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Third Hankel determinant for q-analogue of symmetric starlike connected to q-exponential function

Yasir Hamayun[a], Niamat Ullah[a], Rashid Khan[a], Khurshid Ahmad[a], Muhammad Ghaffar Khan[b,*], Bilal Khan[c,*]

[a] *Government Post Graduate College Dargai, Pakistan.*

[b] *Institute of Numerical Sciences, Kohat university of science and technology, Kohat, Pakistan.*

[c] *School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, Peoples Republic of China.*

Abstract

By making use of the concept of basic (or q-) calculus, a subclass of q-starlike functions with reference to symmetric points, which is associated with the q-exponential function, is introduced in the open unit disc. Further, we derived upper bounds for the third-order Hankel determinant for the defined class. For the validity of our results, relevant connections with those in earlier works are also pointed out.

Keywords: Analytic function, subordination, differential operator, exponential function, Fekete-Szegö functional problems.

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1. Introduction

Let $\mathcal{A}$ indicate the family of all analytic mappings $\Lambda(\omega)$ of the form

$$\Lambda(\omega) = \omega + \sum_{j=2}^{\infty} \sigma_j \omega^j, \tag{1.1}$$

defined on $\mathbb{U}$. By the symbol $\mathcal{S}$, we denote the subfamily of $\mathcal{A}$, which consists of the univalent mappings. Further, for any two given analytic functions, namely Λ_1 and Λ_2, we call Λ_1 is subordinate to Λ_2 if there occurs an analytic mapping $w(\omega)$ that fulfills the conditions

$$w(\omega) = 0 \quad \text{and} \quad |w(\omega)| < 1,$$

such that

$$\Lambda_1(\omega) = \Lambda_2(w(\omega)).$$

*Corresponding author

Email addresses: yasirmaths138@gmail.com (Yasir Hamayun), niamatullahnmt817@gmail.com (Niamat Ullah), rsd.khan9676@gmail.com (Rashid Khan), khurshidahmad410@gmail.com (Khurshid Ahmad), ghaffarkhan020@gmail.com (Muhammad Ghaffar Khan), bilalmaths789@gmail.com (Bilal Khan)

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The mapping $f \in \mathcal{S}$ is a starlike mapping if it fulfills

$$\Re\frac{\omega\Lambda'(\omega)}{\Lambda(\omega)} > 0 \qquad (\omega \in \mathbb{U}).$$

We identify this family with $\mathcal{S}^*$. In the past, many authors introduced and studied numerous subfamilies of family $\mathcal{S}^*$, we refer the reader to see ([24, 25]). Likewise, Srivastava et al. [28] introduced the family $\mathcal{S}^*_{\exp}$, defined as

$$\mathcal{S}^*_{\exp} := \left\{\Lambda \in \mathcal{A} : \frac{\omega\Lambda'(\omega)}{\Lambda(\omega)} \prec e^{\omega} \quad (\omega \in \mathbb{U})\right\}.$$

This function class $\mathcal{S}^*_{\exp}$ was investigated further by Zhang et al. [35].

Now let's review some crucial definitions and concepts of the q-calculus, that are helpful in our research. Throughout the paper, we hypothesise that $0 < q < 1$ and

$$\mathbb{N} := \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}.$$

Definition 1.1 ([7, 8])**.** Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^{\lambda}}{1-q}, & \lambda \in \mathbb{C}, \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \dots + q^{n-1}, & \lambda = n \in \mathbb{N}. \end{cases}$$

Definition 1.2 ([7, 8])**.** Let $q \in (0, 1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n-1} [k]_q, & n \in \mathbb{N}. \end{cases}$$

Definition 1.3 ([7, 8])**.** In terms of q-numbers, the Jackson q-exponential function e_q^{ω} is defined by

$$e_q^{\omega} = \sum_{n=0}^{\infty} \frac{\omega^n}{[n]_q!}.$$

Definition 1.4 ([7, 8])**.** The q-derivative (or q-difference) of a function f of the form (1.1) is denoted by D_q and defined in a given subset of $\mathbb{C}$ by

$$D_q f(\omega) = \begin{cases} \frac{f(\omega) - f(q\omega)}{(1-q)\omega}, & \omega \neq 0, \\ f'(0), & \omega = 0. \end{cases}$$

When $q \longrightarrow 1-$, the difference operator D_q approaches to the ordinary differential operator. That is

$$\lim_{q \longrightarrow 1^-} (D_q f)(\omega) = f'(\omega).$$

In the study of complex analysis known as geometric function theory, the operator D_q plays a very important part in this theory. Ismail et al. [6] was the first to use the operator D_q and give the q-extension of the class of starlike functions in a systematic way. But historically, Srivastava introduced the q-calculus's fundamental use in the framework of geometric function theory and also used the basic (or q-) hypergeometric functions in a book chapter (see [29, pp. 347 et seq.]). Srivastava's papers [30, 31] further motivate the use of the q-analysis in geometric function theory, as well as expose the triviality of the so-called $(\mathfrak{p}, q)$-analysis involving an insignificant and redundant parameter $\mathfrak{p}$ (see [30, 31]). Motivated by the works of Srivastava [30, 31], many well-known authors have contributed to the development of this field. For example, some higher-order q-derivatives have been used to define certain new subclasses of multivalent functions, and some of their entrusting properties have been studied in [2, 13, 26]. Also, more recently, some subclasses of analytic and bi-univalent functions that also involved certain q-polynomials have been studied by many authors; see for example [5, 9]. For some recent studies on q-polynomials, conjugations with the analytic and bi-univalent functions can be found in [1, 10–12, 14, 15, 27, 34]. Recently, Srivastava et al. [32] (see also [16]) used the q-derivative operator to define a new subclass of starlike functions related to the q-exponential function, given as

$$\mathcal{S}^*_{\exp}(q) := \left\{ f \in \mathcal{A} : \frac{\omega(D_q f)(\omega)}{f(\omega)} \prec e_q^{\omega}, \quad \omega \in \mathbb{U} \right\}.$$

As of right now, we're introducing a new subclass of the class $\mathcal{S}^*_{\exp}(q)$ of q-starlike functions related to the exponential function e_q^{ω} in the following way.

Definition 1.5. A function $f \in \mathcal{S}$ is a member of functions class $\mathcal{S}^*_{\exp}(q)$, if it satisfies the criteria

$$\frac{2\omega D_q f(\omega)}{f(\omega) - f(-\omega)} \prec e_q^{\omega} \quad (\omega \in \mathbb{U}).$$

Remark 1.6. We see that

$$\lim_{q \to 1-} \mathcal{S}^*_{\exp}(q) = \mathcal{S}^*_{\exp},$$

where $\mathcal{S}^*_{\exp}$ is the familiar function class that is introduced recently by Mendiratta et al. [19] and further investigated by Zhang et al. [35].

The $\tilde{q}^{\text{th}}$ order Hankel determinant for the family of mappings $\Lambda \in \mathcal{A}$, for $q \geqslant 1$ and $j \geqslant 1$ is defined as follows (see [21]):

$$H_{\tilde{q}}(j) = \begin{vmatrix} \sigma_j & \sigma_{j+1} & \cdot & \cdot & \cdot & \sigma_{j+\tilde{q}-1} \\ \sigma_{j+1} & \sigma_{j+2} & & & & \sigma_{j+\tilde{q}} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \sigma_{j+\tilde{q}-1} & \sigma_{j+q} & \cdot & \cdot & \cdot & \sigma_{j+2(\tilde{q}-1)} \end{vmatrix}.$$

The above defined determent has been investigated and studied by many authors and is an important rule in many areas of mathematical sciences and related subjects. For example, in [20], for the functions class S the author Noor given and obtained the rate of growth of $H_{\tilde{q}}(n)$ as $n \longrightarrow \infty$. For many other different choices of the parameters $\tilde{q}$ and n, one can get some more incredible outputs, for example, the Fekete-Szegö functional $H_2(1) := \sigma_3 - \sigma_2^2$ can be get if we choose $\tilde{q} = 2$ and $n = 1$. Furthermore, the generalized Fekete-Szegö functional is given by $\sigma_3 - \mu\sigma_2^2$, where μ is either real or comlex. For $\tilde{q} = n = 2$, we have second order Hankel deerminant $H_2(2) := \sigma_2 \, \sigma_4 - \sigma_3^2$. Also, another type of second order Hankel deerminant is obtained by taking $\tilde{q} = 2$ and $n = 3$, mathematically written as $H_2(3) := \sigma_3 \, \sigma_5 - \sigma_4^2$. The estimations of the sharp bound for these $H_{\tilde{q}}(n)$ is obtained by many authors for various sub-classes of A ([3, 23, 33]). If we set $\tilde{q} = 3$ and $n = 1$ we get the $|H_3(1)|$, the third order Hankel deerminant, which is given by

$$H_3(1) = \sigma_3(\sigma_2\sigma_4 - \sigma_3^2) - \sigma_4(\sigma_4 - \sigma_2\sigma_3) + \sigma_5(\sigma_3 - \sigma_2^2).$$

2. Preliminary results

Each of the following lemma will be needed in our present investigation.

Lemma 2.1 ([17, 18]). *If* $p(\omega) = 1 + p_1\omega + p_2\omega^2 + \cdots \in \mathcal{P}$, *then* $2p_2 = p_1^2 + x\left(4 - p_1^2\right)$ *for some* x, $|x| \leqslant 1$, *and*

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right) p_1 x - \left(4 - p_1^2\right) p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)\omega,$$

for some ω, $|\omega| \leqslant 1$.

Lemma 2.2 ([4]). *Let* $p \in \mathcal{P}$, *then*

$$|p_n| \leqslant 2 \quad (n = 1, 2, 3, \ldots)$$

and the inequality is sharp.

Lemma 2.3 ([22]). *If* $p \in \mathcal{P}$, *then*

$$\left|p_2 - \nu p_1^2\right| \leqslant 2 \max\{1, |2\nu - 1|\}.$$

3. Main results

We begin with solving the first Hankel determinant $H_2(1)$ for the functions association with the family $\Lambda \in \mathcal{S}^*_{\exp}(q)$.

Theorem 3.1. *If* $\Lambda \in \mathcal{S}^*_{\exp}(q)$ *and have the form* (1.1), *then*

$$\left|\sigma_3 - \sigma_2^2\right| \leqslant \frac{1}{q(q+1)}.$$

Proof. If $\Lambda \in \mathcal{S}^*_{\exp}(q)$, then it follows from definition that

$$\frac{2\omega\left(D_q\Lambda\right)(\omega)}{\Lambda(\omega) - \Lambda(-\omega)} \prec e_q^{\omega}. \tag{3.1}$$

Define a function

$$p(\omega) = \frac{1 + w(\omega)}{1 - w(\omega)} = 1 + p_1\omega + p_2\omega^2 + \cdots.$$

It is clear that $p \in \mathcal{P}$. Then

$$w(\omega) = \frac{p(\omega) - 1}{p(\omega) + 1}.$$

Now from (3.1) we have

$$\frac{2\omega\left(D_q\Lambda\right)(\omega)}{\Lambda(\omega) - \Lambda(-\omega)} = e_q^{w(\omega)}.$$

Now

$$\begin{aligned}
e_q^{w(\omega)} &= 1 + \frac{w(\omega)}{[1]_q!} + \frac{(w(\omega))^2}{[2]_q!} + \frac{(w(\omega))^3}{[3]_q!} + \cdots \\
&= 1 + \frac{1}{2}p_1\omega + \left[\frac{p_2}{2} - \frac{qp_1^2}{4(1+q)}\right]\omega^2 + \left[\frac{p_3}{2} + \frac{qp_1p_2}{2(1+q)} + \frac{q^3p_1^3}{8(1+q)(1+q+q^2)}\right]\omega^3 \\
&\quad + \frac{1}{16(1+q)^2(1+q+q^2)(1+q^2)}[8p_4(q^6 + 3q^5 + 5q^4 + 6q^3 + 5q^2 + 1) \\
&\quad + 6p_1^2p_2(q^6 + q^5 + q^4 + q^3) - 8p_1p_3(q^6 + 2q^5 + 3q^3 + 2q^2 + q) \\
&\quad - 4p_2^2(q^6 + 2q^5 + 3q^4 + 3q^3 + 2q^2 + q) + p_1^4(q^4 - q^6 + q^2)]\omega^4 + \ldots.
\end{aligned} \tag{3.2}$$

On the other hand

$$\begin{aligned}
\frac{2\omega\left(D_q\Lambda\right)(\omega)}{\Lambda(\omega) - \Lambda(-\omega)} &= 1 + (1+q)\sigma_2\omega + q(1+q)\sigma_3\omega^2 + [(1+q+q^2+q^3)\sigma_4 - (1+q)\sigma_2\sigma_3]\omega^3 \\
&\quad + [q(1+q+q^2+q^3)\sigma_5 - q(1+q)\sigma_3^2]\omega^4 + \cdots.
\end{aligned} \tag{3.3}$$

Equating the coefficients of like powers of ω, ω^2, and ω^3 of equations (3.2) and (3.3), respectively, we have

$$\sigma_2 = \frac{1}{2(1+q)}p_1, \tag{3.4}$$

$$\sigma_3 = \frac{1}{2q(1+q)}p_2 - \frac{1}{4(1+q)^2}p_1^2, \tag{3.5}$$

$$\begin{aligned}\sigma_4 &= \frac{1}{2(1+q+q^2+q^3)}p_3 + \frac{(1-2q^2)}{4q(1+q)(1+q+q^2+q^3)}p_1p_2 \\ &+ \frac{q^3(1+q)-(1+q+q^2)}{8(1+q)^2(1+q+q^2)(1+q+q^2+q^3)}p_1^3.\end{aligned} \tag{3.6}$$

$$\begin{aligned}\sigma_5 = [&\frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{2q(1+q^2)(1+q)^2(1+q+q^2)(1+q+q^2+q^3)}p_4 \\ &- \frac{(q^6+2q^5+3q^4+3q^3+2q^2+q)}{2q(1+q^2)(1+q)^2(1+q+q^2)(1+q+q^2+q^3)}p_1p_3 \\ &+ \frac{(q+3q^2+2q^3+2q^4+2q^5-q^6-q^7)}{16(1+q^2)(1+q)^2(1+q+q^2)(1+q+q^2+q^3)}p_1^4].\end{aligned}$$

Since from (3.4) and (3.5), we have

$$\begin{aligned}\left|\sigma_3-\sigma_2^2\right| &= \left|\frac{1}{2q(1+q)}p_2 - \frac{1}{4(1+q)^2}p_1^2 - \frac{1}{4(1+q)^2}p_1^2\right| \\ &= \left|\frac{1}{2q(1+q)}p_2 - \frac{1}{2q(1+q)^2}p_1^2\right| = \frac{1}{2q(1+q)}\left|p_2 - \frac{q}{(1+q)}p_1^2\right|,\end{aligned}$$

using Lemma 2.3, we thus know that

$$\left|\sigma_3-\sigma_2^2\right| \leqslant \frac{1}{2q(1+q)}2\max\{1\} = \frac{1}{q(1+q)}, \qquad \left|\sigma_3-\sigma_2^2\right| \leqslant \frac{1}{q(1+q)}.$$

Hence the desired result is achieved. □

Theorem 3.2. *If $\Lambda \in \mathcal{S}^*_{\exp}(q)$ and have the form (1.1), then we have*

$$|\sigma_2\sigma_3-\sigma_4| \leqslant \frac{\mathcal{E}_1(q)+\mathcal{E}_2(q)+\mathcal{E}_3(q)+\mathcal{E}_4(q)+\mathcal{O}(q)\sqrt{\tau(q)}}{729(1+q)^3(q^2+1)(q^3+2q^2+\frac{5}{3}q+1)^3(q^4-2q^3-6q-6q-4)},$$

where

$$\begin{aligned}\tau(q) &= \frac{\begin{pmatrix}(q2+q+1)(q^{11}+2q^{10}-7q^9-37q^8-80q^7-188q^6 \\ -420q^5-652q^4-724q^3-553q^2-270q-81)(1+q)^3\end{pmatrix}}{q^4-2q^3-6q^2-6q-4}, \\ \mathcal{E}_1(q) &= -4q^{21}-16q^{20}+22q^{19}+264q^{18}+646q^{17}+370q^{16}, \\ \mathcal{E}_2(q) &= -849q^{15}+999q^{14}+10918q^{13}+13919q^{12}-44002q^{11}, \\ \mathcal{E}_3(q) &= -230267q^{10}-551188q^9-900053q^8-1107344q^7, \\ \mathcal{E}_4(q) &= -1066190q^6-813504q^5-490037q^4-228328q^3-78660q^2-18225q-2187, \\ \mathcal{O}(q) &= 4q^{15}-50q^{13}-92q^{12}+56q^{11}+558q^{10}+770q^9 \\ &\quad -1506q^8-7950q^7-16956q^6-23602q^5-23462q^4-17168q^3-9056q^2-3204q-648.\end{aligned} \tag{3.7}$$

Proof. From (3.3), (3.4), and (3.6), and using values of σ_2, σ_3, and σ_4, we have

$$|\sigma_2\sigma_3-\sigma_4| = \left|\frac{3q^2(1+q)p_1p_2}{4q(1+q)^2(1+q+q^2+q^3)} - \frac{p_3}{2(1+q+q^2)} - \frac{q^2(2+2q+q^2)p_1^3}{8(1+q)(1+q+q^2)(1+q+q^2+q^3)}\right|.$$

Substituting for p_2 and p_3 from Lemma 2.1, we obtain

$$|\sigma_2\sigma_3-\sigma_4| = \left|\frac{(4-p_1^2)p_1x^2}{8(1+q+q^2+q^3)} - \frac{q(2+q-q^2)(4-p_1^2)p_1x}{8q(1+q)^2(1+q+q^2+q^3)} - \frac{(4-p_1^2)(1-|x|^2)\omega}{4(1+q+q^2+q^3)} + \frac{(q^5+2q^4+q^3-q-1)p_1^3}{8(1+q)^3(1+q+q^2)(1+q+q^2+q^3)}\right|.$$

Assume that $|x| = t \in [0,1]$, $p_1 = p \in [0,2]$. Then, using the triangle inequality, we deduce that

$$|\sigma_2\sigma_3-\sigma_4| \leqslant: \frac{(4-p^2)pt^2}{8(1+q+q^2+q^3)} + \frac{(2+q-q^2)(4-p^2)pt}{8(1+q)^2(1+q+q^2+q^3)} + \frac{(4-p^2)}{4(1+q+q^2+q^3)} + \frac{(q^5+2q^4+q^3-q-1)p^3}{8(1+q)^3(1+q+q^2)(1+q+q^2+q^3)}.$$

Now define

$$F_q(p,t) =: \frac{(4-p^2)pt^2}{8(1+q+q^2+q^3)} + \frac{(2+q-q^2)(4-p^2)pt}{8(1+q)^2(1+q+q^2+q^3)} + \frac{(4-p^2)}{4(1+q+q^2+q^3)} + \frac{(q^5+2q^4+q^3-q-1)p^3}{8(1+q)^3(1+q+q^2)(1+q+q^2+q^3)}.$$

Differentiating $F_q(p,t)$, we have

$$\frac{\partial F_q}{\partial t} = \frac{(4-p^2)pt}{4(1+q+q^2+q^3)} + \frac{(2+q-q^2)(4-p^2)p}{8(1+q)^2(1+q+q^2+q^3)}$$

and with elementary calculus it can be easily shown that $\frac{\partial F_q}{\partial t} \geqslant 0$, for $t > 0$; implying that $F_q(p,t)$ is increasing function on the close interval $[0,1]$ about t. Thus the $F_q(p,t)$ obtained maximum value at $t = 1$, which is

$$\max_{0\leqslant t\leqslant 1} F_q(p,t) = F_q(p,1) = \frac{(4-p^2)p}{8(1+q+q^2+q^3)} + \frac{(2+q-q^2)(4-p^2)p}{8(1+q)^2(1+q+q^2+q^3)} + \frac{(4-p^2)}{4(1+q+q^2+q^3)} + \frac{(q^5+2q^4+q^3-q-1)p^3}{8(1+q)^3(1+q+q^2)(1+q+q^2+q^3)}.$$

Setting

$$G_q(p) = \frac{(4-p^2)p}{8(1+q+q^2+q^3)} + \frac{(2+q-q^2)(4-p^2)p}{8(1+q)^2(1+q+q^2+q^3)} + \frac{(4-p^2)}{4(1+q+q^2+q^3)} + \frac{(q^5+2q^4+q^3-q-1)p^3}{8(1+q)^3(1+q+q^2)(1+q+q^2+q^3)},$$

thus

$$G_q'(p) = \frac{3(4-p^2)}{8(1+q)(1+q+q^2+q^3)} - \frac{p^2}{(1+q)(1+q+q^2+q^3)} - \frac{p}{(1+q+q^2+q^3)} + \frac{3(q^5+2q^4+q^3-q-1)p^3}{8(1+q)^3(1+q+q^2)(1+q+q^2+q^3)}.$$

Let $G_q'(p) = 0$, then root

$$p = r = \frac{-(2q^6+8q^5+14q^4+14q^3+8q^2+2q)+2\sqrt{\tau(q)}}{9q^4+27q^3+33q^2+24q+9},$$

where $\tau(q)$ is given by (3.7). Consequently the function $G_q(p)$ at

$$p = r = \frac{-(2q^6 + 8q^5 + 14q^4 + 14q^3 + 8q^2 + 2q) + 2\sqrt{\tau(q)}}{9q^4 + 27q^3 + 33q^2 + 24q + 9}$$

attains its maximum value, which is

$$|\sigma_2\sigma_3 - \sigma_4| \leqslant G_q(r) = \frac{\mathcal{E}_1(q) + \mathcal{E}_2(q) + \mathcal{E}_3(q) + \mathcal{E}_4(q) + \mathcal{O}(q)\sqrt{\tau(q)}}{729(1+q)^3(q^2+1)(q^3+2q^2+\frac{5}{3}q+1)^3(q^4-2q^3-6q-6q-4)}.$$

This completes the required proof Theorem 3.2. □

Theorem 3.3. *If* $\Lambda \in \mathcal{S}^*_{\exp}(q)$ *and have the form* (1.1), *then we have*

$$\left|\sigma_2\sigma_4 - \sigma_3^2\right| \leqslant \frac{1 + 2q + 2q^2 + q^3}{q^2(1+q)^2(1+q+q^2)}.$$

Proof. Let $\Lambda \in \mathcal{S}^*_{\exp}(q)$, then form (3.3), (3.4), and (3.6), we have

$$\left|\sigma_2\sigma_4 - \sigma_3^2\right| = \frac{p_1p_3}{4(1+q)(1+q+q^2+q^3)} + \frac{(1-2q^2)p_1^2p_2}{8q(1+q)^2(1+q+q^2+q^3)} + \frac{[q^3(1+q)-(1+q+q^2)]p_1^4}{16q(1+q)^3(1+q+q^2)(1+q+q^2+q^3)} - \frac{p_2^2}{4q^2(1+q)^2} + \frac{2p_2p_1^2}{8q(1+q)^3} - \frac{p_1^4}{8q(1+q)^4}\Bigg|.$$

Substituting for p_2 and p_3 and using Lemma 2.1, we obtain

$$\begin{aligned}\left|\sigma_2\sigma_4 - \sigma_3^2\right| = \Bigg|&\frac{q(1-q+2q^3)(4-p_1^2)p_1^2x}{16q^2(1+q)^3(1+q+q^2+q^3)} + \frac{(4-p_1^2)p_1^2x^2}{16(1+q)(1+q+q^2+q^3)} + \frac{(4-p_1^2)^2x^2}{16q^2(1+q)^2}\\ &+\frac{(4-p_1^2)(1-|x|^2)p_1\omega}{8(1+q)(1+q+q^2+q^3)} + \frac{(4q^7+9q^6+7q^5-5q^3-6q^2-5q-2)p_1^4}{16q^2(1+q)^4(1+q+q^2)(1+q+q^2+q^3)}\Bigg|.\end{aligned} \tag{3.8}$$

We let $p_1 = p$ and assume also without restriction that $p \in [0, 2]$. Thus implementation of the triangle inequality on (3.8), with $|x| = t \in [0, 1]$, we obtain

$$\begin{aligned}\left|\sigma_2\sigma_4 - \sigma_3^2\right| \leqslant \Bigg|&\frac{q(1-q+2q^3)(4-p^2)p^2t}{16q^2(1+q)^3(1+q+q^2+q^3)} + \frac{(4-p^2)p^2t^2}{16(1+q)(1+q+q^2+q^3)}\\ &+\frac{(4-p^2)^2t^2}{16q^2(1+q)^2} + \frac{(4-p^2)(1-|x|^2)p\omega}{8(1+q)(1+q+q^2+q^3)} + \frac{(4q^7+9q^6+7q^5-5q^3-6q^2-5q-2)p^4}{16q^2(1+q)^4(1+q+q^2)(1+q+q^2+q^3)}\Bigg|.\end{aligned}$$

Assume that

$$\begin{aligned}F_q(p,t) = &\frac{q(1-q+2q^3)(4-p^2)p^2t}{16q^2(1+q)^3(1+q+q^2+q^3)} + \frac{(4-p^2)p^2t^2}{16(1+q)(1+q+q^2+q^3)} + \frac{(4-p^2)^2t^2}{16q^2(1+q)^2}\\ &+\frac{(4-p^2)}{4(1+q)(1+q+q^2+q^3)} + \frac{(4q^7+9q^6+7q^5-5q^3-6q^2-5q-2)p^4}{16q^2(1+q)^4(1+q+q^2)(1+q+q^2+q^3)}.\end{aligned}$$

Differentiating $F_q(p,t)$, we have

$$\frac{\partial F_q}{\partial t} = \frac{q(1-q-2q^3)(4-p^2)p^2}{16q^2(1+q)^3(1+q+q^2+q^3)} + \frac{(4-p^2)p^2t}{8(1+q)(1+q+q^2+q^3)} + \frac{(4-p^2)^2t}{8q^2(1+q)^2} \geqslant 0,$$

which implies that $F_q(p,t)$ increases on the closed interval $[0, 1]$ about t. That is, $F_q(p,t)$ has a maximum value at $t = 1$, which is

$$\max_{0\leqslant t\leqslant 1} F_q(p,t) = F_q(p,1)$$

$$= \frac{q(1-q+2q^3)(4-p^2)p^2}{16q^2(1+q)^3(1+q+q^2+q^3)} + \frac{(4-p^2)p^2}{16(1+q)(1+q+q^2+q^3)}$$
$$+ \frac{(4-p^2)^2}{16q^2(1+q)^2} + \frac{(4-p^2)}{4(1+q)(1+q+q^2+q^3)} + \frac{(4q^7+9q^6+7q^5-5q^3-6q^2-5q-2)p^4}{16q^2(1+q)^4(1+q+q^2)(1+q+q^2+q^3)}.$$

Setting

$$G_q(p) = \frac{[q^2(1+q)^2+q(1-q+2q^3)](4-p^2)p^2}{16(1+q)(1+q+q^2+q^3)} + \frac{(4-p^2)^2}{16q^2(1+q)^2}$$
$$+ \frac{(4-p^2)}{4(1+q)(1+q+q^2+q^3)} + \frac{(4q^7+9q^6+7q^5-5q^3-6q^2-5q-2)p^4}{16q^2(1+q)^4(1+q+q^2)(1+q+q^2+q^3)},$$

then

$$G'_q(p) = \frac{[q+2q^3+3q^4](4-p^2)p}{8(1+q)(1+q+q^2+q^3)} - \frac{(4-p^2)p}{4q^2(1+q)^2}$$
$$- \frac{p}{2(1+q)(1+q+q^2+q^3)} + \frac{(5q^7+10q^6+4q^5-8q^4-14q^3-14q^2-11q-4)p^3}{8q^2(1+q)^4(1+q+q^2)(1+q+q^2+q^3)}.$$

If we set $G'_q(p)=0$, then the only root is $p=0$. After that some simple computations, we can conclude that $G''_q(p)\leqslant 0$, at $p=0$, i.e.,

$$\left|\sigma_2\sigma_4-\sigma_3^2\right| \leqslant G_q(0) = \frac{1+2q+2q^2+q^3}{q^2(1+q)^2(1+q+q^2)}.$$

That is our desired proof of our Theorem 3.3. □

Corollary 3.4 ([35]). *Let the function f given by* (1.1) *be a member of the class* $f\in\mathcal{S}^*_{\exp}$. *Then*

$$\left|\sigma_2\sigma_4-\sigma_3^2\right| \leqslant \frac{7}{12}.$$

Theorem 3.5. *Let* $f\in\Lambda\in\mathcal{S}^*_{\exp}(q)$ *and of the form* (1.1), *then*

$$H_3(1) \leqslant \frac{\psi_0(q)\,\psi_1(q)\sqrt{\tau(q)}+\psi_2(q)+\psi_3(q)+\psi_4(q)+\psi_5(q)}{729(q^2+q+1)(q^2+1)^2(q^3+2q^2+\frac{5}{3}q+1)^3(1+q)^6q^3(q^4-2q^3-6q^2-6q-4)},$$

where

$$\psi_0(q) = 8q^2\left(q^4-2q^3-6q^2-6q-4\right)\left(q^5+3q^4+4q^3+2q^2-\frac{1}{2}\right),$$
$$\psi_1(q) = q^{11}+2q^{10}-\frac{5}{2}q^9-10q^8-5q^7+\frac{125}{2}q^6+\frac{435}{2}q^5+\frac{727}{2}q^4+\frac{799}{2}q^3+296q^2+\frac{279}{2}q+\frac{81}{2},$$
$$\psi_2(q) = -8q^{28}-56q^{27}-84q^{26}+516q^{25}+2988q^{24}+6091q^{23}+6033q^{22}+28670q^{21},$$
$$\psi_3(q) = +159306q^{20}+318669q^{19}-498907q^{18}-5182973q^{17}-18366014q^{16}-43982764q^{15},$$
$$\psi_4(q) = -81308728q^{14}-122578343q^{13}-155110960q^{12}-167516301q^{11}-155884481q^{10},$$
$$\psi_5(q) = -125566814q^9-87600428q^8-52751338q^7$$
$$-27208165q^6-11861408q^5-4278627q^4-1234089q^3-268515q^2-39366q-2916.$$

Proof. Since

$$H_3(1) \leqslant |\sigma_3|\left|\left(\sigma_2\sigma_4-\sigma_3^2\right)\right| + |\sigma_4|\left|(\sigma_2\sigma_3-\sigma_4)\right| + |\sigma_5|\left|\left(\sigma_1\sigma_3-\sigma_2^2\right)\right|,$$

using the fact that $\sigma_1=1$, along with Theorems 3.1, 3.2, 3.3, and Lemma 2.2, we get the desired result. □

4. Conclusion

In our present investigation, by making use of the principle of subordination between analytic functions, we have successfully defined and studied certain subfamilies of q-analogue symmetric starlike functions subordinate to q-virsion of exponential functions. Our major results are stated and proved as Theorems 3.1-3.5. These general results are motivated essentially by their several special cases and consequences, some of which are pointed out in this presentation.

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