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Coupling homogenization and large deviations, with applications to nonlocal parabolic partial differential equations

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Abstract

Consider the following nonlocal integro-differential operator of Lévy-type $\mathcal{L}^{\alpha}_{\varepsilon,\delta}$ given by

$$\mathcal{L}^{\alpha}_{\varepsilon,\delta} f(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x + \varepsilon\sigma\left(\frac{x}{\delta}, y\right)\right) - f(x) - \varepsilon\sigma^i\left(\frac{x}{\delta}, y\right)\partial_i f(x)\mathbf{1}_B(y)\right]\nu^{\alpha}_{\varepsilon}(dy) + \left[\left(\frac{\varepsilon}{\delta}\right)^{\alpha-1} b^i_0\left(\frac{x}{\delta}\right) + b^i_1\left(\frac{x}{\delta}\right)\right]\partial_i f(x),$$

related to stochastic differential equations driven by multiplicative isotropic α-stable Lévy noise ($1 < \alpha < 2$). We study by using homogenization theory the behavior of $u^{\varepsilon,\delta} : \mathbb{R}^d \longrightarrow \mathbb{R}$ of double perturbed Kolmogorov, Petrovskii and Piskunov (KPP)-type with periodic coefficients varying over length scale δ and nonlinear reaction term of scale $1/\varepsilon$,

$$\begin{cases} \frac{\partial u^{\varepsilon,\delta}}{\partial t}(t,x) = \mathcal{L}^{\alpha}_{\varepsilon,\delta} u^{\varepsilon,\delta}(t,x) + \frac{1}{\varepsilon} f\left(\frac{x}{\delta}, u^{\varepsilon,\delta}(t,x)\right), & x \in \mathbb{R}^d,\ 0 < t, \\ u^{\varepsilon,\delta}(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1}$$

The behavior is required as ε, δ both tend to 0. Our homogenization method is probabilistic. Since δ and ε go at the same rate, we may apply the large deviations principle with homogenized coefficients.

Keywords: Homogenization, large deviations, nonlocal parabolic PDE, SDE with jumps, Feynman-Kac formula.

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1. Introduction

In mechanics, physics or bio-materials engineering, most observed phenomena are characterized by a large number of spatial scales. The mathematical complexity associated with describing these phenomena increases exponentially with the number of scales. Under certain conditions, if the size of the microstructures encountered is appreciably small compared to the characteristic distance of the field of study, asymptotic analysis is used to study the behavior of the material (or medium) using the infinitesimal parameter δ representing the ratio between these two typical lengths (see, for example, [1, 5–7] and the papers therein). This process is quite common in dynamical systems, but is rather complicated when a taxis term is involved in the model itself. In chemotaxis mechanisms, the scenario is to take into account the interactions between the cells and their environment, while simplifying the equations to

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obtain approximate models that describe the collective movement of the cells (see, for instance [8, 15–17]). Homogenization is a mathematical and physical concept used to simplify the study of complex systems by replacing them with simpler models that are easier to analyze. Clearly, the large deviation principle (LDP) allows us to investigate the asymptotic behavior of large stochastic systems, and several variants can be found in the literature. For our purposes, in the framework studied by Freidlin and Wentzell [13], the theory of *small perturbations of dynamical systems*, the principal term of the operator is perturbed by a small parameter ε. The combinatorial effects of homogenization and large deviations is a classical problem which goes back to Baldi [2] at the end of the 20'th century. Such a problem has been most extensively investigated by Freidlin and Sowers [12] in stochastic differential equations (SDEs) on the whole of $\mathbb{R}^d$. The basic LDP calculations of these two papers involve deriving the Varadhan formula and identifying the barrier. There are several generalizations of the Varadhan formula, see for example Baxendale and Stroock [4], but all require some form of differentiability. Inspired by [2, 12], our aim in this paper is highly motivated by the consideration to combine the two principles in a compatible way, for a class of semilinear parabolic partial differential equations (PDEs).

We first give the rate function $S_{0,t}$ of the large deviations, in fact since δ and ε go at the same rate to zero, this function is expressed by the homogenized coefficients of the PDE (1), next we express the solution of PDE (1) by the use of backward stochastic differential equations (BSDEs) in [3] and the Feynman-Kac formula, then we consider an auxiliary solution via $\varepsilon \log u^{\varepsilon,\delta}$ by applying techniques due to Pradeilles [19]. The limit of this auxiliary solution helps us to find the limit of $u^{\varepsilon,\delta}$ when both ε, δ tend to zero. We show in the end that there exists a function V^* (which depends on $S_{0,t}$) such that $u^{\varepsilon,\delta}$ tends to zero if $(t,x) \in \{V^* < 0\}$ and tends to 1 in the interior of $\{V^* = 0\}$.

We organize the paper as follows. In Section 2, we present some general assumptions and definitions. Section 3 contains the results of large deviations principle. In Section 4, we study the behavior of the solution of the PDE (1).

2. Preliminaries

In this paper, we use Einstein's convention that the repeated indices in a product will be summed automatically. By B_r we means the open ball in $\mathbb{R}^d$ centering at the origin with radius $r > 0$, we shall omit the subscript when the radius is one. We denote by $\mathcal{C}^k$ $\left(\mathcal{C}_b^k\right)$ with integer $k \geqslant 0$ the space of (bounded) continuous functions possessing (bounded) derivatives of orders not greater than k. We shall explicitly write out the domain if necessary. Denote by $\mathcal{C}_b\left(\mathbb{R}^d\right) := \mathcal{C}_b^0\left(\mathbb{R}^d\right)$, it is a Banach space with the supremum norm $\|f\|_0 = \sup_{x\in\mathbb{R}^d} |f(x)|$. The space $\mathcal{C}_b^k\left(\mathbb{R}^d\right)$ is a Banach space endowed with the norm $\|f\|_k = \|f\|_0 + \sum_{j=1}^k \left\|\nabla^{\otimes j} f\right\|$. For a noninteger $\lambda > 0$, the Hölder spaces $\mathcal{C}^\lambda$ $(\mathcal{C}_b^\lambda)$ are defined as the subspaces of $\mathcal{C}^{\lfloor\lambda\rfloor}$ $(\mathcal{C}_b^{\lfloor\lambda\rfloor})$ consisting of functions whose $\lfloor\lambda\rfloor$-th order partial derivatives are locally Hölder continuous (uniformly Hölder continuous) with exponent $\lambda - \lfloor\lambda\rfloor$. These two spaces $\mathcal{C}^{\lfloor\lambda\rfloor}$ and $\mathcal{C}_b^{\lfloor\lambda\rfloor}$ obviously coincide when the underlying domain is compact. The space $\mathcal{C}_b^{\lfloor\lambda\rfloor}\left(\mathbb{R}^d\right)$ is a Banach space endowed with the norm $\|f\|_\lambda = \|f\|_{\lfloor\gamma\rfloor} + [\nabla^{\lfloor\gamma\rfloor} f]_{\lambda-\lfloor\lambda\rfloor}$, where the seminorm $[\cdot]_{\lambda'}$ with $0 < \lambda' < 1$ is defined as $[f]_{\lambda'} := \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\lambda'}}$ (this seminorm can also be defined for the case $\lambda' = 1$, which is exactly the Lipschitz seminorm). In the sequel, the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ will be used frequently. Denote by $\mathcal{D} := \mathcal{D}\left(\mathbb{R}_+; \mathbb{T}^d\right)$ the space of all $\mathbb{T}^d$-valued càdlàg functions on $\mathbb{R}_+$, equipped with the Skorokhod topology. We shall always identify the periodic function on $\mathbb{R}^d$ of period 1 with its restriction on $\mathbb{T}^d$.

For simplicity, we can organize all of this by setting $\delta_\varepsilon := \delta$, where $\lim_{\varepsilon\to 0} \delta_\varepsilon = 0$.

(H.1) We assume that $\lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{\varepsilon} = \gamma > 0$.

Let $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geqslant 0}\right)$ be a filtered probability space endowed with a Poisson random measure $N^{\alpha,\varepsilon^{-1}}$ on $\mathbb{R}^d\backslash\{0\} \times \mathbb{R}_+$ with jump intensity measure $\nu_\varepsilon^\alpha(dy) = \frac{1}{\varepsilon}\nu^\alpha(dy) = \frac{\varepsilon^{-1}dy}{|y|^{d+\alpha}}$, where $1 < \alpha < 2$, $\varepsilon > 0$.

Denote by $\tilde{N}$ the associated compensated Poisson random measure, that is,

$$\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds) := N^{\alpha,\varepsilon^{-1}}(dyds) - \nu_\varepsilon^\alpha(dy)ds.$$

We assume that the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies the usual $\mathbb{P}$-null conditions. Let $L^{\alpha,\varepsilon^{-1}} = \left\{L_t^{\alpha,\varepsilon^{-1}}\right\}_{t\geqslant 0}$ be a d-dimensional isotropic α-stable Lévy process given by

$$L_t^{\alpha,\varepsilon^{-1}} := \int_0^t \int_{B\setminus\{0\}} y\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds) + \int_0^t \int_{B^c} yN^{\alpha,\varepsilon^{-1}}(dyds).$$

Given $\varepsilon > 0, x \in \mathbb{R}^d$, consider the following:

$$\begin{cases} dX_t^{\varepsilon,\delta_\varepsilon} = \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}} b_0\left(\frac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dt + b_1\left(\frac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dt + \varepsilon\sigma\left(\frac{X_{t-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, dL_t^{\alpha,\varepsilon^{-1}}\right), \\ X_0^{\varepsilon,\delta_\varepsilon} = x, \end{cases} \tag{2.1}$$

or more precisely,

$$\begin{aligned} X_t^{\varepsilon,\delta_\varepsilon} = x &+ \int_0^t \left[\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}} b_0\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) + b_1\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)\right]ds + \int_0^t \int_{B\setminus\{0\}} \sigma\left(\frac{X_{s-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, y\right)\varepsilon\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds) \\ &+ \int_0^t \int_{B^c} \sigma\left(\frac{X_{s-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, y\right)\varepsilon N^{\alpha,\varepsilon^{-1}}(dyds). \end{aligned}$$

Before continuing, we list some general assumptions for the PDE (1) and the nonlocal SDE (2.1). We consider $u_0 \in \mathcal{C}_b\left(\mathbb{R}^d\right)$ and we set $\sup_{x\in\mathbb{R}^d} u_0(x) = \overline{u}_0 < \infty$. Letting $U_0 = \left\{x \in \mathbb{R}^d : u_0(x) > 0\right\}$, since u_0 is continuous we have $\overline{\overset{\circ}{U}_0} = \overline{U}_0$. We assume that $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is periodic in each direction with respect to the first argument, and it verifies:

- $\forall x \in \mathbb{R}^d,\ f(x,1) = 0$;
- there exists $c \in \mathcal{C}_b^\beta(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ such that $f(x,y) = c(x,y)\cdot y$, with

$$c(x,y) > 0, \forall x \in \mathbb{R}^d,\ y \in [0,1) \cup \mathbb{R}_-^*, \quad \text{and} \quad c(x,y) \leqslant 0, \forall x \in \mathbb{R}^d,\ y > 1.$$

And we assume that

$$\max c(x,y) = c(x) = c(x,0) > 0,\ \forall x \in \mathbb{R}^d.$$

(H.2) i) $(b_0, b_1, u_0) : \mathbb{R}^{3d} \longrightarrow \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$ are all periodic of period 1 in each component.
ii) $x \mapsto (\sigma(x,\cdot), c(x,\cdot))$ is periodic of period 1 in each component.
iii) b_0, b_1, c are of class $\mathcal{C}_b^\beta$ with β satisfying : $1 - \frac{\alpha}{2} < \beta < 1$.
iv) The initial functions u_0 is continuous.

The function $\sigma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies the following conditions.

(H.3) i) $\forall x \in \mathbb{R}^d$, the function $y \mapsto \sigma(x,y)$ is of class $\mathcal{C}^2$.
ii) There exists a constant $C > 0$, such that for any $x_1, x_2, y \in \mathbb{R}^d$, $|\sigma(x_1,y) - \sigma(x_1,y)| \leqslant C|x_1 - x_2||y|$.
iii) The oddness condition: for all $x, y \in \mathbb{R}^d,\ \sigma(x,-y) = -\sigma(x,y)$.
iv) There exists a positive bounded measurable $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ such that for all $x, y \in \mathbb{R}^d$, $\phi(x)^{-1}|y| \leqslant \sigma(x,y) \leqslant \phi(x)|y|$.

Let us introduce the linear operator $\mathcal{A}^{\sigma,\nu^\alpha}$ defined as

$$\mathcal{A}^{\sigma,\nu^\alpha}f(x) := \int_{\mathbb{R}^d\setminus\{0\}}\Big[f\left(x+\sigma\left(x,y\right)\right)-f(x)-\sigma^i\left(x,y\right)\partial_i f(x)\mathbf{1}_B(y)\Big]\nu^\alpha(dy), \quad x\in\mathbb{R}^d.$$

By virtue of the oddness condition and the symmetry of the jump intensity measure ν^α, we can rewrite the operator $\mathcal{A}^{\sigma,\nu^\alpha}$ as (see [14]):

$$\mathcal{A}^{\sigma,\nu^\alpha}f(x) := \int_{\mathbb{R}^d\setminus\{0\}}\Big[f\left(x+z\right)-f(x)-z^i\partial_i f(x)\mathbf{1}_B(z)\Big]\nu^{\sigma,\alpha}(x,dz), \quad x\in\mathbb{R}^d,$$

where the kernel $\nu^{\sigma,\alpha}$ is given by

$$\nu^{\sigma,\alpha}(x,A) = \int_{\mathbb{R}^d\setminus\{0\}}\mathbf{1}_A\left(\sigma(x,y)\right)\nu^\alpha(dy), \quad A\in\mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right).$$

Next, to move the SDE (2.1) to the torus $\mathbb{T}^d$, we define $\tilde{X}_t^{\varepsilon,\delta_\varepsilon} := \frac{1}{\delta_\varepsilon}X^{\varepsilon,\delta_\varepsilon}_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}$, via the canonical quotient map $\pi:\mathbb{R}^d\longrightarrow\mathbb{R}^d/\mathbb{Z}^d$. It is not difficult to check that

$$\begin{cases} d\tilde{X}_t^{\varepsilon,\delta_\varepsilon} = \left[b_0\left(\tilde{X}_t^{\varepsilon,\delta_\varepsilon}\right)+\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}b_1\left(\tilde{X}_t^{\varepsilon,\delta_\varepsilon}\right)\right]dt+\frac{\varepsilon}{\delta_\varepsilon}\sigma\left(\tilde{X}_{t-}^{\varepsilon,\delta_\varepsilon},\frac{\delta_\varepsilon}{\varepsilon}dL_t^\alpha\right), \\ \tilde{X}_0^{\varepsilon,\delta_\varepsilon} = \frac{x}{\delta_\varepsilon}, \end{cases} \tag{2.2}$$

where

$$L_t^\alpha := \int_0^t\int_{B\setminus\{0\}}y\tilde{N}^\alpha(dyds)+\int_0^t\int_{B^c}yN^\alpha(dyds),$$

and with $\left\{\frac{\varepsilon}{\delta}L^{\alpha,\varepsilon^{-1}}_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}\right\}\equiv\left\{\frac{\varepsilon}{\delta_\varepsilon}L^\alpha_{(\delta_\varepsilon/\varepsilon)^\alpha t}\right\}\overset{d}{:=}\{L_t^\alpha\}$ by the self-similarity. We shall also consider the limit SDE (2.2), namely

$$d\tilde{X}_t = b_0\left(\tilde{X}_t\right)dt+\gamma^{\alpha-1}b_1\left(\tilde{X}_t\right)dt+\overline{\sigma}_\gamma\left(\tilde{X}_{t-},dL_t^\alpha\right), \quad \tilde{X}_0 = x,$$

where, $\overline{\sigma}_\gamma(\cdot,y) := \frac{1}{\gamma}\sigma\left(\cdot,\gamma y\right)$ is the point-wise limit as $\varepsilon\downarrow 0$ of $\frac{\varepsilon}{\delta_\varepsilon}\sigma\left(\cdot,\frac{\delta_\varepsilon}{\varepsilon}y\right)$.

We need a stronger convergence as follows.

(H.4) $\forall y\in\mathbb{R}^d$, $\frac{1}{\eta}\sigma(x,\eta y)\longrightarrow\frac{1}{\gamma}\sigma(x,\gamma y)$ uniformly in $x\in\mathbb{R}^d$, as $\eta\to\gamma$.

Let $\mathcal{L}_\gamma^\alpha$ be the linear integro-partial differential operator given by

$$\mathcal{L}_\gamma^\alpha := \mathcal{A}^{\overline{\sigma}_\gamma,\nu^\alpha}+\left(b_0+\gamma^{\alpha-1}b_1\right)\cdot\nabla.$$

By requirement there exists a $\mathcal{L}_\gamma^\alpha$-Feller process on $\mathbb{R}^d$ and by periodicity assumption on the coefficients such a process induces a process $\tilde{X}$ which is a strong Markov process on the torus $\mathbb{T}^d$, moreover the $\mathcal{L}^\alpha$-process is ergodic (see [14]). We denote by μ_γ its unique invariant measure on $\left(\mathbb{T}^d,\mathcal{B}\left(\mathbb{T}^d\right)\right)$.

Thanks to [14, Proposition 4.11 (equation without second member)], there is a unique periodic solution $\varphi\in\mathcal{C}^{\alpha+\beta}$ of the Poisson equation

$$\mathcal{L}_\gamma^\alpha\varphi+b_0 = \int_{\mathbb{T}^d}b_0(x)\mu_\gamma(dx) \quad \text{such that} \quad \int_{\mathbb{T}^d}\varphi(x)\mu_\gamma(dx) = 0.$$

Now we set

$$\begin{aligned}
\overline{C} &:= \int_{\mathbb{T}^d}c(x)\mu_\gamma(dx),\\
\overline{\Sigma}_\varphi &:= \int_{\mathbb{T}^d}\left[\left(I+\nabla\varphi\right)\left(\gamma^{1-\alpha}b_0+b_1\right)(x)+\gamma^{1-\alpha}\mathcal{A}^{\overline{\sigma}_\gamma,\nu^\alpha}\varphi(x)\right]\mu_\gamma(dx),\\
\overline{\nu}_\mu(A) &:= \int_{\mathbb{R}^d\setminus\{0\}}\int_{\mathbb{T}^d}\mathbf{1}_A\left(\sigma(x,y)\right)\mu(dx)\nu^\alpha(dy), \quad A\in\mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right).
\end{aligned}$$

3. Large deviation principle

The theory of large deviations is concerned with events A for which probability $\mathbb{P}\left(X^{\varepsilon,\delta_\varepsilon} \in A\right)$ converges to zero exponentially fast as $\varepsilon \to 0$ (see [9]). The exponential decay rate of such probabilities is typically expressed in terms of a rate function $\mathcal{J}$ mapping $\mathbb{R}^d$ into $[0,+\infty]$. Our method allows us to characterize the LDP by analysing the logarithmic moment generating function [9, Chap. 2.3]. Initially the corresponding rate function is identified as the Legendre transform of the limit (when it exists) of the logarithmic moment generating function defined as:

$$\lim_{\varepsilon\to 0} g^{\varepsilon}_{t,x}(\theta) := \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left\{\exp\left(\frac{1}{\varepsilon}\langle \theta, X^{\varepsilon,\delta_\varepsilon}_t\rangle\right)\right\}.$$

Let $\varphi \in \mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$. If we set

$$\hat{X}^{\varepsilon,\delta_\varepsilon}_t := X^{\varepsilon,\delta_\varepsilon}_t + \delta_\varepsilon\left[\varphi\left(\frac{X^{\varepsilon,\delta_\varepsilon}_t}{\delta_\varepsilon}\right) - \varphi\left(\frac{x}{\delta_\varepsilon}\right)\right],$$

then we have by Itô's formula

$$\begin{aligned}
\hat{X}^{\varepsilon,\delta_\varepsilon}_t &= x + \int_0^t (I+\nabla\varphi_\varepsilon)\left[\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}} b_{0_\varepsilon} + b_{1_\varepsilon}\right]\left(X^{\varepsilon,\delta_\varepsilon}_s\right) ds + \frac{\delta_\varepsilon}{\varepsilon}\int_0^t \mathcal{A}^{\varepsilon\sigma_\varepsilon,\nu^\alpha}\varphi_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s\right) ds \\
&\quad + \delta_\varepsilon \int_0^t \int_{\mathbb{R}^d\setminus\{0\}}\left[\varphi_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s + \varepsilon\sigma_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_{s-}, y\right)\right) - \varphi_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s\right)\right]\tilde{N}^{\alpha,\varepsilon^{-1}}(dy ds) \\
&\quad + \int_0^t \varepsilon\sigma_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_{s-}, dL^{\alpha,\varepsilon^{-1}}_s\right),
\end{aligned}$$

where $\zeta_\varepsilon(x) = \zeta\left(\frac{x}{\delta_\varepsilon}\right)$ for $\zeta(x)$ in $\{b_0(x), b_1(x), \phi b(x), \nabla\phi, \sigma(x,\cdot)\}$. Note that $\nu^\alpha(\gamma A) = \gamma^{-\alpha}\nu^\alpha(A)$, $A \in \mathcal{B}\left(\mathbb{R}\setminus\{0\}\right)$. Before proceeding, let us define for all $z \in \mathbb{T}^d$,

$$H^{\varepsilon,\varphi}(z,\cdot) := \varphi\Big(z + \frac{\varepsilon}{\delta_\varepsilon}\sigma(z, \frac{\delta_\varepsilon}{\varepsilon}\cdot)\Big) - \varphi(z), \quad \mathcal{Q}^{\varepsilon,\varphi}(z) := \mathcal{A}^{\frac{\varepsilon}{\delta_\varepsilon}\sigma(\cdot,(\delta_\varepsilon/\varepsilon)\cdot),\nu^\alpha}\varphi(z) - \mathcal{A}^{\overline{\sigma}_\gamma,\nu^\alpha}\varphi(z).$$

Now, by Girsanov's formula, we have

$$\begin{aligned}
g^{\varepsilon}_{t,x}(\theta) = \langle\theta, x\rangle + \varepsilon\log\tilde{\mathbb{E}}\Bigg\{ &\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta, (I+\nabla\varphi)\, b_1(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s)\rangle ds\right) \\
&\times\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta, \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\Big[(I+\nabla\varphi)\, b_0 + \mathcal{A}^{\overline{\sigma}_\gamma,\nu^\alpha}\varphi\Big](\tilde{X}^{\varepsilon,\delta_\varepsilon}_s)\rangle ds\right) \\
&\times\exp\left(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\varphi}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right) ds - \frac{\delta_\varepsilon}{\varepsilon}\left[\varphi\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)t}\right) - \varphi\left(\frac{x}{\delta_\varepsilon}\right)\right]\right) \\
&\times\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\frac{\delta_\varepsilon}{\varepsilon}\langle\theta, H^{\varepsilon,\varphi}(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y)\rangle} - 1 - \frac{\delta_\varepsilon}{\varepsilon}\langle\theta, H^{\varepsilon,\varphi}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\mathbf{1}_B(y)\rangle\right\}\nu^\alpha(dy)ds\right) \\
&\times\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\langle\theta, \sigma(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y)\rangle} - 1 - \langle\theta, \sigma(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y)\mathbf{1}_B(y)\rangle\right\}\nu^\alpha(dy)ds\right)\Bigg\},
\end{aligned} \tag{3.1}$$

where $\tilde{\mathbb{E}}$ is the expectation operator with respect to the probability $\tilde{\mathbb{P}}$ defined as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \exp\left(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta, H^{\varepsilon,\varphi}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{s-}, y\right)\rangle\tilde{N}^{\alpha,(\delta_\varepsilon/\varepsilon)^\alpha}(dy ds) + \frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\langle\theta, \sigma\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{s-}, dL^\alpha_s\right)\rangle\right)$$

$$\times \exp\left(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\frac{\delta_\varepsilon}{\varepsilon}\langle\theta,H^{\varepsilon,\varphi}(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},y)\rangle}-1-\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\varphi}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\mathbf{1}_B(y)\right\rangle\right\}\nu^\alpha(dy)ds\right)$$
$$\times \exp\left(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\langle\theta,\sigma(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},y)\rangle}-1-\left\langle\theta,\sigma\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\mathbf{1}_B(y)\right\rangle\right\}\nu^\alpha(dy)ds\right).$$

Let us set, for all $z\in\mathbb{T}^d$, for all $\theta\in\mathbb{R}^d$:

$$\begin{aligned}\Phi^{\varepsilon,\varphi}(z,\theta):=&\left\langle\theta,(I+\nabla\varphi)\,b_1\,(z)\,\right\rangle+\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\left\langle\theta,\left[(I+\nabla\varphi)\,b_0+\mathcal{A}^{\overline{\sigma}_\gamma,\nu^\alpha}\varphi\right](z)\,\right\rangle\\&+\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\langle\theta,\sigma(z,y)\rangle}-1-\left\langle\theta,\sigma(z,y)\mathbf{1}_B(y)\right\rangle\right\}\nu^\alpha(dy)\\&+\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\frac{\delta_\varepsilon}{\varepsilon}\langle\theta,H^{\varepsilon,\varphi}(z,y)\rangle}-1-\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta,H^{\varepsilon,\varphi}(z,y)\mathbf{1}_B(y)\right\rangle\right\}\nu^\alpha(dy).\end{aligned}$$

Let $\Psi_\theta^\varepsilon\in\mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$ be the unique solution of

$$\mathcal{L}_\gamma^\alpha\Psi_\theta^\varepsilon(z)+\Phi^{\varepsilon,\varphi}(z,\theta)=\int_{\mathbb{T}^d}\Phi^{\varepsilon,\varphi}(z,\theta)\mu_\gamma(dz)$$

satisfying $\int_{\mathbb{T}^d}\Psi_\theta^\varepsilon(z)\mu_\gamma(dz)=0$. Such a solution Ψ_θ^ε must exist again by the assumptions on the coefficients and the Fredholm alternative. So applying Itô's formula to $\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\Psi_\theta^\varepsilon\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}\right)$, we have

$$\begin{aligned}\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\Phi^{\varepsilon,\varphi}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},\theta\right)=&\,t\cdot\int_{\mathbb{T}^d}\Phi^{\varepsilon,\varphi}(z,\theta)\mu_\gamma(dz)+\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\left[\Psi_\theta^\varepsilon\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)}\right)-\Psi_\theta^\varepsilon\left(\frac{x}{\delta_\varepsilon}\right)\right]\\&+\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\Psi_\theta^\varepsilon}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon}\right)ds-\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\left(\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}-\gamma^{\alpha-1}\right)\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\nabla\Psi_\theta^\varepsilon b_1\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon}\right)ds\\&-\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}H^{\varepsilon,\Psi_\theta^\varepsilon}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\tilde{N}^\alpha(dyds).\end{aligned}$$

Then putting above equation into (3.1), we obtain

$$\begin{aligned}g_{t,x}^\varepsilon(\theta)=&\langle\theta,x\rangle+t\cdot\int_{\mathbb{T}^d}\Phi^{\varepsilon,\varphi}(z,\theta)\mu_\gamma(dz)\\&+\varepsilon\log\hat{\mathbb{E}}\Bigg\{\exp\left(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\hat{b}}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon}\right)ds-\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\left(\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}-\gamma^{\alpha-1}\right)\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\nabla\Psi_\theta^\varepsilon b_1\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon}\right)ds\right)\\&\times\exp\left(-\frac{\delta_\varepsilon}{\varepsilon}\left[\varphi\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)t}\right)-\varphi\left(\frac{x}{\delta_\varepsilon}\right)\right]+\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\left[\Psi_\theta^\varepsilon\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)t}\right)-\Psi_\theta^\varepsilon\left(\frac{x}{\delta_\varepsilon}\right)\right]\right)\\&\times\exp\left(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\Psi_\theta^\varepsilon}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon}\right)ds\right)\\&\times\exp\left(\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^\alpha H^{\varepsilon,\Psi_\theta^\varepsilon}(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},y)}-1-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}H^{\varepsilon,\Psi_\theta^\varepsilon}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},y\right)\mathbf{1}_B(y)\right\}\nu^\alpha(dy)ds\right)\Bigg\},\end{aligned}\tag{3.2}$$

where $\hat{\mathbb{E}}$ is the expectation operator with respect to the probability $\hat{\mathbb{P}}$ defined as

$$\begin{aligned}\frac{d\hat{\mathbb{P}}}{d\tilde{\mathbb{P}}}:=&\exp\left(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}H^{\varepsilon,\Psi_\theta^\varepsilon}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},\frac{\delta_\varepsilon}{\varepsilon}y\right)\tilde{N}^\alpha(dyds)\right)\\&\times\exp\left(\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^\alpha H^{\varepsilon,\Psi_\theta^\varepsilon}(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},\frac{\delta_\varepsilon}{\varepsilon}y)}-1-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}H^{\varepsilon,\Psi_\theta^\varepsilon}\left(\tilde{X}_s^{\varepsilon,\delta_\varepsilon},\frac{\delta_\varepsilon}{\varepsilon}y\right)\mathbf{1}_B(y)\right\}\nu^\alpha(dy)ds\right).\end{aligned}$$

Since the coefficients are bounded, we first notice that

$$\sup_{z\in\mathbb{T}^d}\left\{\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\left[\Psi_\theta^\varepsilon(z_t)-\Psi_\theta^\varepsilon\left(\frac{x}{\delta_\varepsilon}\right)\right]-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\left(\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}-\gamma^{\alpha-1}\right)\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\nabla\Psi_\theta^\varepsilon b_1(z_s)\,ds\right)\right.$$
$$\left.\times\exp\left(-\frac{\delta_\varepsilon}{\varepsilon}\left[\varphi(z_t)-\varphi\left(\frac{x}{\delta_\varepsilon}\right)\right]\right)\right\}\leqslant\exp\left\{\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}K_1+\frac{\delta_\varepsilon}{\varepsilon}K_2+\frac{1}{\varepsilon}\left(\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}-\gamma^{\alpha-1}\right)K_3\right\}. \tag{3.3}$$

Now we recall an elementary result.

Lemma 3.1 ([14]). *Let $0<\lambda\leqslant 1$ and $f\in\mathcal{C}_b^{1+\lambda}(\mathbb{R}^d)$. For any $x,u,v\in\mathbb{R}^d$, it holds that*

$$\left|f(x+u)-f(x+v)-(u-v)\cdot\nabla f(x)\right|\leqslant\frac{1}{1+\lambda}[\nabla f]_\lambda\,|u-v|^{1+\lambda}.$$

We let $r=\sqrt[1/\beta]{\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}}$ that will be chosen for B_r. We have (see [14, Lemma 5.3], where the following stochastic integral term, is determined via the characteristics of semi-martingales) for all $\varphi\in\mathcal{C}^{\alpha+\beta}(\mathbb{T}^d)$:

$$\begin{aligned}\left|\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\varphi}(z_s)\,ds\right|\leqslant\frac{t}{\varepsilon}\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\int_{B_r\setminus\{0\}}&\left|\left[\varphi\left(z+\frac{\varepsilon}{\delta_\varepsilon}\sigma\Big(z,\frac{\delta_\varepsilon}{\varepsilon}y\Big)\right)-\varphi\left(z+\frac{1}{\gamma}\sigma\Big(z,\gamma y\Big)\right)\right.\right.\\&\left.\left.-\left(\frac{\varepsilon}{\delta_\varepsilon}\sigma^i\Big(z,\frac{\delta_\varepsilon}{\varepsilon}y\Big)-\frac{1}{\gamma}\sigma^i\Big(z,\gamma y\Big)\right)\partial_i\varphi(z)\right]\right|\nu^\alpha(dy)\\&+t\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\int_{B_r^c}\left|\left[\varphi\left(z+\frac{\varepsilon}{\delta_\varepsilon}\sigma\Big(z,\frac{\delta_\varepsilon}{\varepsilon}y\Big)\right)-\varphi\left(z+\frac{1}{\gamma}\sigma\Big(z,\gamma y\Big)\right)\right]\right|\nu^\alpha(dy)\\:=\ &t\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\Big(I_1^\varepsilon(z)+I_2^\varepsilon(z)\Big).\end{aligned}$$

It follows from Lemma 3.1 that:

- $$\begin{aligned}I_1^\varepsilon(z)&=\frac{\|\varphi\|_{\alpha+\beta}}{\alpha+\beta}\int_{B_r\setminus\{0\}}\left|\frac{\varepsilon}{\delta_\varepsilon}\sigma\Big(z,\frac{\delta_\varepsilon}{\varepsilon}y\Big)-\frac{1}{\gamma}\sigma\Big(z,\gamma y\Big)\right|^{\alpha+\beta}\nu^\alpha(dy)\\&\leqslant\frac{2^{\alpha+\beta}}{\alpha+\beta}\|\varphi\|_{\alpha+\beta}(\phi(z))^{\alpha+\beta}\int_{B_r\setminus\{0\}}|y|^{\alpha+\beta}\,\nu^\alpha(dy)\leqslant\frac{2^{\alpha+\beta}}{\alpha+\beta}\left|\mathbb{S}^{d-1}\right|\|\varphi\|_{\alpha+\beta}(\phi(z))^{\alpha+\beta}\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}};\end{aligned}$$
- $$I_2^\varepsilon(z)=2\|\varphi\|_0\int_{B^c_{r^{\beta\alpha}}}\left|\frac{\varepsilon}{\delta_\varepsilon}\sigma\Big(z,\frac{\delta_\varepsilon}{\varepsilon}y\Big)-\frac{1}{\gamma}\sigma\Big(z,\gamma y\Big)\right|\nu^\alpha(dy)\leqslant\frac{2}{1-\alpha}\left|\mathbb{S}^{d-1}\right|\|\varphi\|_0\|\phi(z)\|\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}.$$

Thus

$$\sup_{z\in\mathbb{T}^d}\left\{\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\mathcal{Q}^{\varepsilon,\varphi}(z_s)\,ds\right\}\longrightarrow t\gamma\left|\mathbb{S}^{d-1}\right|\left(\frac{2^{\alpha+\beta}}{\alpha+\beta}\|\varphi\|_{\alpha+\beta}(\phi(z))^{\alpha+\beta}+\frac{2}{1-\alpha}\|\varphi\|_0\|\phi(z)\|_0\right). \tag{3.4}$$

On the other hand, using a similar estimate once again, for all $\varphi\in\mathcal{C}^{\alpha+\beta}(\mathbb{T}^d)$ we have:

$$\begin{aligned}J_\varepsilon^\varphi(z)&:=\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\setminus\{0\}}\left\{e^{\frac{\delta_\varepsilon}{\varepsilon}H^{\varepsilon,\varphi}(z_s,y)}-1-\frac{\delta_\varepsilon}{\varepsilon}H^{\varepsilon,\varphi}\Big(z_s,y\Big)\mathbf{1}_B(y)\right\}\nu^\alpha(dy)ds\\&\leqslant\frac{t}{\varepsilon(\alpha+\beta)}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{1+\beta}\left[J_1^{\varepsilon,\varphi}(z)+J_2^{\varepsilon,\varphi}(z)\right],\end{aligned}$$

with

- $$\begin{aligned} J_1^{\varepsilon,\varphi}(z) &= \int_{B_r\setminus\{0\}} \left|\varphi\Big(z+\frac{\varepsilon}{\delta_\varepsilon}\sigma\big(z,\frac{\delta_\varepsilon}{\varepsilon}y\big)\Big)-\varphi(z)\right|^{\alpha+\beta} \nu^\alpha(dy) \\ &\leqslant \|\varphi\|_{\alpha+\beta}\,(\phi(z))^{\alpha+\beta}\int_{B_r\setminus\{0\}}|y|^{\alpha+\beta}\,\nu^\alpha(dy) \leqslant \left|\mathbb{S}^{d-1}\right|\|\varphi\|_{\alpha+\beta}\,(\phi(z))^{\alpha+\beta}\,\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}; \end{aligned}$$
- $$J_2^{\varepsilon,\varphi}(z) = \int_{B_r^c} \left|\varphi\Big(z+\frac{\varepsilon}{\delta_\varepsilon}\sigma\big(z,\frac{\delta_\varepsilon}{\varepsilon}y\big)\Big)-\varphi(z)\right|^{\alpha+\beta} \nu^\alpha(dy) \leqslant \frac{2^{\alpha+\beta}}{\beta}\left|\mathbb{S}^{d-1}\right|\|\varphi\|_0\,\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}.$$

Then

$$\sup_{z\in\mathbb{T}^d}\left\{J_\varepsilon^\varphi(z)\right\} \longrightarrow t\left|\mathbb{S}^{d-1}\right|\gamma^\alpha\left(\frac{\gamma^{1+\beta}}{\alpha+\beta}\|\varphi\|_{\alpha+\beta}\,(\phi(z))^{\alpha+\beta}+\frac{2^{\alpha+\beta}}{\beta}\left|\mathbb{S}^{d-1}\right|\|\varphi\|_0\right). \tag{3.5}$$

Now, let us set

$$\mathcal{J}(\theta) := \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)}\inf_{\varphi\in\mathbb{T}^d}\int_{\mathbb{R}^d\setminus\{0\}}\left(e^{\langle\theta,y\rangle}-1+\langle\theta,\overline{\Sigma}_\varphi-y\mathbf{1}_B\rangle\right)\overline{\nu}_\mu(dy).$$

From (3.2)-(3.5), we observe that

$$\lim_{\varepsilon\to 0} g_{t,x}^\varepsilon(\theta) = \langle\theta,x\rangle + t\mathcal{J}(\theta).$$

Let $\overline{\mathcal{J}}$ denote the Fenchel-Legendre transform of $\mathcal{J}$. Then we have

$$\overline{\mathcal{J}}(\theta) := \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)}\inf_{\varphi\in\mathbb{T}^d}\int_{\mathbb{R}^d\setminus\{0\}}\rho\left(\frac{\left|\theta-(\overline{\Sigma}_\varphi-y\mathbf{1}_B)\right|}{|y|}\right)\overline{\nu}_\mu(dy),$$

where $\rho(r) := r\log r - r + 1, r\in\mathbb{R}_+^*$.

We now have all the tools we need to state our main results (see [4, Corollary 1.12]).

Theorem 3.2. *Fix $T>0$ and assume* **(H.1)**-**(H.4)** *hold true. Then for every $x\in\mathbb{R}^d$, the family $\left\{X^{\varepsilon,\delta_\varepsilon}:\varepsilon>0\right\}$ of $\mathbb{R}^d$-valued random variables has a large deviations principle with good rate function*

$$I_{T,x}(z) := T\overline{\mathcal{J}}\Big(\frac{z-x}{T}\Big).$$

Next, let us consider

$$S_{0,T}(\varphi) := \begin{cases} \int_0^T \overline{\mathcal{J}}(\dot{\varphi}(s))\,ds, & \text{if } \varphi\in\mathcal{D}\left([0,T],\mathbb{R}^d\right) \text{ and } \varphi(0)=x, \\ +\infty, & \text{else.} \end{cases}$$

Since the function $\mathcal{J}$ is convex we can show that

$$\inf_{\substack{\varphi\in\mathcal{D}\left([0,T],\mathbb{R}^d\right)\\ \varphi(0)=x,\ \varphi(T)=z}}\int_0^T\overline{\mathcal{J}}(\dot{\varphi}(s))\,ds := T\overline{\mathcal{J}}\left(\frac{z-x}{T}\right).$$

So we express the path space-LDP

Corollary 3.3. *Assume* **(H.1)**-**(H.4)** *hold true. Then the family $\left\{X^{\varepsilon,\delta_\varepsilon}\right\}_{\varepsilon>0}$ of $\mathcal{D}\left([0,T];\mathbb{R}^d\right)$-valued random variables has a large deviations principle with good rate function $S_{0,T}(\varphi)$ for all $\varphi\in\mathcal{D}\left([0,T];\mathbb{R}^d\right)$.*

Therefore, we can establish the analogue of the Varadhan's Lemma (see [9]).

Remark 3.4. Let D be a Borel subset on $\mathcal{D}\left([0,t];\mathbb{R}^d\right)$ and c be an element of $\mathcal{C}^{\alpha+\beta}\left(\mathbb{R}^d\right)$. Then we have

$$\liminf_{\varepsilon\downarrow 0}\varepsilon\log\mathbb{E}\left[\mathbf{1}_D\left(X^{\varepsilon,\delta_\varepsilon}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}\right] \geqslant \overline{C}t - \inf_{\phi\in\mathring{D}} S_{0,t}(\phi),$$

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log\mathbb{E}\left[\mathbf{1}_D\left(X^{\varepsilon,\delta_\varepsilon}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}\right] \leqslant \overline{C}t - \inf_{\phi\in\overline{D}} S_{0,t}(\phi).$$

4. Convergence

Let us consider the progressive measurable process $(Y^{\varepsilon,\delta_\varepsilon}, U^{\varepsilon,\delta_{\varepsilon'}})$ solution of the BSDE:

$$\begin{cases} Y_t^{\varepsilon,\delta_\varepsilon} = u_0(X_t^{\varepsilon,\delta_\varepsilon}) + \frac{1}{\varepsilon}\int_s^t f\left(\frac{X_r^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, Y_r^{x,\varepsilon,\delta_\varepsilon}\right) dr - \int_s^t U_r^{\varepsilon,\delta_\varepsilon} dL_r^\alpha, \ 0 \leqslant s \leqslant t, \\ \sqrt{\mathbb{E}\int_s^t \int_{\mathbb{R}^d\setminus\{0\}} U_r^{\varepsilon,\delta_\varepsilon}(y)^2 \nu^\alpha(dy) dr} < \infty. \end{cases}$$

By [3, 18], we have for all $(t,x) \in [0,+\infty[\times \mathbb{R}^d$, the solution $u^{\varepsilon,\delta_\varepsilon}(t,x)$ of the PDE (1) is of the form

$$Y_0^{x,\varepsilon,\delta_\varepsilon} = u^{\varepsilon,\delta_\varepsilon}(t,x),$$

and the Feynman-Kac formula implies that the solution of PDE (1) obeys

$$u^{\varepsilon,\delta_\varepsilon}(t,x) = \mathbb{E}\left\{u_0\left(X_t^{\varepsilon,\delta_\varepsilon}\right)\exp\left(\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, Y_s^{\varepsilon,\delta_\varepsilon}\right) ds\right)\right\}.$$

Remark 4.1.

- If $\overline{u}_0 \leqslant 1$, then $\forall \varepsilon > 0, \ 0 \leqslant Y_s^{\varepsilon,\delta_\varepsilon} \leqslant 1, \quad d\mathbb{P} \times ds \text{ a.s.}.$
- On the other and, if $c(x,y) \leqslant \kappa(y) < 0, \ (x,y) \in \mathbb{R}^d \times]1,+\infty[$, where κ is Lipschitz continuous, then $\limsup_{\varepsilon\to 0} Y_t^{\varepsilon,\delta_\varepsilon} \leqslant 1$ uniformly in any compact set of $]0,+\infty[\times \mathbb{R}^d$.

To prove this, we will use similar results proved in [19].
Before continuing, let us introduce $v^{\varepsilon,\delta_\varepsilon(t,x)} = \varepsilon \log u^{\varepsilon,\delta_\varepsilon}(t,x)$, and let us set

$$\begin{aligned} \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) := \int_{\mathbb{R}^d\setminus\{0\}} \Bigg[& e^{\left\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}\left(t, x+\varepsilon\sigma\left(\frac{x}{\delta}, y\right)\right)\right\}} - 1 - \sigma\left(\frac{x}{\delta}, y\right) \partial_i v^{\varepsilon,\delta_\varepsilon}(t,x) \mathbf{1}_B(y) \\ & - \left\{ v^{\varepsilon,\delta_\varepsilon}\left(t, x+\varepsilon\sigma\left(\frac{x}{\delta}, y\right)\right) - v^{\varepsilon,\delta_\varepsilon}(t,x) \right\} \Bigg] \nu^\alpha(dy). \end{aligned}$$

Then, we observe that $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is a viscosity solution of:

$$\begin{cases} \frac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) = \mathcal{L}^\alpha_{\varepsilon,\delta_\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x) + \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) + c\left(\frac{x}{\delta_\varepsilon}, \exp\left\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x)\right\}\right), & x \in \mathbb{R}^d, \ t > 0, \\ v^{\varepsilon,\delta_\varepsilon}(0,x) = \varepsilon \log\left(u_0(x)\right), & x \in U_0, \\ \lim_{t\to 0} v^{\varepsilon,\delta_\varepsilon}(t,x) = -\infty, & x \in \mathbb{R}^d \setminus U_0. \end{cases} \tag{4.1}$$

Let us define a distance in $\mathbb{R}_+ \times \mathbb{R}^d$, for $(t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$d\{(t,x),(s,y)\} = \max\Big\{|t-s|, |x-y|\Big\},$$

and let us set

$$\begin{aligned} u^*(t,x) &= \limsup_{\eta\to 0}\Big\{v^{\varepsilon,\delta_\varepsilon}(s,y) : \ \varepsilon \leqslant \eta, \ (s,y) \in B_\eta(t,x)\Big\}, \\ v^*(t,x) &= \liminf_{\eta\to 0}\Big\{v^{\varepsilon,\delta_\varepsilon}(s,y) : \ \varepsilon \leqslant \eta, \ (s,y) \in B_\eta(t,x)\Big\}. \end{aligned}$$

Theorem 4.2. u^* *and* v^* *are sub and super viscosity solutions of:*

$$\begin{cases} \max_w \left(\frac{\partial w}{\partial t}(t,x) - \mathcal{H}^{\mathbf{Id}_y, \overline{\nu}^*} \nabla w(t,x) - \overline{\Sigma}_* \cdot \nabla w(t,x) - \overline{C}\right) = 0, & x \in \mathbb{R}^d, t > 0, \\ w(0,x) = 0, & x \in U_0, \\ \lim_{t\to 0} w(t,x) = -\infty, & x \in \mathbb{R}^d \setminus U_0, \end{cases}$$

where

$$\overline{\Sigma}_* := \inf_{\varphi\in\mathbb{T}^d} \overline{\Sigma}_\varphi, \quad \mathcal{H}^{\mathbf{Id}_y,\overline{\nu}^*} w := \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{R}^d\setminus\{0\}} \Big\{e^{\langle w,y\rangle} - 1 - \langle w,y\rangle \mathbf{1}_B(y)\Big\} \overline{\nu}_\mu(dy).$$

Proof. We use similar techniques as in Evans [10, 11]. Let us prove that u^* is a viscosity subsolution. The function $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is viscosity solution of

$$\frac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) - \mathcal{L}^{\alpha}_{\varepsilon,\delta_\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x) - \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) - c\left(\frac{x}{\delta_\varepsilon}, \exp\left\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x)\right\}\right) = 0. \tag{4.2}$$

We notice that

$$\lim_{\varepsilon\to 0} \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} w = \mathcal{H}^{\sigma,\nu^\alpha}\nabla w.$$

Now, let Φ be a smooth function, (t_0,x_0) be a strict local maximum of $v^{\varepsilon,\delta_\varepsilon} - \Phi$, and $\psi \in \mathcal{C}^\beta(\mathbb{T}^d)$ be a periodic function solution of the following Poisson equation,

$$\begin{aligned}&\mathcal{L}^{\alpha}_{\gamma}\psi(z) + (I+\nabla\varphi)\, b_1(z) D\Phi(t_0,x_0) + \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}\Phi(t_0,x_0) + c(z)\\ &\quad + \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\left[(I+\nabla\varphi)\, b_0 + \mathcal{A}^{\overline{\sigma},\nu^\alpha}\varphi\right](z) D\Phi(t_0,x_0) = \mathcal{H}^{\varepsilon,\mathbf{Id}_y,\overline{\nu}_\mu}\Phi(t_0,x_0) + \overline{\Sigma}_\varphi\cdot\nabla D\Phi(t_0,x_0) - \overline{C}.\end{aligned}$$

We consider now the perturbed test function

$$\Phi^\varepsilon(t,x) = \Phi(t,x) + \delta_\varepsilon\varphi\left(\frac{x}{\delta_\varepsilon}\right) D\Phi(t,x) + \frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\psi\left(\frac{x}{\delta_\varepsilon}\right). \tag{4.3}$$

Then we have

$$\begin{aligned}\frac{\partial\Phi^\varepsilon(t,x)}{\partial t} &= \frac{\partial\Phi(t,x)}{\partial t} + \delta_\varepsilon\varphi\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial t} D\Phi(t,x),\\ D\Phi^\varepsilon(t,x) &= (I+\nabla\varphi)\left(\frac{x}{\delta_\varepsilon}\right) D\Phi(t,x) + \delta_\varepsilon\varphi\left(\frac{x}{\delta_\varepsilon}\right) D^2\Phi(t,x) - \frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}} D\psi\left(\frac{x}{\delta_\varepsilon}\right).\end{aligned}$$

There exists a sequence $(t_\varepsilon,x_\varepsilon)$ local maximum of $v^{\varepsilon,\delta_\varepsilon} - \Phi^\varepsilon$ converging towards (t_0,x_0). If we set $z_\varepsilon = \frac{x_\varepsilon}{\delta_\varepsilon}$, getting ε small enough and putting everything together in (4.2), we have

$$\begin{aligned}&\frac{\partial\Phi}{\partial t}(t_0,x_0) - \mathcal{L}^{\alpha}_{\gamma}\psi(z) - \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}\Phi(t_0,x_0) - (I+\nabla\varphi)\, b_1(z) D\Phi(t_0,x_0) - c(z)\\ &\quad + \mathcal{A}^{\overline{\sigma}_\gamma,\nu^\alpha}\psi(z) + \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\mathcal{A}^{\overline{\sigma}_\gamma,\nu}\varphi(z) D\Phi(t_0,x_0) - \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\left(\gamma^{\alpha-1} - \frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}\right)\nabla\varphi b_1(z) D\Phi(t_0,x_0)\\ &\quad - \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\left[(I+\nabla\varphi)\, b_0(z) + \mathcal{A}^{\overline{\sigma}_\gamma,\nu}\varphi(z)\right] D\Phi(t_0,x_0) + o(1) \leqslant 0.\end{aligned}$$

So, from (4.3) we remark

$$\mathcal{A}^{\overline{\sigma}_\gamma,\nu^\alpha}\psi(z) = -\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\mathcal{A}^{\overline{\sigma},\nu}\varphi(z) D\Phi(t_0,x_0) + \frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}\mathcal{A}^{\overline{\sigma}_\gamma,\nu}\left[\Phi^\varepsilon(t,x) - \Phi(t,x)\right].$$

One can observe that

$$\sup_{x\in\mathbb{R}^d}\left\{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}\mathcal{A}^{\overline{\sigma}_\gamma,\nu}\left[\Phi^\varepsilon(t,x) - \Phi(t,x)\right]\right\} \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

Hence, we deduce

$$\frac{\partial D\Phi}{\partial t}(t_0,x_0) - \mathcal{H}^{\mathbf{Id}_y,\overline{\nu}_\mu} D\Phi(t_0,x_0) - \overline{\Sigma}_\varphi\cdot D\Phi(t_0,x_0) - \overline{C} \leqslant 0.$$

Let us now consider v^*. Let (t_0,x_0) be such that $\overline{v}(t_0,x_0) < 0$. Let $\Phi \leqslant v^*$ be a smooth function such that $\Phi(t_0,x_0) = \overline{v}(t_0,x_0)$ and (t_0,x_0) is a strict local maximum of $\Phi - v^*$. Consider the same perturbed function test Φ^ε as above. Hence, there exists a sequence $(t_\varepsilon,x_\varepsilon)$ that locally maximizes $\Phi^\varepsilon - v^{\varepsilon,\delta_\varepsilon}$ and converges towards (t_0,x_0). By analogy,

$$\frac{\partial D\Phi}{\partial t}(t_0,x_0) - \mathcal{H}^{\mathbf{Id}_y,\overline{\nu}_\mu} D\Phi(t_0,x_0) - \overline{\Sigma}_\varphi\cdot D\Phi(t_0,x_0) - \overline{C} \geqslant 0. \qquad \square$$

Let us now introduce some notations

$$\rho^2(t,x,y) := \inf\Big\{S_{0,t}(\varphi): \ \varphi(0)=x, \ \varphi(t)=y\Big\}, \quad \rho^2(t,x,U_0) := \inf_{y\in U_0} \rho^2(t,x,y).$$

From this, we have following.

Remark 4.3 ([19]). Let u^* and v^* be respectively the sub- and supper-viscosity solutions of PDE (4.1). Assume that for all $(t,x) \in]0,\infty[\times \mathbb{R}^d$,

$$-\rho^2(t,x,U_0) \leqslant v^*(t,x) \leqslant u^*(t,x) \leqslant \min\Big(\overline{C}t - \rho^2(t,x,U_0); 0\Big).$$

Then we have $v^* \geqslant u^*$.

Now, let $\mathcal{O}$ be an open subset in $\mathbb{R}\times\mathbb{R}^d$, define the function τ on $\mathbb{R}\times\mathcal{D}\left([0,\infty]\times\mathbb{R}^d\right)$ values into $[0,\infty]$,

$$\tau = \tau_{\mathcal{O}}(t,\phi) = \inf\{s : (t-s, \phi(s)) \in \mathcal{O}\}.$$

Take Θ the set of Markov functions τ. Let $V^*(t,x)$, $t>0, x\in\mathbb{R}^d$ be the function:

$$V^*(t,x) = \inf_{\tau\in\Theta} \sup_{\{\phi\in\mathcal{D}([0,t],\mathbb{R}^d),\phi(0)=x,\phi(t)\in U_0\}} \Big\{\overline{C}\tau - S_{0,\tau}(\phi)\Big\}.$$

Hence, we have the uniform convergence as follows.

Remark 4.4 ([19]). For $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log u^{\varepsilon,\delta_\varepsilon}(t,x) = V^*(t,x).$$

Consider the partitions $\mathcal{M}$ and $\mathcal{E}$ of $\mathbb{R}_+\times\mathbb{R}^d$:

$$\mathcal{M} = \Big\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d; V^*(t,x)=0\Big\}, \quad \mathcal{E} = \Big\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d; V^*(t,x)<0\Big\}.$$

We have following.

Theorem 4.5. *By our assumptions,*

$$\lim_{\varepsilon\downarrow 0} u^{\varepsilon,\delta_\varepsilon}(t,x) = \begin{cases} 0 & \textit{uniformly from any compact } \mathcal{K} \textit{ of } \mathcal{E}, \\ 1 & \textit{uniformly from any compact } \mathcal{K}' \textit{ of } \overset{\circ}{\mathcal{M}}. \end{cases}$$

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