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Coefficient estimates for a class of bi-univalent functions involving Mittag-Leffler type Borel distribution

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Abstract

In recent years, many new subclasses of analytic and bi-univalent functions have been studied and examined from different viewpoints and prospectives. In this article, we introduce new subclass of analytic and bi-univalent functions based on Mittag-Leffler type Borel distribution associated with the Gegenbauer polynomials. Furthermore we obtain estimates for $|a_2|$, $|a_3|$, and $|a_4|$ coefficients and Fekete-Szegö inequality for this functions class. Providing specific values to parameters involved in our main results, we get some new results.

Keywords: Analytic function, bi-univalent function, Gegenbaure polynomials, coefficient estimates, subordination, Fekete-Szegö functional problems.

2020 MSC: 30C45, 30C50, 30C80.

1. Introduction and Definitions

Let $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$ be a unit disk and let $\mathcal{A}$ denote the class of analytic functions (symmetric under rotation) of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (z \in \mathcal{U}). \tag{1.1}$$

Let $\mathcal{S} \subset \mathcal{A}$ be the class of analytic function in $\mathcal{U}$. Let s_1 and s_2 are analytic functions in open unit disc $\mathcal{U}$, then the function s_1 is subordinated to s_2 symbolically denoted as $s_1(z) \prec s_2(z)$, $z \in \mathcal{U}$, if there occur an analytic function $w(z)$ with properties that $w(0) = 0$ and $|w(z)| < 1$, such that suppose ω holomorphic in $\mathcal{U}$, with $s_1(z) = s_2(w(z))$. If the function $s_2(z)$ is univalent in $\mathcal{U}$ then above condition is equivalent to $s_1(z) \prec s_2(z) \Leftrightarrow s_1(0) = s_2(0)$ and $s_1(\mathcal{U}) \subset s_2(\mathcal{U})$.

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It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

and

$$f^{-1}(f(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqslant \frac{1}{4}\right),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent function in $\mathcal{U}$ given by (1.1). Example of functions in the class Σ are

$$\frac{z}{1-z}, \qquad \log \frac{1}{1-z}, \qquad \text{and} \quad \log \sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is a member of class Σ. Other common examples of functions in $\mathcal{U}$ are such as

$$\frac{2z-z^2}{2} \qquad \text{and} \qquad \frac{z}{1-z^2}$$

which are also not members of Σ.

Lewin [31] investigated a bi-univalent functions class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [10] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [35], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. The coefficient for each of the Taylor-Maclaurin coefficients $|a_2|$ $(n \geqslant 3,\ n \in \mathbb{N})$ is presumably still an open problem.

Similar to the familiar subclass $\mathcal{S}^*(\zeta)$ and $\mathcal{K}(\zeta)$ of starlike and convex functions of order $\zeta\,(0 \leqslant \zeta < 1)$, respectively, Brannan and Taha [11] (see also [47]) introduced certain subclasses of the bi-univalent function class Σ, $\mathcal{S}^*_\Sigma(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ of bi-starlike functions and bi-convex functions of order $\zeta\,(0 \leqslant \zeta < 1)$, respectively. For each of the function classes $\mathcal{S}^*_\Sigma(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ they found non-sharp bounds on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For some interesting functions and characterization of the class Σ, see [2, 12, 16, 19].

In 1933, Fekete and Szegö [20] obtained a sharp bound of the functional $\eta a_2^2 - a_3$, with real $\eta\ (0 \leqslant \eta < 1)$ for a univalent function f. Since the problem of finding sharp bounds for this functional of any compact family of function $f \in \mathcal{A}$ with any complex η known as the classical Fekete-Szegö problems or inequality. Some recent development for different subclasses of analytic functions on Fekete-Szegö problems can be found in [1, 22, 23, 39, 42]. Here we also passing to remark that, many well-known authors have studied different subclasses of analytic and bi-univalent functions form different aspects. For example in univalent function theory in conjunction with q-calculus the works given in [29, 36, 37, 53] are worth mentioning.

In 1903, Mittag-Leffler [33] introduced the function called Mittag-Leffler function which can be denoted by $\mathcal{E}_\mu(z)$. We have the Mittag-Leffler function define as follows:

$$\mathcal{E}_\mu(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\mu j + 1)} \quad \text{and} \quad \mathcal{E}_{\mu,\delta}(z) = \frac{1}{\Gamma(\delta)} + \sum_{j=2}^{\infty} \frac{z^{j-1}}{\Gamma(\mu(j-1)+\delta)},$$

where

$$\Re(\mu) > 0 \quad \text{and} \quad \Re(\delta) > 0 \qquad (z, \delta, \mu \in \mathbb{C}).$$

After then Wiman [51] introduced the function $\mathcal{E}_{\mu,\delta}$ which produces the one of Mittag-Leffler function and can be define as follows:

$$\mathcal{E}_{\mu,\delta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\mu j + \delta)}, \tag{1.3}$$

where

$$\Re(\mu) > 0 \quad \text{and} \quad \Re(\delta) > 0 \qquad (z, \delta, \mu \in \mathbb{C}).$$

As a special instance of the function $\mathcal{E}_{\mu,\delta}(z)$, it could be seen that it comprises a lot of well-known functions. The Mittag-Leffler function naturally appears in the solution of fractional order differential and integral equations, particularly in the study of fractional generalizations of kinetic equations, random walks, Levy flights, super diffusive transport, and complex systems [7, 8, 17, 21, 47]. The Mittag-Leffler function $\mathcal{E}_{\mu,\delta}(z)$ does not belong to the Mittag-Leffler family $\mathcal{A}$. As a result, the normalization of Mittag-Leffler functions are given as follows:

$$\mathcal{E}_{\mu,\delta}(z) = z\Gamma(\delta)\mathcal{E}_{\mu,\delta}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(\delta)}{\Gamma(\mu(j-1)+\delta)}. \tag{1.4}$$

The same is true for the complex parameters μ, δ and $z \in \mathcal{C}$. We will focus our attention in this work on the situation with real-valued μ, δ and $z \in \mathcal{C}$.

If a discrete random variable t has the values $1, 2, 3, \ldots$, it is said to have a Borel distribution with the probability

$$\frac{e^{-\gamma}}{1!}, \quad \frac{2\gamma e^{-2\gamma}}{2!}, \quad \frac{9\gamma^2 e^{-3\gamma}}{3!}, \ldots,$$

where γ is referred to as the parameter. Wanas and Khuttar [49] recently presented the Borel distribution (BD), which has a probability mass function of

$$L(t=\sigma) = \frac{e^{-\gamma\sigma}(\sigma\gamma)^{\sigma-1}}{\sigma!}, \quad \sigma = 1, 2, \ldots.$$

Wanas and Khuttar [49] introduced the $\mathcal{M}(\gamma; z)$ series, whose coefficients are Borel distribution (BD)

$$\mathcal{M}(\gamma; z) = z + \sum_{j=2}^{\infty} \frac{e^{-\gamma(j-1)}[\gamma(j-1)]^{j-2}}{(j-1)!} z^j, \qquad (0 < \gamma \leqslant 1), \tag{1.5}$$

probabilities. In 2021, Murugusundaramoorthy and El-Deeb [34] defined the Mittag-Leffler-type Borel distribution as follows:

$$\mathcal{P}(\gamma, \mu, \delta; \sigma) = \frac{(\gamma\sigma)^{\sigma-1}}{\mathcal{E}_{\mu,\delta}(z)(\gamma\sigma)\Gamma(\mu\sigma+\delta)},$$

where $\mathcal{E}_{\mu,\delta}(z)$ is given in (1.3) and $\sigma = 0, 1, 2, \ldots$. Making use of (1.4) and (1.5), we define the Mittag-Leffler-type Borel distribution series as follows:

$$\mathcal{Q}(\gamma, \mu, \delta)(z) = z + \sum_{j=2}^{\infty} \frac{e^{-\gamma(j-1)}[\gamma(j-1)]^{j-2}(\gamma(j-1))!}{(j-1)!\mathcal{E}_{\mu,\delta}(\gamma(j-1))\Gamma(\mu(j-1)+\delta)} z^j, \quad (0 < \gamma \leqslant 1).$$

$$\mathcal{Q}(\gamma, \mu, \delta)f(z) = f(z) * \mathcal{Q}(\gamma, \mu, \delta)(z) = z + \sum_{j=2}^{\infty} \frac{e^{-\gamma(j-1)}[\gamma(j-1)]^{j-2}(\gamma(j-1))!}{(j-1)!\mathcal{E}_{\mu,\delta}(\gamma(j-1))\Gamma(\mu(j-1)+\delta)} a_j z^j = z + \sum_{j=2}^{\infty} \Omega_j a_j z^j,$$

where

$$\Omega_j = \frac{e^{-\gamma(j-1)}[\gamma(j-1)]^{j-2}(\gamma(j-1))!}{(j-1)!\mathcal{E}_{\mu,\delta}(\gamma(j-1))\Gamma(\mu(j-1)+\delta)}, \tag{1.6}$$

and $\delta, \mu \in \mathcal{C}$, $\Re(\mu) > 0$, $\Re(\delta) > 0$, $0 < \gamma \leqslant 1$, and $z \in \mathcal{C}$. Gegenbauer polynomials, also known as ultra-spherical polynomials $G_j^{(\upsilon)}(t)$, are orthogonal polynomials with regard to the weight function $(1-t^2)^{\upsilon-1/2}$ on the interval $[1, 1]$. They are particular examples of Jacobi polynomials and generalize Legendre and Chebyshev polynomials. They were given the name Leopold Gegenbauer. The generating function of polynomials

$$H(t, z) = \frac{1}{(1-2tz+z^2)^{\upsilon}} = \sum_{j=0}^{\infty} G_j^{(\upsilon)}(t) z^j \tag{1.7}$$

can be used to define them. The following recurrence relation is satisfied by these polynomials:

$$G_0^{(\upsilon)}(t)=1,\quad G_1^{(\upsilon)}(t)=2\upsilon t,\quad jG_j^{(\upsilon)}(t)=2t(j+\upsilon-1)G_{j-1}^{(\upsilon)}(t)-(j+2\upsilon-2)G_{j-2}^{(\upsilon)}(t).$$

Gegenbauer polynomials are specific solutions to the differential equation:

$$(1+t^2)y''-(2\upsilon+1)ty'+j(j+2\upsilon)y=0.$$

The equation becomes the Legendre equation when $\upsilon=1/2$ and the Gegenbauer polynomials become Legendre polynomials. When $\upsilon=1$, the equation becomes a Chebyshev differential equation and the Gegenbauer polynomials become second-order Chebyshev polynomials.

It could also be seen that symmetry is a key tool in analyzing functions of several variables. For example, the harmonic homogeneous polynomials, which are invariant under the group of rotations fixing the North Pole on the unit sphere in are essentially the same as Gegenbauer polynomials.

In the framework of potential theory and harmonic analysis, the Gegenbauer polynomials naturally emerge as extensions of Legendre polynomials. In the field of mathematical physics, the Gegenbauer polynomial appears to be intriguing and relevant. Many writers have recently investigated Gegenbauer polynomials in the context of mathematical physics (see [3, 9, 13, 25, 38, 40, 43]). Many scholars have recently started investigating bi-univalent functions related to orthogonal polynomials, with a few to name [4, 5, 15, 30, 32, 48, 50].

As far as we know, there is minimal work related with bi-univalent functions in the literature for the Gegenbauer polynomial. The major objective of this work is to begin an investigation into the characteristics of bi-univalent functions linked with Gegenbauer polynomials.

We define the Mittag-Leffler-type Borel distribution subordinating with Gegenbauer polynomials as defined in (1.7), inspired on earlier work by Amourah et al. [6] and El-Deeb et al. [14]. For this subclass of bi-univalent function class Σ, we get estimates for initial Taylor-Maclaurin coefficients as well as the Fekete-Szego inequalities. In addition, we present results for novel bi-univalent function classes that we establish in this paper.

Definition 1.1 ([3])**.** Let $H(t,z)$ be defined as follows:

$$H(t,z)=1+\sum_{j=1}^{\infty}G_j^{(\upsilon)}(t)z^j.$$

A function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,\eta;\upsilon,t)$ if the following subordination condition are fulfilled:

$$e^{i\phi}\left[\frac{((1-\nu)z)^{1-\beta}((\mathcal{Q}(\gamma,\mu,\delta)f(z))'^{\eta}}{(\mathcal{Q}(\gamma,\mu,\delta)f(z)-\mathcal{Q}(\gamma,\mu,\delta)f(z\nu))^{1-\beta}}\right]\prec H(t,z)\cos\phi+i\sin\phi \tag{1.8}$$

and

$$e^{i\phi}\left[\frac{((1-\nu)\omega)^{1-\beta}((\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'^{\eta}}{(\mathcal{Q}(\gamma,\mu,\delta)g(\omega)-\mathcal{Q}(\gamma,\mu,\delta)g(\omega\nu))^{1-\beta}}\right]\prec H(t,\omega)\cos\phi+i\sin\phi, \tag{1.9}$$

where $\delta,\mu,\in\mathcal{C}$, $z,w\in\mathcal{U}$, $\Re(\mu)>0$, $\Re(\delta)>0$, $0<\gamma\leqslant 1$, $\beta\geqslant 0$, $\eta\geqslant 1$, $|\nu|\leqslant 1$ but $\nu\neq 1$, $\phi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and the function g is given by (1.2).

By choosing special values for β,ν and η, the class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,\eta;\upsilon,t)$ yields some interesting new classes given below.

Example 1.2. For $\phi=0$, we have the new class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(0,\beta,\nu,\eta;\upsilon,t)$. The class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(0,\beta,\nu,\eta;\upsilon,t)$ consists of the functions of $f\in\Sigma$ fulfilled

$$\frac{((1-\nu)z)^{1-\beta}(\mathcal{Q}(\gamma,\mu,\delta)f(z))'}{(\mathcal{Q}(\gamma,\mu,\delta)f(z)-\mathcal{Q}(\gamma,\mu,\delta)f(z\nu))^{1-\beta}}\prec H(t,z)$$

and
$$\frac{((1-\nu)\omega)^{1-\beta}(\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'}{(\mathcal{Q}(\gamma,\mu,\delta)g(\omega)-\mathcal{Q}(\gamma,\mu,\delta)g(\omega\nu))^{1-\beta}} \prec H(t,\omega).$$

Example 1.3. For $\eta = 1$, we have the new class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,1;\upsilon,t)$. The class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,1;\upsilon,t)$ consists of the functions of $f \in \Sigma$ fulfilled
$$e^{i\phi}\left[\frac{((1-\nu)z)^{1-\beta}(\mathcal{Q}(\gamma,\mu,\delta)f(z))'}{(\mathcal{Q}(\gamma,\mu,\delta)f(z)-\mathcal{Q}(\gamma,\mu,\delta)f(z\nu))^{1-\beta}}\right] \prec H(t,z)\cos\phi + i\sin\phi$$
and
$$e^{i\phi}\left[\frac{((1-\nu)\omega)^{1-\beta}(\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'}{(\mathcal{Q}(\gamma,\mu,\delta)g(\omega)-\mathcal{Q}(\gamma,\mu,\delta)g(\omega\nu))^{1-\beta}}\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$

Example 1.4. For $\beta = 0$, we have the new class
$$\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,0,\nu,\eta;\upsilon,t) = \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\nu,\eta;\upsilon,t).$$
For this class $f \in \Sigma$ if
$$e^{i\phi}\left[\frac{(1-\nu)z((\mathcal{Q}(\gamma,\mu,\delta)f(z))'^{\eta}}{\mathcal{Q}(\gamma,\mu,\delta)f(z)-\mathcal{Q}(\gamma,\mu,\delta)f(z\nu)}\right] \prec H(t,z)\cos\phi + i\sin\phi$$
and
$$e^{i\phi}\left[\frac{(1-\nu)\omega((\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'^{\eta}}{\mathcal{Q}(\gamma,\mu,\delta)g(\omega)-\mathcal{Q}(\gamma,\mu,\delta)g(\omega\nu)}\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$
Furthermore:

1. Choosing $\eta = 1$ in the class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\nu,1;\upsilon,t)$. For this class $f \in \Sigma$ if
$$e^{i\phi}\left[\frac{(1-\nu)z(\mathcal{Q}(\gamma,\mu,\delta)f(z))'}{\mathcal{Q}(\gamma,\mu,\delta)f(z)-\mathcal{Q}(\gamma,\mu,\delta)f(z\nu)}\right] \prec H(t,z)\cos\phi + i\sin\phi$$
and
$$e^{i\phi}\left[\frac{(1-\nu)\omega(\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'}{\mathcal{Q}(\gamma,\mu,\delta)g(\omega)-\mathcal{Q}(\gamma,\mu,\delta)g(\omega\nu)}\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$
2. Choosing $\eta = 1$ and fixing $\nu = -1$ in the class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,-1;\upsilon,t)$ we have new class
$$\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,-1;\upsilon,t) = \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi;\upsilon,t).$$
For this class $f \in \Sigma$ if
$$e^{i\phi}\left[\frac{2z(\mathcal{Q}(\gamma,\mu,\delta)f(z))'}{\mathcal{Q}(\gamma,\mu,\delta)f(z)-\mathcal{Q}(\gamma,\mu,\delta)f(-z)}\right] \prec H(t,z)\cos\phi + i\sin\phi$$
and
$$e^{i\phi}\left[\frac{2\omega(\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'}{\mathcal{Q}(\gamma,\mu,\delta)g(\omega)-\mathcal{Q}(\gamma,\mu,\delta)g(-\omega)}\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$
3. Choosing $\nu = 0$ in the class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,0,\eta;\upsilon,t)$ we have the new class
$$\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,0,\eta;\upsilon,t) = \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\eta;\upsilon,t).$$
For this class $f \in \Sigma$ if
$$e^{i\phi}\left[\frac{z((\mathcal{Q}(\gamma,\mu,\delta)f(z))'^{\eta}}{\mathcal{Q}(\gamma,\mu,\delta)f(z)}\right] \prec H(t,z)\cos\phi + i\sin\phi$$
and
$$e^{i\phi}\left[\frac{\omega((\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'^{\eta}}{\mathcal{Q}(\gamma,\mu,\delta)g(\omega)}\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$

4. Choosing $\eta = 1$ in the class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi, 1; \upsilon, t)$ we have the new class
$$\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi, 1; \upsilon, t) = \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi; \upsilon, t).$$
For this class $f \in \Sigma$ if
$$e^{i\phi}\left[\frac{z(\mathcal{Q}(\gamma,\mu,\delta)f(z))'}{\mathcal{Q}(\gamma,\mu,\delta)f(z)}\right] \prec H(t,z)\cos\phi + i\sin\phi$$
and
$$e^{i\phi}\left[\frac{\omega(\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'}{\mathcal{Q}(\gamma,\mu,\delta)g(\omega)}\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$

Example 1.5. For $\beta = 1$, we have the new class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\eta;\upsilon,t)$. The class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\eta;\upsilon,t)$ consists of the functions of $f \in \Sigma$ satisfying
$$e^{i\phi}\left[((\mathcal{Q}(\gamma,\mu,\delta)f(z))'^{\eta}\right] \prec H(t,z)\cos\phi + i\sin\phi \quad \text{and} \quad e^{i\phi}\left[((\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'^{\eta}\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$

1. Choosing $\eta = 1$ in the class $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\eta;\upsilon,t)$ we have the new class
$$\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi, 1; \upsilon, t) = \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi; \upsilon, t).$$
For this class $f \in \Sigma$ if
$$e^{i\phi}\left[(\mathcal{Q}(\gamma,\mu,\delta)f(z))'\right] \prec H(t,z)\cos\phi + i\sin\phi$$
and
$$e^{i\phi}\left[(\mathcal{Q}(\gamma,\mu,\delta)g(\omega))'\right] \prec H(t,\omega)\cos\phi + i\sin\phi.$$

Remark 1.6. Setting $\nu = 0$, $\eta = 1$, and $\upsilon = \frac{1}{2}$, we have the class introduced and studied by El-Deeb et al. [14].

2. Coefficients estimates

Unless otherwise mentioned, we will focus our research on real-valued μ, δ, and let $\delta, \mu, \in \mathbb{C}$, $z, w \in \mathcal{U}$, $\Re(\mu) > 0$, $\Re(\delta) > 0$, $0 < \gamma \leqslant 1$, $\beta \geqslant 0$, $\eta \geqslant 1$, $|\nu| \leqslant 1$ but $\nu \neq 1$, $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The bound for initial coefficients of functions in $\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,\eta;\upsilon,t)$ are given below.

Lemma 2.1. *If $S(z) = s_1 z + s_2 z^2 + s_3 z^3 + \cdots$, $s_1 \neq 0$ is holomorphic and satisfies $|s(z)| < 1$ on the unit disk $\mathcal{U}$, then for each $0 < r < 1$, $|s'(z)| < 1$ and $|s(re^{i\phi})| < 1$ unless $s(z) = re^{i\phi}$ for some real number ϕ.*

Theorem 2.2. *Let $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,\eta;\upsilon,t)$, $\delta, \mu, \in \mathbb{C}$, $z, w \in \mathcal{U}$, $\Re(\mu) > 0$, $\Re(\delta) > 0$, $0 < \gamma \leqslant 1$, $\beta \geqslant 0$, $\eta \geqslant 1$, $|\nu| \leqslant 1$ but $\nu \neq 1$, $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then*
$$|b_2| \leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_1] - 2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2\right|}},$$
$$|b_3| \leqslant \left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{|[(\beta-1)(1+\nu)+2\eta]\Omega_2|}\right)^2 + \frac{G_1^{(\upsilon)}(t)\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|},$$
$$|b_4| \leqslant \frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{|2[(\beta-1)(1+\nu)+2\eta][(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_2\Omega_3|} - \frac{M(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_2^3\Omega_4|}$$

$$+\frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3\Omega_2^3|}+\frac{(G_1^{(\upsilon)}(t)+2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t))\cos\phi}{|2[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4|},$$

where

$$\begin{aligned}
\Phi_1 &= (\beta-1)(1+\nu)\left[(\beta-2)(1+\nu)+4\eta\right]\Omega_2^2+4\eta(\eta-1)\Omega_2^2+(2(\beta-1)(1+\nu+\nu^2)+6\eta)\Omega_3,\\
M &= 5[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4+2\Phi_2\Omega_2\Omega_3+\Phi_3\Omega_2^3,\\
\Phi_2 &= (\beta-1)(1+\nu+\nu^2)[(\beta-2)(1+\nu)+2\eta]+3\eta[(\beta-1)(1+\nu)+2(\eta-1)],\\
\Phi_3 &= \frac{1}{3}(\beta-1)(\beta-2)(1+\nu)^2[(\beta-3)(1+\nu)+6\eta]+\frac{4}{3}\eta(\eta-1)[3(\beta-1)(1+\nu)+2(\eta-2)],
\end{aligned}$$

and the coefficients Ω_j *are as fixed in* (1.6).

Proof. Taking it from (1.8) and (1.9), we have

$$e^{i\phi}\left[\frac{((1-\nu)z)^{1-\beta}((\mathfrak{Q}(\gamma,\mu,\delta)f(z))'^{\eta}}{(\mathfrak{Q}(\gamma,\mu,\delta)f(z)-\mathfrak{Q}(\gamma,\mu,\delta)f(z\nu))^{1-\beta}}\right]\prec H(t,z)\cos\phi+i\sin\phi \tag{2.1}$$

and

$$e^{i\phi}\left[\frac{((1-\nu)\omega)^{1-\beta}((\mathfrak{Q}(\gamma,\mu,\delta)g(\omega))'^{\eta}}{(\mathfrak{Q}(\gamma,\mu,\delta)g(\omega)-\mathfrak{Q}(\gamma,\mu,\delta)g(\omega\nu))^{1-\beta}}\right]\prec H(t,\omega)\cos\phi+i\sin\phi, \tag{2.2}$$

where $H(t,z)$ and $H(t,\omega)$ are generating function for Gegenbauer polynomial with the following power series

$$H(t,z)=1+G_1^{(\upsilon)}(t)z+G_2^{(\upsilon)}(t)z^2+G_3^{(\upsilon)}(t)z^3+\cdots,\quad (z\in\mathfrak{U}),$$

and

$$H(t,\omega)=1+G_1^{(\upsilon)}(t)\omega+G_2^{(\upsilon)}(t)\omega^2+G_3^{(\upsilon)}(t)\omega^3+\cdots,\quad (\omega\in\mathfrak{U}).$$

For some holomorphic x and y such that

$$x(0)=y(0)=0\quad\text{and}\quad |x(z)|<1,|y(\omega)|<1.$$

Also we know that $|x_j|\leqslant 1$, $|y_j|\leqslant 1$ for every $z,\omega\in\mathfrak{U}$ and $j\in\mathbb{N}$. By equating the coefficients in (2.1) and (2.2), we have

$$e^{i\phi}[(\beta-1)(1+\nu)+2\eta]\Omega_2a_2=x_1G_1^{(\upsilon)}(t)\cos\phi, \tag{2.3}$$

$$\begin{aligned}
&e^{i\phi}\Bigg[\Omega_3a_3((\beta-1)(1+\nu+\nu^2)+3\eta)+\Omega_2^2a_2^2\\
&\qquad\cdot\left(\frac{1}{2}(\beta-1)(\beta-2)(1+\nu)^2+2\eta(\beta-1)(1+\nu)+2\eta(2\eta-1)\right)\Bigg]\\
&\quad=x_2G_1^{(\upsilon)}(t)\cos\phi+x_1^2G_2^{(\upsilon)}(t)\cos\phi,
\end{aligned}\tag{2.4}$$

$$\begin{aligned}
&e^{i\phi}\Bigg[\Omega_4a_4\Big((\beta-1)(1+\nu)(1+\nu^2)+4\eta\Big)+\Omega_2\Omega_3a_2a_3\Big((\beta-1)(\beta-2)(1+\nu)(1+\nu+\nu^2)\\
&\qquad+2\eta(\beta-1)(1+\nu+\nu^2)+6\eta(\eta-1)+3\eta(\beta-1)(1+\nu)\Big)\\
&\qquad+\Omega_2^3a_2^3\Big(\frac{1}{6}(\beta-1)(\beta-2)(\beta-3)(1+\nu)(1+\nu)^2+\eta(\beta-1)(\beta-2)(1+\nu)^2
\end{aligned}\tag{2.5}$$

$$
\begin{aligned}
&\quad + 2\eta(\eta-1)(\beta-1)(1+\nu) + \frac{4}{3}\eta(\eta-1)(\eta-2)\Big)\Big]\\
&= x_3 G_1^{(\upsilon)}(t)\cos\phi + 2x_1x_2G_2^{(\upsilon)}(t)\cos\phi + x_1^3G_3^{(\upsilon)}(t)\cos\phi,
\end{aligned}
$$

$$
-e^{i\phi}[(\beta-1)(1+\nu)+2\eta]\Omega_2 a_2 = y_1 G_1^{(\upsilon)}(t)\cos\phi, \tag{2.6}
$$

$$
\begin{aligned}
&e^{i\phi}\Bigg[a_2^2\Bigg(2(\beta-1)(1+\nu+\nu^2)\Omega_3 + 6\eta\Omega_3 + \Omega_2^2\Bigg(\frac{1}{2}(\beta-1)(\beta-2)(1+\nu)^2\\
&\quad + 2\eta(\beta-1)(1+\nu) + 2\eta(2\eta-1)\Bigg)\Bigg) - a_3\Omega_3((\beta-1)(1+\nu+\nu^2)+3\eta)\Bigg] \qquad (2.7)\\
&= y_2G_1^{(\upsilon)}(t)\cos\phi + y_1^2G_2^{(\upsilon)}(t)\cos\phi,
\end{aligned}
$$

and

$$
\begin{aligned}
&e^{i\phi}\Bigg[a_2a_3\Bigg((5(\beta-1)(1+\nu)(1+\nu^2)+20\eta)\Omega_4 + \Bigg((\beta-1)(\beta-2)(1+\nu)\\
&\quad\cdot(1+\nu+\nu^2) + 2\eta(\beta-1)(1+\nu+\nu^2) + 6\eta(\eta-1) + 3\eta(\beta-1)(1+\nu)\Bigg)\Omega_2\Omega_3\Bigg)\\
&\quad - a_2^3\Bigg((5(\beta-1)(1+\nu)(1+\nu^2)+20\eta)\Omega_4 + \Omega_2\Omega_3\Bigg(2(\beta-1)(\beta-2)(1+\nu)(1+\nu+\nu^2)\\
&\quad + 4\eta(\beta-1)(1+\nu+\nu^2) + 6\eta(\beta-1)(1+\nu) + 12\eta(\eta-1)\Bigg) + \Omega_2^3\Bigg(\frac{1}{6}(\beta-1) \qquad (2.8)\\
&\quad\cdot(\beta-2)(\beta-3)(1+\nu)(1+\nu)^2 + \eta(\beta-1)(\beta-2)(1+\nu)^2 + 2\eta(\eta-1)(\beta-1)(1+\nu)\\
&\quad + \frac{4}{3}\eta(\eta-1)(\eta-2)\Bigg)\Bigg) - \Omega_4a_4\Bigg((\beta-1)(1+\nu)(1+\nu^2)+4\eta\Bigg)\Bigg]\\
&= y_3G_1^{(\upsilon)}(t)\cos\phi + 2y_1y_2G_2^{(\upsilon)}(t)\cos\phi + y_1^3G_3^{(\upsilon)}(t)\cos\phi.
\end{aligned}
$$

From (2.3) and (2.6), we get

$$
x_1 = -y_1. \tag{2.9}
$$

We can as well have

$$
2e^{2i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2a_2^2 = (x_1^2+y_1^2)(G_1^{(\upsilon)}(t))^2\cos^2\phi. \tag{2.10}
$$

Now, summation of (2.4) and (2.7), we get

$$
\begin{aligned}
&e^{i\phi}\Bigg[2\Omega_2^2\Bigg(\frac{1}{2}(\beta-1)(\beta-2)(1+\nu)^2 + 2\eta(\beta-1)(1+\nu) + 2\eta(2\eta-1)\Bigg)\\
&\quad + \Bigg(2(\beta-1)(1+\nu+\nu^2)+6\eta\Bigg)\Omega_3\Bigg]a_2^2 = G_1^{(\upsilon)}(t)\cos\phi(x_2+y_2) + G_2^{(\upsilon)}(t)\cos\phi(x_1^2+y_1^2).
\end{aligned}
$$

Making use of (2.10) in the above equation, we have

$$
b_2^2 = \frac{(G_1^{(\upsilon)}(t))^3(x_2+y_2)\cos^2\phi e^{-i\phi}}{(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_1] - 2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2}. \tag{2.11}
$$

Applying Lemma 2.1, we get

$$|b_2| \leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_1]-2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2\right|}}.$$

Next, in order to find the bound on $|b_3|$, by subtracting (2.7) from (2.4), we have

$$2e^{i\phi}((\beta-1)(1+\nu+\nu^2)+3\eta)\Omega_3[b_3-b_2^2]=(x_2-y_2)G_1^{(\upsilon)}(t)\cos\phi. \tag{2.12}$$

Making use of (2.9) and (2.10) we have

$$b_3=\frac{(x_1^2+y_1^2)(G_1^{(\upsilon)}(t))^2\cos^2\phi}{2e^{2i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2}+\frac{(x_2-y_2)G_1^{(\upsilon)}(t)\cos\phi e^{-i\phi}}{2((\beta-1)(1+\nu+\nu^2)+3\eta)\Omega_3}.$$

Applying Lemma 2.1, we have

$$|b_3|\leqslant\left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{|[(\beta-1)(1+\nu)+2\eta]\Omega_2|}\right)^2+\frac{G_1^{(\upsilon)}(t)\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|}.$$

Also, to get the bound on $|b_4|$, we make use of (2.3), (2.5), (2.8) and (2.12), then we have

$$\begin{aligned}b_4=&\frac{5e^{-3i\phi}x_1(x_2-y_2)(G_1^{(\upsilon)}(t))^2\cos^2\phi}{4[(\beta-1)(1+\nu)+2\eta][(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_2\Omega_3}\\&-\frac{Me^{-3i\phi}x_1^3(G_1^{(\upsilon)}(t))^3\cos^3\phi}{2[(\beta-1)(1+\nu)+2\eta]^3[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_2^3\Omega_4}\\&+\frac{5e^{-3i\phi}x_1(x_1^2+y_1^2)(G_1^{(\upsilon)}(t))^3\cos^3\phi}{4[(\beta-1)(1+\nu)+2\eta]^3\Omega_2^3}+\frac{(x_1^3-y_1^3)G_3^{(\upsilon)}(t)\cos\phi e^{-i\phi}}{2[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4}\\&+\frac{(x_3-y_3)G_1^{(\upsilon)}(t)\cos\phi e^{-i\phi}}{2[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4}+\frac{(x_1x_2-y_1y_2)G_2^{(\upsilon)}(t)\cos\phi e^{-i\phi}}{[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4}.\end{aligned}$$

Applying Lemma 2.1, we have

$$\begin{aligned}|b_4|\leqslant&\frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{|2[(\beta-1)(1+\nu)+2\eta][(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_2\Omega_3|}\\&-\frac{M(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_2^3\Omega_4|}\\&+\frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3\Omega_2^3|}+\frac{(G_1^{(\upsilon)}(t)+2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t))\cos\phi}{|2[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4|}.\end{aligned}$$

Thus, it completes the proof. □

3. Fekete-Szegö inequalities

In this section, the Fekete-Szegö Inequalities for the defined functions classes are given.

Theorem 3.1. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,\eta;\upsilon,t)$, $\delta,\mu,\in \mathbb{C}$, $z,w \in \mathcal{U}$, $\Re(\mu) > 0$, $\Re(\delta) > 0$, $0 < \gamma \leqslant 1$, $\beta \geqslant 0$, $\eta \geqslant 1$, $|\nu| \leqslant 1$ *but* $\nu \neq 1$, $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *Then*

$$\left|b_3 - \varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|}, & |\varphi - 1| \leqq D_2, \\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_4]-2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}, & |\varphi - 1| \geqq D_2, \end{cases}$$

where

$$D_2 = \frac{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_4] - 2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}{|2(G_1^{(\upsilon)}(t))^2((\beta-1)(1+\nu+\nu^2)+3\eta)\cos\phi\Omega_3|},$$

$$\Phi_4 = (\beta-1)(1+\nu)[(\beta-2)(1+\nu)+4\eta]\Omega_2^2 + 4\eta(\eta-1)\Omega_2^2 + (2(\beta-1)(1+\nu+\nu^2)+6\eta)\Omega_3,$$

and Ω_j *are given by* (1.6).

Proof. From (2.11) and (2.12), we have

$$\begin{aligned} b_3 - \varphi b_2^2 = G_1^{(\upsilon)}(t)e^{i\phi}\cos\phi\Bigg[&\left(h(\varphi) + \frac{1}{2((\beta-1)(1+\nu+\nu^2)+3\eta)\Omega_3}\right)x_2 \\ &+ \left(h(\varphi) - \frac{1}{2((\beta-1)(1+\nu+\nu^2)+3\eta)\Omega_3}\right)y_2\Bigg], \end{aligned} \tag{3.1}$$

where

$$h(\varphi) = \frac{(1-\varphi)(G_1^{(\upsilon)}(t))^2\cos\phi}{(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_4] - 2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2}.$$

From (3.1), we have

$$\left|b_3 - \varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|}, & 0 < |h(\varphi)| < \frac{1}{|2((\beta-1)(1+\nu+\nu^2)+3\eta)\Omega_3|}, \\ 2|G_1^{(\upsilon)}(t)||h(\varphi|\cos\phi, & |h(\varphi)| \geqslant \frac{1}{|2((\beta-1)(1+\nu+\nu^2)+3\eta)\Omega_3|}, \end{cases}$$

where

$$h(\varphi) = \frac{(1-\varphi)(G_1^{(\upsilon)}(t))^2\cos\phi}{(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_4] - 2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2}.$$

Hence,

$$\left|b_3 - \varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|}, & |\varphi - 1| \leqq D_2, \\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_4]-2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}, & |\varphi - 1| \geqq D_2. \end{cases}$$

Thus, the proof of our Theorem is now completed. □

4. Corollaries and consequences

In this section, we use our major findings to derive each of the new corollaries and consequences as follow.

Corollary 4.1. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,\eta;\frac{1}{2},t)$, *then*

$$|b_2| \leqslant t\cos\phi\sqrt{\frac{2t}{|t^2\cos\phi[\Phi_5] - (3t^2-1)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}},$$

$$
\begin{aligned}
|b_3| &\leqslant \left(\frac{t\cos\phi}{|[(\beta-1)(1+\nu)+2\eta]\Omega_2|}\right)^2+\frac{t\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|},\\
|b_4| &\leqslant \frac{5t^2\cos^2\phi}{|2[(\beta-1)(1+\nu)+2\eta][(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_2\Omega_3|}\\
&\quad -\frac{M_1t^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_2^3\Omega_4|}\\
&\quad +\frac{5t^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3\Omega_2^3|}+\frac{(5t^3+6t^2-t-2)\cos\phi}{|2[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4|},\\
\left|b_3-\varphi b_2^2\right| &\leqslant \begin{cases}\frac{|t|\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|}, & |\varphi-1|\leqq D_3,\\ \frac{2t^3|1-\varphi|\cos^2\phi}{|t^2\cos\phi[\Phi_5]-(3t^2-1)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}, & |\varphi-1|\geqq D_3,\end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_5 &= (\beta-1)(1+\nu)[(\beta-2)(1+\nu)+4\eta]\Omega_2^2+4\eta(\eta-1)\Omega_2^2+(2(\beta-1)(1+\nu+\nu^2)+6\eta)\Omega_3,\\
M_1 &= 5[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4+2\Phi_6\Omega_2\Omega_3+\Phi_7\Omega_2^3,\\
\Phi_6 &= (\beta-1)(1+\nu+\nu^2)[(\beta-2)(1+\nu)+2\eta]+3\eta[(\beta-1)(1+\nu)+2(\eta-1)],\\
\Phi_7 &= \frac{1}{3}(\beta-1)(\beta-2)(1+\nu)^2[(\beta-3)(1+\nu)+6\eta]+\frac{4}{3}\eta(\eta-1)[3(\beta-1)(1+\nu)+2(\eta-2)],
\end{aligned}
$$

and

$$
D_3=\frac{|t^2\cos\phi[\Phi_5]-(3t^2-1)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}{|2t^2((\beta-1)(1+\nu+\nu^2)+3\eta)\cos\phi\Omega_3|}.
$$

Corollary 4.2. *Let* $f\in\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,\eta;1,t)$*, then*

$$
\begin{aligned}
|b_2| &\leqslant 2t\cos\phi\sqrt{\frac{2t}{|2t^2\cos\phi[\Phi_8]-(4t^2-1)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}},\\
|b_3| &\leqslant \left(\frac{2t\cos\phi}{|[(\beta-1)(1+\nu)+2\eta]\Omega_2|}\right)^2+\frac{2t\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|},\\
|b_4| &\leqslant \frac{20t^2\cos^2\phi}{|2[(\beta-1)(1+\nu)+2\eta][(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_2\Omega_3|}\\
&\quad -\frac{8M_2t^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_2^3\Omega_4|}\\
&\quad +\frac{40t^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2\eta]^3\Omega_2^3|}+\frac{(8t^3+8t^2-2t-2)\cos\phi}{|2[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4|},
\end{aligned}
$$

and

$$
\left|b_3-\varphi b_2^2\right| \leqslant \begin{cases}\frac{|2t|\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|}, & |\varphi-1|\leqq D_4,\\ \frac{16t^3|1-\varphi|\cos^2\phi}{|4t^2\cos\phi[\Phi_8]-2(4t^2-1)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}, & |\varphi-1|\geqq D_4,\end{cases}
$$

where

$$
\begin{aligned}
\Phi_8 &= (\beta-1)(1+\nu)[(\beta-2)(1+\nu)+4\eta]\Omega_2^2+4\eta(\eta-1)\Omega_2^2+(2(\beta-1)(1+\nu+\nu^2)+6\eta)\Omega_3,\\
M_2 &= 5[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4+2\Phi_9\Omega_2\Omega_3+\Phi_{10}\Omega_2^3,\\
\Phi_9 &= (\beta-1)(1+\nu+\nu^2)[(\beta-2)(1+\nu)+2\eta]+3\eta[(\beta-1)(1+\nu)+2(\eta-1)],\\
\Phi_{10} &= \frac{1}{3}(\beta-1)(\beta-2)(1+\nu)^2[(\beta-3)(1+\nu)+6\eta]+\frac{4}{3}\eta(\eta-1)[3(\beta-1)(1+\nu)+2(\eta-2)],
\end{aligned}
$$

and

$$D_4 = \frac{|4t^2\cos\phi[\Phi_8] - 2(4t^2-1)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}{|8t^2((\beta-1)(1+\nu+\nu^2)+3\eta)\cos\phi\Omega_3|}.$$

Corollary 4.3. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(0,\beta,\nu,\eta;\upsilon,t)$. *Then*

$$|b_2| \leqslant G_1^{(\upsilon)}(t)\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|(G_1^{(\upsilon)}(t))^2[\Phi_{11}] - 2G_2^{(\upsilon)}(t)[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2\right|}},$$

$$|b_3| \leqslant \left(\frac{G_1^{(\upsilon)}(t)}{|[(\beta-1)(1+\nu)+2\eta]\Omega_2|}\right)^2 + \frac{G_1^{(\upsilon)}(t)}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|},$$

$$\begin{aligned}|b_4| \leqslant{}& \frac{5(G_1^{(\upsilon)}(t))^2}{|2[(\beta-1)(1+\nu)+2\eta][(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_2\Omega_3|}\\ &- \frac{M_3(G_1^{(\upsilon)}(t))^3}{|2[(\beta-1)(1+\nu)+2\eta]^3[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_2^3\Omega_4|}\\ &+ \frac{5(G_1^{(\upsilon)}(t))^3}{|2[(\beta-1)(1+\nu)+2\eta]^3\Omega_2^3|} + \frac{G_1^{(\upsilon)}(t)+2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t)}{|2[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4|},\end{aligned}$$

$$\left|b_3 - \varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|}{|[(\beta-1)(1+\nu+\nu^2)+3\eta]\Omega_3|}, & |\varphi-1| \leqq D_5,\\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|}{|(G_1^{(\upsilon)}(t))^2[\Phi_{11}]-2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}, & |\varphi-1| \geqq D_5,\end{cases}$$

where

$$\begin{aligned}\Phi_{11} &= (\beta-1)(1+\nu)[(\beta-2)(1+\nu)+4\eta]\Omega_2^2 + 4\eta(\eta-1)\Omega_2^2 + (2(\beta-1)(1+\nu+\nu^2)+6\eta)\Omega_3,\\ M_3 &= 5[(\beta-1)(1+\nu)(1+\nu^2)+4\eta]\Omega_4 + 2\Phi_{12}\Omega_2\Omega_3 + \Phi_{13}\Omega_2^3,\\ \Phi_{12} &= (\beta-1)(1+\nu+\nu^2)[(\beta-2)(1+\nu)+2\eta] + 3\eta[(\beta-1)(1+\nu)+2(\eta-1)],\\ \Phi_{13} &= \frac{1}{3}(\beta-1)(\beta-2)(1+\nu)^2[(\beta-3)(1+\nu)+6\eta] + \frac{4}{3}\eta(\eta-1)[3(\beta-1)(1+\nu)+2(\eta-2)],\end{aligned}$$

and

$$D_5 = \frac{|(G_1^{(\upsilon)}(t))^2[\Phi_{11}] - 2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2\eta]^2\Omega_2^2|}{|2(G_1^{(\upsilon)}(t))^2((\beta-1)(1+\nu+\nu^2)+3\eta)\Omega_3|}.$$

Corollary 4.4. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,\beta,\nu,1;\upsilon,t)$. *Then*

$$|b_2| \leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{14}] - 2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2]^2\Omega_2^2\right|}},$$

$$|b_3| \leqslant \left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{|[(\beta-1)(1+\nu)+2]\Omega_2|}\right)^2 + \frac{G_1^{(\upsilon)}(t)\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3]\Omega_3|},$$

$$\begin{aligned}|b_4| \leqslant{}& \frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{|2[(\beta-1)(1+\nu)+2][(\beta-1)(1+\nu+\nu^2)+3]\Omega_2\Omega_3|}\\ &- \frac{M_4(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2]^3[(\beta-1)(1+\nu)(1+\nu^2)+4]\Omega_2^3\Omega_4|}\end{aligned}$$

$$+\frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[(\beta-1)(1+\nu)+2]^3\Omega_2^3|}+\frac{(G_1^{(\upsilon)}(t)+2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t))\cos\phi}{|2[(\beta-1)(1+\nu)(1+\nu^2)+4]\Omega_4|},$$

and

$$\left|b_3-\varphi b_2^2\right|\leqslant\begin{cases}\frac{|G_1^{(\upsilon)}(t)|\cos\phi}{|[(\beta-1)(1+\nu+\nu^2)+3]\Omega_3|}, & |\varphi-1|\leqq D_6,\\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{14}]-2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2]^2\Omega_2^2|}, & |\varphi-1|\geqq D_6,\end{cases}$$

where

$$\begin{aligned}\Phi_{14}&=(\beta-1)(1+\nu)[(\beta-2)(1+\nu)+4]\Omega_2^2+(2(\beta-1)(1+\nu+\nu^2)+6)\Omega_3,\\ M_4&=5[(\beta-1)(1+\nu)(1+\nu^2)+4]\Omega_4+2\Phi_{15}\Omega_2\Omega_3+\Phi_{16}\Omega_2^3,\\ \Phi_{15}&=(\beta-1)(1+\nu+\nu^2)[(\beta-2)(1+\nu)+2]+3[(\beta-1)(1+\nu)],\\ \Phi_{16}&=\frac{1}{3}(\beta-1)(\beta-2)(1+\nu)^2[(\beta-3)(1+\nu)+6],\end{aligned}$$

and

$$D_6=\frac{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{14}]-2G_2^{(\upsilon)}(t)e^{i\phi}[(\beta-1)(1+\nu)+2]^2\Omega_2^2|}{|2(G_1^{(\upsilon)}(t))^2((\beta-1)(1+\nu+\nu^2)+3)\cos\phi\Omega_3|}.$$

Corollary 4.5. *Let* $f\in\mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi,0,\nu,\eta;\upsilon,t)$. *Then*

$$|b_2|\leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{17}]-2G_2^{(\upsilon)}(t)e^{i\phi}[2\eta-(1+\nu)]^2\Omega_2^2\right|}},$$

$$|b_3|\leqslant\left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{|[2\eta-(1+\nu)]\Omega_2|}\right)^2+\frac{G_1^{(\upsilon)}(t)\cos\phi}{|[3\eta-(1+\nu+\nu^2)]\Omega_3|},$$

$$\begin{aligned}|b_4|\leqslant&\frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{|2[2\eta-(1+\nu)][3\eta-(1+\nu+\nu^2)]\Omega_2\Omega_3|}+\frac{(G_1^{(\upsilon)}(t)+2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t))\cos\phi}{|2[4\eta-(1+\nu)(1+\nu^2)]\Omega_4|}\\ &-\frac{M_5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[2\eta-(1+\nu)]^3[4\eta-(1+\nu)(1+\nu^2)]\Omega_2^3\Omega_4|}+\frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[2\eta-(1+\nu)]^3\Omega_2^3|},\end{aligned}$$

and

$$\left|b_3-\varphi b_2^2\right|\leqslant\begin{cases}\frac{|G_1^{(\upsilon)}(t)|\cos\phi}{|[3\eta-(1+\nu+\nu^2)]\Omega_3|}, & |\varphi-1|\leqq D_7,\\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{17}]-2G_2^{(\upsilon)}(t)e^{i\phi}[2\eta-(1+\nu)]^2\Omega_2^2|}, & |\varphi-1|\geqq D_7,\end{cases}$$

where

$$\begin{aligned}\Phi_{17}&=(1+\nu)[2(1+\nu)-4\eta]\Omega_2^2+4\eta(\eta-1)\Omega_2^2+(6\eta-2(1+\nu+\nu^2))\Omega_3,\\ M_5&=5[4\eta-(1+\nu)(1+\nu^2)]\Omega_4+2\Phi_{18}\Omega_2\Omega_3+\Phi_{19}\Omega_2^3,\\ \Phi_{18}&=(1+\nu+\nu^2)[2(1+\nu)-2\eta]+3\eta[2\eta-\nu-3],\\ \Phi_{19}&=2(1+\nu)^2[2\eta-1-\nu]+\frac{4}{3}\eta(\eta-1)[2\eta-3\nu-7],\end{aligned}$$

and

$$D_7=\frac{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{17}]-2G_2^{(\upsilon)}(t)e^{i\phi}[2\eta-(1+\nu)]^2\Omega_2^2|}{|2(G_1^{(\upsilon)}(t))^2(3\eta-(1+\nu+\nu^2))\cos\phi\Omega_3|}.$$

Corollary 4.6. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi, 0, \nu, 1; \upsilon, t)$. *Then*

$$
\begin{aligned}
|b_2| &\leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{20}] - 2G_2^{(\upsilon)}(t)e^{i\phi}[1-\nu]^2\Omega_2^2\right|}},\\
|b_3| &\leqslant \left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{|[1-\nu]\Omega_2|}\right)^2 + \frac{G_1^{(\upsilon)}(t)\cos\phi}{|[2-\nu-\nu^2]\Omega_3|},\\
|b_4| &\leqslant \frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{|2[1-\nu][2-\nu-\nu^2]\Omega_2\Omega_3|} - \frac{M_6(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[1-\nu]^3[4-(1+\nu)(1+\nu^2)]\Omega_2^3\Omega_4|}\\
&\quad + \frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{|2[1-\nu]^3\Omega_2^3|} + \frac{(G_1^{(\upsilon)}(t) + 2G_2^{(\upsilon)}(t) + G_3^{(\upsilon)}(t))\cos\phi}{|2[4-(1+\nu)(1+\nu^2)]\Omega_4|}
\end{aligned}
$$

and

$$
\left|b_3 - \varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|\cos\phi}{|[2-\nu-\nu^2]\Omega_3|}, & |\varphi - 1| \leqq D_8,\\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{20}] - 2G_2^{(\upsilon)}(t)e^{i\phi}[1-\nu]^2\Omega_2^2|}, & |\varphi - 1| \geqq D_8, \end{cases}
$$

where

$$
\begin{aligned}
\Phi_{20} &= 2(1+\nu)[\nu-1]\Omega_2^2 + (4-\nu-\nu^2)\Omega_3,\\
M_6 &= 5[4-(1+\nu)(1+\nu^2)]\Omega_4 + 2\Phi_{21}\Omega_2\Omega_3 + \Phi_{22}\Omega_2^3,\\
\Phi_{21} &= 2\nu(1+\nu+\nu^2) - 3\eta[\nu+1],\\
\Phi_{22} &= 2(1+\nu)^2[1-\nu],\\
D_8 &= \frac{|(G_1^{(\upsilon)}(t))^2\cos\phi[\Phi_{20}] - 2G_2^{(\upsilon)}(t)e^{i\phi}[1-\nu]^2\Omega_2^2|}{|2(G_1^{(\upsilon)}(t))^2(2-\nu-\nu^2)\cos\phi\Omega_3|}.
\end{aligned}
$$

Corollary 4.7. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi, 0, -1, 1; \upsilon, t)$. *Then*

$$
\begin{aligned}
|b_2| &\leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|4(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3 - 8G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2\right|}},\\
|b_3| &\leqslant \left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{2\Omega_2}\right)^2 + \frac{G_1^{(\upsilon)}(t)\cos\phi}{2\Omega_3},\\
|b_4| &\leqslant \frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{16\Omega_2\Omega_3} + \frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{16\Omega_2^3} - \frac{(20\Omega_4 - 4\Omega_2\Omega_3)(G_1^{(\upsilon)}(t))^3\cos^3\phi}{64\Omega_2^3\Omega_4}\\
&\quad + \frac{(G_1^{(\upsilon)}(t) + 2G_2^{(\upsilon)}(t) + G_3^{(\upsilon)}(t))\cos\phi}{8\Omega_4},
\end{aligned}
$$

and

$$
\left|b_3 - \varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|\cos\phi}{2\Omega_3}, & |\varphi - 1| \leqq D_9,\\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|4(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3 - 8G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2|}, & |\varphi - 1| \geqq D_9, \end{cases}
$$

where

$$
D_9 = \frac{|4(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3 - 8G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2|}{|4(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3|}.
$$

Corollary 4.8. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi, 0, 0, 1; \upsilon, t)$. *Then*

$$|b_2| \leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|(G_1^{(\upsilon)}(t))^2\cos\phi[2\Omega_2^2-4\Omega_3]-2G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2\right|}},$$

$$|b_3| \leqslant \left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{\Omega_2}\right)^2+\frac{G_1^{(\upsilon)}(t)\cos\phi}{2\Omega_3},$$

$$|b_4| \leqslant \frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{4\Omega_2\Omega_3}+\frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{2\Omega_2^3} - \frac{(15\Omega_4-6\Omega_2\Omega_3+2\Omega_2^3)(G_1^{(\upsilon)}(t))^3\cos^3\phi}{6\Omega_2^3\Omega_4}+\frac{(G_1^{(\upsilon)}(t)+2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t))\cos\phi}{6\Omega_4},$$

and

$$\left|b_3-\varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|\cos\phi}{2\Omega_3}, & |\varphi-1| \leqq D_{10}, \\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|(G_1^{(\upsilon)}(t))^2\cos\phi[2\Omega_2^2-4\Omega_3]-2G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2|}, & |\varphi-1| \geqq D_{10}, \end{cases}$$

where

$$D_{10} = \frac{|(G_1^{(\upsilon)}(t))^2\cos\phi[2\Omega_2^2-4\Omega_3]-2G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2|}{|4(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3|}.$$

Corollary 4.9. *Let* $f \in \mathcal{W}^{\Sigma}_{\gamma,\mu,\delta}(\phi, \beta, \nu, \eta; \upsilon, t)$. *Then*

$$|b_2| \leqslant G_1^{(\upsilon)}(t)\cos\phi\sqrt{\frac{2G_1^{(\upsilon)}(t)}{\left|6(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3-8G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2\right|}},$$

$$|b_3| \leqslant \left(\frac{G_1^{(\upsilon)}(t)\cos\phi}{2\Omega_2}\right)^2+\frac{G_1^{(\upsilon)}(t)\cos\phi}{3\Omega_3},$$

$$|b_4| \leqslant \frac{5(G_1^{(\upsilon)}(t))^2\cos^2\phi}{12\Omega_2\Omega_3}-\frac{20(G_1^{(\upsilon)}(t))^3\cos^3\phi}{64\Omega_2^3}+\frac{5(G_1^{(\upsilon)}(t))^3\cos^3\phi}{16\Omega_2^3} + \frac{(G_1^{(\upsilon)}(t)+2G_2^{(\upsilon)}(t)+G_3^{(\upsilon)}(t))\cos\phi}{8\Omega_4},$$

and

$$\left|b_3-\varphi b_2^2\right| \leqslant \begin{cases} \frac{|G_1^{(\upsilon)}(t)|\cos\phi}{|3\Omega_3|}, & |\varphi-1| \leqq D_{11}, \\ \frac{2(G_1^{(\upsilon)}(t))^3|1-\varphi|\cos^2\phi}{|6(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3-8G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2|}, & |\varphi-1| \geqq D_{11}, \end{cases}$$

where

$$D_{11} = \frac{|6(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3-8G_2^{(\upsilon)}(t)e^{i\phi}\Omega_2^2|}{|6(G_1^{(\upsilon)}(t))^2\cos\phi\Omega_3|}.$$

Remark 4.10. Choosing $\nu = 0$, $\eta = 1$, and $\upsilon = \frac{1}{2}$ in Theorem 2.2, we get the results of El-Deeb et al. [14].

5. Conclusion

The special functions are particularly applicable in many diverse areas of mathematics and other sciences. In Geometric Function Theory, the usage of specials functions are quite significant. Here in our present investigation we are motivated by the recent research going on and have defined certain new subclasses of analytic and bi-univalent functions linked with Gegenbauer polynomial. We have then obtained some useful results like estimates for the first two Taylor-Maclaurin coefficients and the Fekete-Szegö functional problems for each of our defined function classes. We have showed connections of our main results to those a number of earlier and new works.

In concluding our present investigation, we draw the attention of the interested readers toward the prospect of studying the basic or quantum (or $\mathfrak{q}$-) generalizations of the results which we have developed in this paper. This direction of research is indeed influenced and motivated by a recently-published survey-cum-expository review article by Srivastava [44] (see also [24, 26–28, 41, 46]). However, as already demonstrated by Srivastava (see [44, p. 340] and [45, Section 5, pp. 1511–1512]), the $(\mathfrak{p}, \mathfrak{q})$-variations of the proposed $\mathfrak{q}$-results will lead trivially to inconsequential research, because the forced-in parameter $\mathfrak{p}$ is obviously redundant. Furthermore, in light of Srivastava's more recent expository article [45], the interested readers should be advised *not* to be misled to believe that the so-called k-Gamma function provides a "generalization" of the classical (Euler's) Gamma function. Similar remarks will apply also to all of the usages of the so-called k-Gamma function including (for example) the so-called (k, s)-extensions of the Riemann-Liouville and other operators of fractional integral and fractional derivatives.

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