillator*, Math. Methods Appl. Sci., (2022), 1–17.