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Viscosity approximation method for a variational problem

Ramzi May

Mathematics Department, College of Science, King Faisal University, P.O. 380, Ahsaa 31982, Kingdom of Saudi Arabia.

Abstract

By combining the works of Moudafi [A. Moudafi, J. Math. Anal. Appl., **241** (2000), 46–55] and Iiduka and Takahashi [H. Iiduka, W. Takahashi, Nonlinear Anal., **61** (2005), 341–350], we introduce an iterative process that converges strongly to a particular solution of a variational inequality problem. We also study the stability of the algorithm under relatively small perturbation and we apply the obtained results to the study of a constrained optimization problem and a problem of common fixed points of two nonexpansive mappings. Some numerical experiments are provided to study the affect of some parameters on the speed of the convergence of the considered algorithm.

Keywords: Hilbert spaces, variational inequality problem, nonexpansive mapping, inverse strongly monotone mappings.

2020 MSC: 47H09, 47H05, 47H10, 47J20.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle .,. \rangle$ and associated norm $\|.\|$ and let Q be a nonempty, closed, and convex subset of $\mathcal{H}$. In this paper, we are interested in determining a particular common solution to the variational problem:

$$\text{Find } q \in Q \text{ such that } \langle Aq, x-q \rangle \geqslant 0 \text{ for every } x \in Q, \tag{1.1}$$

and the fixed point problem:

$$\text{Find } q \in Q \text{ such that } Sq = q,$$

where

(A1) $S : Q \to Q$ is a nonexpansive mapping, i.e., $\|Sx - Sy\| \leqslant \|x - y\|$ for every $x, y \in Q$.
(A2) $A : Q \to \mathcal{H}$ is a ν- inverse strongly monotone operator which means that $\nu > 0$ and

$$\langle Ax - Ay, x - y \rangle \geqslant \nu \|Ax - Ay\|^2, \ \forall x, y \in Q.$$

We denote by $F_{ix}(S) = \{x \in Q : Sx = x\}$ the set of fixed points of the operator S and by $S_{VI(A,Q)}$ the set of solutions to the problem (1.1). It is easy to see that the sets $F_{ix}(S)$ and $S_{VI(A,Q)}$ are closed and convex

Email address: rmay@kfu.edu.sa (Ramzi May)

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subsets of $\mathcal{H}$ (see Lemmas 2.3 and 2.4 in Section 2); hence the set

$$\Omega := F_{ix}(S) \cap S_{VI(A,Q)}$$

is also closed and convex subset of $\mathcal{H}$. We assume that

(A3) The set Ω is nonempty.

In order to approximate numerically the elements of the set Ω, Takahashi and Toyoda [9] have introduced the following algorithm

$$\begin{cases} x_1 \in Q, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_Q(x_n - \lambda_n A x_n), \ n \geqslant 1, \end{cases} \tag{1.2}$$

where $P_Q : \mathcal{H} \to Q$ is the metric projection from $\mathcal{H}$ onto Q (see the next section for the definition and some proprieties of he operator P_Q). They have proved that if the sequence $\{(\alpha_n, \lambda_n)\}$ remains in a fixed compact subset of $(0,1) \times (0, 2\nu)$, then any sequence $\{x_n\}$ generated by the process (1.2) converges weakly to some element q_∞ of Ω.

Later, Plutieng and Kumam [8] generalized the previous result by replacing in the algorithm (1.2) the nonexpansive mapping S by S_n, where $\{S_n\}$ is a sequence of a nonexpansive mapping with common fixed points. They proved, under some suitable assumptions on $\{S_n\}$ and the same assumption on the real sequences $\{\alpha_n\}$ and $\{\lambda_n\}$, that the generated sequence $\{x_n\}$ converges weakly to a common fixed point of the mapping $\{S_n\}$, which is a solution of the variational problem (1.1).

To overcome the drawback of the weak convergence and the non specification of the limit point q_∞ in the algorithm (1.2), Iiduka and Tokahashi [5] have considered the following iterative process:

$$\begin{cases} x_1 \in Q, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) S P_Q(x_n - \lambda_n A x_n), \ n \geqslant 1, \end{cases} \tag{1.3}$$

where u is a given element of Q. They established that if $\{\lambda_n\} \in [a, b]$, with $0 < a < b < 2\nu$, $\{\alpha_n\} \in [0, 1]$, $\alpha_n \to 0$, $\sum_{n\geqslant 1} \alpha_n = +\infty$ and $\sum_{n\geqslant 1} (|\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n|) < \infty$, then any sequence $\{x_n\}$ generated by the algorithm (1.3) converges strongly in $\mathcal{H}$ to the closest element of Ω to u.

In 2011, Yao, Liou, and Chen [14] studied two averaged version of the algorithm (1.3). Precisely, they introduced the following two algorithms

$$\begin{cases} x_1 \in Q, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) P_Q(\alpha_n u + (1-\alpha_n) S P_Q\left(x_n - \lambda_n A x_n\right)), \ n \geqslant 1, \end{cases} \tag{1.4}$$

$$\begin{cases} x_1 \in Q, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S P_Q(\alpha_n u + (1-\alpha_n)\left(x_n - \lambda_n A x_n\right)), \ n \geqslant 1. \end{cases} \tag{1.5}$$

They proved that if the sequence $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the same assumptions as in the Theorem of Iiduka and Tokahashi and $\{\beta_n\}$ belongs to a sub-interval $[0, b]$ of $[0, 1)$ and satisfies $\sum_{n\geqslant 1} |\beta_{n+1} - \beta_n| < \infty$, then any sequence $\{x_n\}$ generated by (1.4) or (1.5) converges strongly in $\mathcal{H}$ to the closed element of Ω to u.

Recently, many authors (see for instance [3, 10, 11, 15]), inspired by the viscosity approximation method due to Moudafi [6], have introduced a variant of general algorithms that converge strongly to a specified solution of a variational inequality problems. These algorithms include as a special case the following generalization of the algorithm (1.3):

$$\begin{cases} x_1 \in Q, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_Q(x_n - \lambda_n A x_n), \ n \geqslant 1, \end{cases} \tag{1.6}$$

where

(A4) $f : Q \to Q$ is a contraction with coefficient $\rho \in [0, 1)$, that is

$$\|f(x) - f(y)\| \leqslant \rho \|x - y\| \ \forall x, y \in Q,$$

and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are nonnegative real sequences that satisfy the relation $\alpha_n + \beta_n + \gamma_n = 1$.

The authors studied the case when the sequence $\{\beta_n\}$ stays away from 0 and 1 in the sense $0 \leqslant a \leqq \beta_n \leqq b < 1$ for some constants a and b. They proved, under suitable and mild conditions on the parameters $\{\alpha_n\}$ and $\{\lambda_n\}$, that any sequence generated by the algorithm (1.6) converges strongly to the unique solution q^* of the variational inequality problem

$$\text{Find } q^* \in \Omega \text{ such that } \langle f(q^*) - q^*, x - q^* \rangle \leqslant 0, \ \forall x \in \Omega. \tag{1.7}$$

In the present work, we study the limit case of the algorithm (1.6), where $\beta_n = 0$ for all n. Precisely, we consider the process

$$\begin{cases} x_1 \in Q, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_Q(x_n - \lambda_n A x_n), \ n \geqslant 1, \end{cases} \tag{1.8}$$

and we prove, under the same conditions on the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ as in the result of Iiduka and Takahashi mentioned above, that the algorithm (1.8) still converges strongly to the solution $q^\star$ of the problem (1.7). Moreover, we establish the strong convergence of the implicit version of (1.8). In fact, we prove that if $\lambda : (0, 1] \to [a, b]$, with $[a, b] \subset (0, 2\nu)$, then for every $t \in (0, 1]$, there exists a unique x_t in Q such that

$$x_t = t\, f(x_t) + (1 - t)\, SP_Q(x_t - \lambda(t) A x_t). \tag{1.9}$$

Then we show that x_t converges strongly in $\mathcal{H}$ to q^* as $t \to 0^+$.

Throughout this paper, we always assume that all the assumptions (A1)-(A4) are satisfied.

The sequel of the paper is organized as follows. In Section 2, we recall some well-known results from convex analysis that will be useful in the proof of the main results of the paper. In Section 3, we establish the strong convergence of the implicit algorithm (1.9). In Section 4, we study the strong convergence of the explicit algorithm (1.8). Then we prove its stability of the process under the effect of small computational errors and we apply the obtained results to the study of a constrained optimization problem and the problem of common fixed points of two nonexpansive mappings. Section 5 is devoted to the study of the effect of the sequence $\{\alpha_n\}$ on the convergence rate of a particular example of the perturbed version of the algorithm (1.8) through some numerical experiments.

2. Preliminaries

In this section, we recall some results that will be helpful in the next sections. Most of these results are classical and can be found in any good convex analysis book as [1, 4, 7]. Let us first recall the definition of the metric projection onto a nonempty, closed, and convex subset of $\mathcal{H}$.

Lemma 2.1 ([1, Theorem 3.14])**.** *Let* K *be a nonempty, closed, and convex subset of* $\mathcal{H}$*. For every* $x \in \mathcal{H}$*, there exists a unique* $P_K(x) \in Q$ *such that*

$$\|x - P_K(x)\| \leqslant \|x - y\|, \ \ \forall y \in K.$$

The operator $P_K : \mathcal{H} \to K$ *is called the metric projection onto* K.

The following lemma gathers some classical and important properties of the projection operator P_K.

Lemma 2.2 ([1, Corollary 4.18])**.** *Let* K *be a nonempty, closed, and convex subset of* $\mathcal{H}$.

(1) *For every* $x \in \mathcal{H}$, $P_K(x)$ *is the unique element of* K *which satisfies*

$$\langle P_K(x) - x, P_K(x) - y \rangle \leqslant 0, \text{ for every } y \in K.$$

(2) *The operator* $P_K : \mathcal{H} \to K$ *is firmly nonexpansive, i.e.,*

$$\langle P_K(x) - P_K(y), x - y \rangle \geqslant \|P_K(x) - P_K(y)\|^2, \text{ for all } x, y \in \mathcal{H}. \tag{2.1}$$

In particular

$$\|P_K(x) - P_K(y)\| \leqslant \|x - y\|, \text{ for all } x, y \in \mathcal{H}. \tag{2.2}$$

Lemma 2.3 ([1, Theorem 3.13])**.** *Let* C *be a closed convex and nonempty subset of* $\mathcal{H}$. *If* $T : C \to C$ *is a nonexpansive mapping, then* $F_{ix}(T) = \{x \in C : Tx = x\}$ *is a closed and convex subset of* $\mathcal{H}$.

Lemma 2.4. *Let* $\lambda \in (0, 2\nu]$. *Then the following assertions hold true.*

(i) *For every* $x, y \in Q$,

$$\|(x - \lambda Ax) - (y - \lambda Ay)\|^2 \leqslant \|x - y\|^2 - \lambda(2\nu - \lambda) \|Ax - Ay\|^2. \tag{2.3}$$

(ii) *The operator* $\Theta_\lambda := P_Q \circ (I - \lambda A) : Q \to Q$ *is nonexpansive and* $F_{ix}(\Theta_\lambda) = S_{VI(A,Q)}$.

(iii) $S_{VI(A,Q)}$ *is a closed and convex subset of* $\mathcal{H}$.

Proof.

(i) Let $x, y \in Q$. A simple computation gives

$$\begin{aligned}\|(x - \lambda Ax) - (y - \lambda Ay)\|^2 &= \|x - y\|^2 - 2\lambda\langle Ax - Ay, x - y\rangle + \lambda^2 \|Ax - Ay\|^2 \\ &\leqslant \|x - y\|^2 - \lambda(2\nu - \lambda) \|Ax - Ay\|^2.\end{aligned}$$

(ii) Combining (2.2) and (2.3) yields immediately that Θ_λ is nonexpansive. Now let $q \in Q$. Clearly $q \in S_{VI(A,Q)}$ if and only if

$$\langle q - (q - \lambda Aq), q - x \rangle \leqslant 0, \quad \forall x \in Q,$$

which, thanks to the first assertion of Lemma 2.2, is equivalent to $q = P_Q(q - \lambda Aq) = \Theta_\lambda(q)$.

The last assertion (iii) follows directly from (ii) and Lemma 2.3. □

The next result is a particular case of the well-known demi-closedness principle.

Lemma 2.5 ([1, Corollary 4.18])**.** *Let* C *be a closed convex and nonempty subset of* $\mathcal{H}$, $T : C \to C$ *a nonexpansive mapping, and* $\{x_n\}$ *a sequence in* C . *If* $\{x_n\}$ *converges weakly in* $\mathcal{H}$ *to some* $\bar{x}$ *and* $\{x_n - T(x_n)\}$ *converges strongly in* $\mathcal{H}$ *to* 0, *then* $\bar{x} \in F_{ix}(T)$.

The last result of this section is a powerful lemma which is a generalization due to Xu [12] of a lemma firstly proved by Berstrekas ([2, Lemma 1.5.1]).

Lemma 2.6. *Let* $\{a_n\}$ *be a sequence of non negative real numbers such that:*

$$a_{n+1} \leqslant (1 - \gamma_n)a_n + \gamma_n r_n + \delta_n, \; n \geqslant 0, \tag{2.4}$$

where $\{\gamma_n\} \in [0, 1]$ *and* $\{r_n\}$ *and* $\{\delta_n\}$ *are three real sequences such that:*

(1) $\sum_{n=0}^{+\infty} \gamma_n = +\infty$;
(2) $\sum_{n=0}^{+\infty} |\delta_n| < +\infty$;
(3) $\limsup_{n \to +\infty} r_n \leqslant 0$.

Then the sequence $\{a_n\}$ *converges to* 0.

Proof. We give here a proof different from the original one due to Xu [12]. Set $s_n = \sum_{k=n}^{+\infty} \delta_k$, $e_n = a_n + s_n$, and $\beta_n = \max\{r_n, 0\} + s_n$. Using the fact that $\delta_n = s_n - s_{n+1}$, we easily obtain from (2.4) that

$$e_{n+1} \leqslant (1-\gamma_n)e_n + \gamma_n\beta_n,\ n \geqslant 0. \tag{2.5}$$

Let $\varepsilon > 0$. Since $\beta_n \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leqslant \frac{\varepsilon}{2}$ for every $n \geqslant n_0$. Let us suppose that $e_n \geqslant \varepsilon$ for every $n \geqslant n_0$. Hence, from (2.5), we infer that, for every $n \geqslant n_0$,

$$e_n - e_{n+1} \geqslant \gamma_n(e_n - \beta_n) \geqslant \frac{\varepsilon}{2}\gamma_n,$$

which implies that $\sum_{n \geqslant n_0} \gamma_n < \infty$. This is a contradiction. Then there exists $n_1 \geqslant n_0$ such that $e_{n_1} \leqslant \varepsilon$. Therefore

$$e_{n_1+1} \leqslant (1-\gamma_{n_1})\varepsilon + \gamma_{n_1}\frac{\varepsilon}{2} \leqslant \varepsilon.$$

And so on we get $e_n \leqslant \varepsilon$ for every $n \geqslant n_1$. Hence $e_n \to 0$ as $n \to \infty$, which clearly implies that $a_n \to 0$ as $n \to \infty$ since $s_n \to 0$ as $n \to \infty$. □

We close this section by proving that the variational problem (1.7) mentioned in the introduction has a unique solution.

Lemma 2.7. *The problem* (1.7) *has a unique solution* q^**. Moreover,* q^* *is the unique fixed point of the contraction* $P_\Omega \circ f : \Omega \to \Omega$.

Proof. Let us first recall that the set Ω is nonempty, closed, and convex subset of $\mathcal{H}$. Then from the variational characterization of the metric projection P_Ω (see the first assertion of Lemma 2.2), the problem (1.7) is equivalent to the identity $q^* = P_\Omega(f(q^*))$. Hence, the existence and the uniqueness of q^* follow from the classical Banach fixed point and the fact that the application $P_\Omega \circ f : \Omega \to \Omega$ is a contraction. □

3. The convergence of the implicit algorithm (1.9)

The following section is devoted to the proof of the strong convergence of the implicit algorithm (1.9).

Theorem 3.1. *Let* a *and* b *be two reals such that* $0 < a < b < 2\nu$ *and let* $\lambda : (0,1) \to [a,b]$ *be a mapping. Then, for every* $t \in (0,1)$*, there exists a unique* $x_t \in Q$ *such that*

$$x_t = tf(x_t) + (1-t)SP_Q(x_t - \lambda(t)Ax_t).$$

Moreover $\{x_t\}$ *converges strongly in* $\mathcal{H}$ *as* $t \to 0^+$ *to the unique solution* q^* *of the variational problem* (1.7).

The following simple lemma, which is an immediate consequence of the second assertion of Lemma 2.4, will be very useful in the proof of the previous theorem and also in the proof of the main result of the next section.

Lemma 3.2. *Let* $t \in [0,1]$ *and* $\mu \in (0, 2\nu]$*. Then the application* $T_{t,\mu} : Q \to Q$ *defined by*

$$T_{t,\mu}(x) = tf(x) + (1-t)SP_Q(x - \mu Ax),$$

satisfies

$$\|T_{t,\mu}(x) - T_{t,\mu}(y)\| \leqslant (1-\sigma t)\,\|x-y\|\,,\ \forall x, y \in Q,$$

where $\sigma = 1 - \rho$.

Now we are in position to prove Theorem 3.1.

Poof of Theorem 3.1. Let $t \in (0,1]$. According to Lemma 3.2 and the classical Banach fixed point theorem, there exists a unique $x_t \in Q$ such that $x_t = T_{t,\lambda(t)}(x_t)$. Let us now prove that the family $\{x_t\}_{0<t\leqslant 1}$ is bounded in $\mathcal{H}$. Let $q \in \Omega$. In view of the last assertion of Lemma 2.4,

$$T_{t,\lambda(t)}(q) = t\, f(q) + (1-t)Sq = t\, f(q) + (1-t)q. \tag{3.1}$$

Hence, Lemma 3.2 yields

$$\|x_t - q\| \leqslant \left\|T_{t,\lambda(t)}(x_t) - T_{t,\lambda(t)}(q)\right\| + t\,\|f(q)-q\| \leqslant (1-\sigma t)\,\|x_t - q\| + t\,\|f(q)-q\|,$$

which implies

$$\sup_{0<t\leqslant 1} \|x_t - q\| \leqslant \frac{1}{\sigma}\,\|f(q)-q\|.$$

Therefore $\{x_t\}_{0<t\leqslant 1}$ is a bounded family in $\mathcal{H}$.

In the sequel, in order to simplify the notations, we will use M to denote a real constant independent of $t \in (0,1]$ that may change from line to another. Moreover, $\varepsilon(t)$ will simply denotes a real quantity that converges to 0 as the variable t tends to 0. By the way, let us recall here this simple result that will be often implicitly used in the sequel: since $\{x_t\}_{0<t\leqslant 1}$ is a bounded in $\mathcal{H}$, then for every Lipschitz continuous function $g : Q \to \mathcal{H}$ the family $\{g(x_t)\}_{0<t\leqslant 1}$ is also bounded in $\mathcal{H}$.

For $t \in (0,1]$, we set $z_t = P_Q(x_t - \lambda(t)Ax_t)$. Let $q \in \Omega$. Clearly, by using the classical identity

$$\|tu + (1-t)v\|^2 \leqslant t\,\|u\|^2 + (1-t)\,\|v\|^2, \quad \forall u, v \in \mathcal{H}, \tag{3.2}$$

the fact that S and P_Q are nonexpansive operators, and Lemma 2.4, we obtain

$$\begin{aligned}
\|x_t - q\|^2 &\leqslant t\,\|f(x_t) - q\|^2 + (1-t)\,\|Sz_t - q\|^2 \\
&\leqslant tM + \|z_t - q\|^2 \\
&= tM + \left\|P_Q(x_t - \lambda(t)Ax_t) - P_Q(q - \lambda(t)Aq)\right\|^2 \\
&\leqslant tM + \|(x_t - \lambda(t)Ax_t) - (q - \lambda(t)Aq)\|^2 \\
&\leqslant tM + \|x_t - q\|^2 - \lambda(t)(2\nu - \lambda(t))\,\|Ax_t - Aq\|^2 .
\end{aligned} \tag{3.3}$$

We then deduce that

$$a(2\nu - b)\,\|Ax_t - Aq\|^2 \leqslant tM.$$

Therefore, by using the fact that the operator P_Q is firmly nonexpansive (see (2.1)), we get

$$\begin{aligned}
\|z_t - q\|^2 &= \left\|P_Q(x_t - \lambda(t)Ax_t) - P_Q(q - \lambda(t)Aq)\right\|^2 \\
&\leqslant \langle z_t - q, (x_t - \lambda(t)Ax_t) - (q - \lambda(t)Aq)\rangle \\
&\leqslant \langle z_t - q, x_t - q\rangle + \lambda(t)\,\|z_t - q\|\,\|Ax_t - Aq\| \\
&= \langle z_t - q, x_t - q\rangle + \varepsilon(t) = \frac{1}{2}\left(\|z_t - q\|^2 + \|x_t - q\|^2 - \|x_t - z_t\|^2\right) + \varepsilon(t).
\end{aligned}$$

The last inequality implies

$$\|z_t - q\|^2 \leqslant \|x_t - q\|^2 - \|x_t - z_t\|^2 + 2\varepsilon(t).$$

Hence, by combining this inequality with the estimate (3.3), we deduce that

$$\|x_t - z_t\|^2 \leqslant tM + 2\epsilon(t),$$

which implies

$$x_t - z_t \to 0 \text{ as } t \to 0^+. \tag{3.4}$$

The last inequality in turn implies that

$$x_t - Sx_t \to 0 \text{ as } t \to 0^+. \tag{3.5}$$

Indeed,

$$\begin{aligned}\|x_t - Sx_t\| &\leqslant \|x_t - Sz_t\| + \|Sx_t - Sz_t\| \\ &= \left\|T_{t,\lambda(t)}(x_t) - Sz_t\right\| + \|Sx_t - Sz_t\| \\ &\leqslant t\,\|f(x_t) - Sz_t\| + \|x_t - z_t\| \leqslant tM + \|x_t - z_t\|.\end{aligned}$$

Now we are in position to prove the following key result:

$$\kappa := \limsup_{t\to 0^+} \langle f(q^*) - q^*, x_t - q^*\rangle \leqslant 0, \tag{3.6}$$

where q^* is the unique solution of the variational problem (1.7). From the definition of κ, there exists a sequence $\{t_n\}$ in $(0,1]$ converging to 0 such that

$$\kappa = \lim_{n\to+\infty} \langle f(q^*) - q^*, x_{t_n} - q^*\rangle.$$

On the other hand, since the family $\{x_{t_n}\}_{0<t\leqslant 1}$ is a bounded subset of the closed and convex subset Q of $\mathcal{H}$, we can assume, up to a subsequence, that $\{x_{t_n}\}$ converges weakly in $\mathcal{H}$ to some $x_\infty \in Q$. Therefore we have

$$\kappa = \langle f(q^*) - q^*, x_\infty - q^*\rangle.$$

Hence, in order to prove that $\kappa \leqslant 0$, we just need to verify that $x_\infty \in \Omega$. Firstly, from Lemma 2.5 and (3.5), we have $x_\infty \in F_{ix}(S)$. Secondly, up to a subsequence, we can assume that the real sequence $\{\lambda_{t_n}\}$ converges to some real λ^* which belongs to $(0, 2\nu)$. Let $\Theta_{\lambda^*} = P_Q \circ (I - \lambda^* A)$ be the nonexpansive operator introduced in Lemma 2.5. Since $z_n = \Theta_{\lambda_n}(x_{t_n})$, we have

$$\begin{aligned}\|x_{t_n} - \Theta_{\lambda^*}(x_{t_n})\| &\leqslant \|x_{t_n} - z_{t_n}\| + \|\Theta_{\lambda_n}(x_{t_n}) - \Theta_{\lambda^*}(x_{t_n})\| \\ &\leqslant \|x_{t_n} - z_{t_n}\| + |\lambda_{t_n} - \lambda^*|\,\|Ax_{t_n}\| \leqslant \|x_{t_n} - z_{t_n}\| + |\lambda_{t_n} - \lambda^*|\,M.\end{aligned}$$

Hence, by combining (3.4) and Lemma 2.5, we deduce that x_∞ is a fixed point of Θ_{λ^*}. Thus, thanks to the second assertion of Lemma 2.4, we deduce that $x_\infty \in VI(A, Q)$. The claim (3.6) is then proved.

Let us finally prove that $x_t \to q^*$ in $\mathcal{H}$ as t goes to 0^+. Let $t \in (0,1]$. First, from the identity (3.1), we have

$$x_t - q^* = u + v,$$

with

$$u = T_{t,\lambda(t)}(x_t) - T_{t,\lambda(t)}\,(q^*)\, v = t(f(q^*) - q^*).$$

Thus, by applying the inequality

$$\|u+v\|^2 \leqslant \|u\|^2 + 2\langle v, u+v\rangle$$

and Lemma 3.2, we get the inequality

$$\|x_t - q^*\|^2 \leqslant (1-\sigma t)^2\,\|x_t - q^*\|^2 + 2t\langle f(q^*) - q^*, x_t - q^*\rangle,$$

which implies

$$\|x_t - q^*\|^2 \leqslant \frac{\sigma t}{2}\,\|x_t - q^*\|^2 + \frac{1}{\sigma}\langle f(q^*) - q^*, x_t - q^*\rangle \leqslant tM + \frac{1}{\sigma}\langle f(q^*) - q^*, x_t - q^*\rangle.$$

Then, by letting $t \to 0^+$ and using (3.6), we obtain the desired result. □

4. The convergence of the explicit algorithm (1.8)

In this section, we study the strong convergence property of the process (1.8). Precisely, we prove the following theorem.

Theorem 4.1. *Let* $\{\alpha_n\} \in (0,1]$ *and* $\{\lambda_n\} \in (0,2\nu]$ *be two real sequences such that:*

(i) $\alpha_n \to 0$ *and* $\sum_{n=0}^{+\infty} \alpha_n = +\infty$;
(ii) $0 < \liminf_{n\to+\infty} \lambda_n \leqslant \limsup_{n\to+\infty} \lambda_n < 2\nu$;
(iii) $\frac{\alpha_{n+1}-\alpha_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\alpha_{n+1}-\alpha_n| < \infty$;
(iv) $\frac{\lambda_{n+1}-\lambda_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\lambda_{n+1}-\lambda_n| < \infty$.

Then for every initial data $x_1 \in Q$, *the sequence* $\{x_n\}$ *generated by the iterative process*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) SP_Q(x_n - \lambda_n A x_n), \; n \geqslant 1,$$

converges strongly in $\mathcal{H}$ *to* q^*, *the unique solution of the variational problem* (1.7).

Proof. Since we are only interested in the study of the asymptotic behavior of the sequence $\{x_n\}$, we can replace hypothesis (ii) by the stronger one: there exist two real a and b in $(0,2\nu)$ such that the sequence $\{\lambda_n\}$ is in $[a,b]$.

For every $n \in \mathbb{N}$, we set $T_n := T_{\alpha_n,\lambda_n}$, where T_{α_n,λ_n} is the mapping defined by Lemma 3.2. First, we will prove that the sequence $\{x_n\}$ is bounded in $\mathcal{H}$. Let $q \in \Omega$. Thanks to the identity (3.1) and Lemma 3.2, we have

$$\begin{aligned} \|x_{n+1}-q\| &\leqslant \|T_n(x_n) - T_n(q)\| + \alpha_n \|f(q)-q\| \\ &\leqslant (1-\sigma\alpha_n)\|x_n - q\| + \alpha_n \|f(q)-q\| \leqslant \max\{\|x_n-q\|, \frac{1}{\sigma}\|f(q)-q\|\}. \end{aligned}$$

We then deduce by induction that

$$\|x_n - q\| \leqslant \max\{\|x_0-q\|, \frac{1}{\sigma}\|f(q)-q\|\}, \;\; \forall n \in \mathbb{N}.$$

Therefore $\{x_n\}$ is bounded in $\mathcal{H}$. Hence, for every Lipschitz function $g : Q \to \mathcal{H}$, the sequence $\{g(x_n)\}$ is also bounded in $\mathcal{H}$.

From here, as we have done in the proof of Theorem 3.1, M will denote a constant independent of n and $\{\varepsilon_n\}$ a real sequence that converges to 0. M and $\{\varepsilon_n\}$ may change from line to an other.

Let us now show that the sequence $\{\Delta x_n := x_{n+1} - x_n\}$ converges strongly to 0. For every $n \in \mathbb{N}$, we clearly have

$$\begin{aligned} \|\Delta x_n\| &\leqslant \|T_n(x_n) - T_n(x_{n-1})\| + \|T_n(x_{n-1}) - T_{n-1}(x_{n-1})\| \\ &\leqslant (1-\sigma\alpha_n)\|\Delta x_{n-1}\| + M\left[|\Delta\alpha_{n-1}| + |\Delta\lambda_{n-1}|\right], \end{aligned}$$

where

$$\Delta\alpha_n := \alpha_{n+1} - \alpha_n,$$

and

$$\Delta\lambda_n := \lambda_{n+1} - \lambda_n.$$

Hence, by applying Lemma 2.6, we deduce that

$$\|\Delta x_n\| \to 0 \text{ as } n \to \infty. \tag{4.1}$$

For every $n \in \mathbb{N}$, we set $z_n = P_Q(x_n - \lambda_n A x_n)$. Let $q \in \Omega$. As we have proceeded in the proof of Theorem 3.1, by using the classical identity (3.2) with $t = \alpha_n$, the fact that S and P_Q are nonexpansive operators, and Lemma 2.4, we get

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leqslant \alpha_n \|f(x_n) - q\|^2 + (1-\alpha_n)\|Sz_n - q\|^2 \\ &\leqslant \varepsilon_n + \|z_n - q\|^2 \\ &= \varepsilon_n + \left\|P_Q(x_n - \lambda_n A x_n) - P_Q(q - \lambda_n A q)\right\|^2 \\ &\leqslant \varepsilon_n + \|(x_n - \lambda_n A x_n) - (q - \lambda_n A q)\|^2 \\ &\leqslant \varepsilon_n + \|x_n - q\|^2 - \lambda_n(2\nu - \lambda_n)\|Ax_n - Aq\|^2. \end{aligned} \tag{4.2}$$

Therefore, we have

$$\begin{aligned} a(2\nu - b)\|Ax_n - Aq\|^2 &\leqslant \varepsilon_n + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &= \varepsilon_n - \langle \Delta x_n, x_{n+1} + x_n - 2q\rangle \\ &\leqslant \varepsilon_n + \|\Delta x_n\|\,\|x_{n+1} + x_n - 2q\| \leqslant \varepsilon_n + M\|\Delta x_n\|. \end{aligned}$$

Hence, thanks to (4.1), we deduce that

$$Ax_n - Aq \to 0 \text{ as } n \to \infty.$$

Therefore, by using the fact that the operator P_Q is firmly nonexpansive (see (2.1)), we get

$$\begin{aligned} \|z_n - q\|^2 &= \left\|P_Q(x_n - \lambda_n A x_n) - P_Q(q - \lambda_n A q)\right\|^2 \\ &\leqslant \langle z_n - q, (x_n - \lambda_n A x_n) - (q - \lambda_n A q)\rangle \\ &\leqslant \langle z_n - q, x_n - q\rangle + \lambda_n \|z_n - q\|\,\|Ax_n - Aq\| \\ &= \langle z_n - q, x_n - q\rangle + \varepsilon_n = \frac{1}{2}\left(\|z_n - q\|^2 + \|x_n - q\|^2 - \|z_n - x_n\|^2\right) + \varepsilon_n. \end{aligned}$$

Thus, we obtain

$$\|z_n - q\|^2 \leqslant \|x_n - q\|^2 - \|z_n - x_n\|^2 + \varepsilon_n.$$

Inserting this inequality into (4.2) yields

$$\|z_n - x_n\|^2 \leqslant \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \varepsilon_n = -\langle \Delta x_n, x_{n+1} + x_n - 2q\rangle + \varepsilon_n \leqslant M\|\Delta x_n\| + \varepsilon_n.$$

Hence, by using (4.1), we deduce that

$$x_n - z_n \to 0 \text{ as } n \to \infty.$$

Therefore, by proceeding exactly as in the proof of Theorem 3.1, we first infer that

$$x_n - Sx_n \to 0 \text{ as } n \to \infty,$$

then we deduce the key result:

$$\limsup_{n\to\infty} \langle f(q^*) - q^*, x_n - q^*\rangle \leqslant 0. \tag{4.3}$$

Let us finally prove that the sequence $\{x_n\}$ converges strongly in $\mathcal{H}$ to q^*. For every $n \in \mathbb{N}$,

$$x_{n+1} - q^* = T_n(x_n) - T_n(q^*) + \alpha_n(f(q^*) - q^*).$$

Hence, by using the inequality

$$\|u + v\|^2 \leqslant \|u\|^2 + 2\langle v, u + v\rangle,$$

with

$$u = T_n(x_n) - T_n(q^*), \qquad v = \alpha_n(f(q^*) - q^*),$$

we obtain

$$\begin{aligned}\|x_{n+1} - q^*\|^2 &\leqslant \|T_n(x_n) - T_n(q^*)\|^2 + 2\alpha_n\langle f(q^*) - q^*, x_{n+1} - q^*\rangle \\ &\leqslant (1-\sigma\alpha_n)^2\|x_n - q^*\|^2 + 2\alpha_n\langle f(q^*) - q^*, x_{n+1} - q^*\rangle \\ &\leqslant (1-2\sigma\alpha_n)\|x_n - q^*\|^2 + \alpha_n\left[2\langle f(q^*) - q^*, x_{n+1} - q^*\rangle + M\alpha_n\right].\end{aligned}$$

Therefore, by applying Lemma 2.6 and using the key result (4.3), we deduce that the sequence $\{x_n\}$ converges strongly in $\mathcal{H}$ to q^*. The proof is then achieved. □

Now we are going to prove that the algorithm (1.8) is stable under relatively small perturbations. Precisely, we establish the following result.

Theorem 4.2. *Let $\{\alpha_n\} \in (0,1]$, $\{\lambda_n\} \in [0, 2\nu]$, and $\{e_n\} \in \mathcal{H}$ be three sequences such that:*

(i) $\alpha_n \to 0$ *and* $\sum_{n=0}^{+\infty} \alpha_n = +\infty$;
(ii) $0 < \liminf_{n\to+\infty} \lambda_n \leqslant \limsup_{n\to+\infty} \lambda_n < 2\nu$;
(iii) $\frac{\alpha_{n+1}-\alpha_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iv) $\frac{\lambda_{n+1}-\lambda_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
(v) $\frac{\|e_n\|}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} \|e_n\| < \infty$.

Then every sequence $\{x_n\}$ in Q satisfying

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_Q(x_n - \lambda_n A x_n) + e_n, \ n \geqslant 1,$$

converges strongly in $\mathcal{H}$ to q^* the unique solution of the variational problem (1.7).

Proof. Let $\{y_n\}$ the sequence defined by

$$\begin{cases} y_1 = x_1, \\ y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)SP_Q(y_n - \lambda_n A y_n), \ n \geqslant 1. \end{cases}$$

Let $n \geqslant 1$. By invoking Lemma 3.2, we obtain

$$\|x_{n+1} - y_{n+1}\| \leqslant \|T_{\alpha_n,\lambda_n}(x_n) - T_{\alpha_n,\lambda_n}(y_n)\| + \|e_n\| \leqslant (1-\sigma\alpha_n)\|x_n - y_n\| + \|e_n\|,$$

where $\sigma = 1 - \rho$. Therefore, by applying Lemma 2.4, we deduce that

$$\|x_n - y_n\| \to 0 \text{ as } n \to \infty,$$

which implies that $\{x_n\}$ converges strongly in $\mathcal{H}$ to q^* since, from Theorem 4.1, the sequence $\{y_n\}$ converges strongly in $\mathcal{H}$ to q^*. □

As a first direct application of Theorem 4.2, we have the following result, which improves and generalizes [13, Theorem 5.2].

Corollary 4.3. *Let $\varphi : Q \longrightarrow \mathcal{H}$ be a continuously differentiable convex function such that its gradient $\nabla\varphi : Q \longrightarrow \mathcal{H}$ is Lipschitz with coefficient $L > 0$. We assume that the set $F_{ix}(S) \cap \arg\min_Q \varphi$ is nonempty, where $\arg\min_Q \varphi = \{q \in Q : \varphi(q) \leqslant \varphi(x), \ \forall x \in Q\}$. Let $\{\alpha_n\} \in (0,1]$, $\{\lambda_n\} \in [0, \frac{2}{L}]$, and $\{e_n\} \in \mathcal{H}$ be three sequences such that:*

(i) $\alpha_n \to 0$ *and* $\sum_{n=0}^{+\infty} \alpha_n = +\infty$;

(ii) $0 < \liminf_{n\to+\infty} \lambda_n \leqslant \limsup_{n\to+\infty} \lambda_n < \frac{2}{L}$;
(iii) $\frac{\alpha_{n+1}-\alpha_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\alpha_{n+1}-\alpha_n| < \infty$;
(iv) $\frac{\lambda_{n+1}-\lambda_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\lambda_{n+1}-\lambda_n| < \infty$;
(v) $\frac{\|e_n\|}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} \|e_n\| < \infty$.

Then every sequence $\{z_n\} \in Q$ *satisfying*

$$z_{n+1} = \alpha_n f(z_n) + (1-\alpha_n) SP_Q(z_n - \lambda_n \nabla\varphi(z_n)) + e_n, \; n \geqslant 1, \tag{4.4}$$

converges strongly in $\mathcal{H}$ *to* q^* *the unique element of* $F_{ix}(S) \cap \arg\min_Q \varphi$ *satisfying the variational inequality*

$$\langle f(q^*) - q^*, x - q^* \rangle \leqslant 0 \tag{4.5}$$

for all $x \in F_{ix}(S) \cap \arg\min_Q \varphi$.

Proof. The proof follows directly from Theorem 4.2. In fact, according to the famous Baillon-Haddad Theorem ([7, Theorem 3.13]), the operator $\nabla\varphi : Q \to \mathcal{H}$ is $\frac{1}{L}$ inverse strongly monotone and, from the classical varational characterization of constrained convex problem solutions ([4, Theorem 3.13]), we have

$$S_{VI(\nabla\varphi,Q)} = \{q \in Q : \langle \nabla\varphi(q), x-q \rangle \geqslant 0, \; \forall x \in Q\} = \arg\min_Q \varphi.$$

□

A second direct application of Theorem 4.2 is the following result concerning the problem of common fixed point of two nonexpansive mappings.

Corollary 4.4. *Let* $T_1, T_2 : Q \longrightarrow Q$ *be two nonexpansive mappings such that the set* $F_{1,2} := F_{ix}(T_1) \cap F_{ix}(T_2)$ *is nonempty. Let* $\{\alpha_n\} \in (0,1]$, $\{\lambda_n\} \in [0,1]$, *and* $\{e_n\} \in \mathcal{H}$ *be three sequences such that:*

(i) $\alpha_n \to 0$ *and* $\sum_{n=0}^{+\infty} \alpha_n = +\infty$;
(ii) $0 < \liminf_{n\to+\infty} \lambda_n \leqslant \limsup_{n\to+\infty} \lambda_n < 1$;
(iii) $\frac{\alpha_{n+1}-\alpha_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\alpha_{n+1}-\alpha_n| < \infty$;
(iv) $\frac{\lambda_{n+1}-\lambda_n}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} |\lambda_{n+1}-\lambda_n| < \infty$;
(v) $\frac{\|e_n\|}{\alpha_n} \to 0$ *or* $\sum_{n=0}^{+\infty} \|e_n\| < \infty$.

Then every sequence $\{x_n\} \in Q$ *satisfying*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_1((1-\lambda_n)x_n + \lambda_n T_2(x_n)) + e_n, \; n \geqslant 1, \tag{4.6}$$

converges strongly in $\mathcal{H}$ *to* q^* *the unique element of the set* $F_{1,2}$ *that satisfies the variational inequality*

$$\langle f(q^*) - q^*, x - q^* \rangle \leqslant 0 \tag{4.7}$$

for all $x \in F_{1,2}$.

Proof. The proof follows directly from the application of Theorem 4.2 with $S = T_1$ and $A = I - T_2$ and the use of the following facts.

(1) For every $x, y \in H$, we have

$$\begin{aligned} 2\langle Ax - Ay, x-y \rangle &= \|Ax - Ay\|^2 + \|x-y\|^2 - \|(Ax-x)-(Ay-y)\|^2 \\ &= \|Ax-Ay\|^2 + \|x-y\|^2 - \|T_2x - T_2y\|^2 \geqslant \|Ax-Ay\|^2, \end{aligned}$$

which means that A is is a ν- inverse strongly monotone operator with $\nu = \frac{1}{2}$.

(2) From the application of the second assertion of Lemma 2.4 with $\lambda = 1$, we have $S_{VI(A,Q)} = F_{ix}(P_Q \circ T_2) = F_{ix}(T_2)$.

(3) For every $n \geqslant 1$,

$$SP_Q(x_n - \lambda_n Ax_n) = T_1 P_Q((1-\lambda_n)x_n + \lambda_n T_2(x_n)) = T_1((1-\lambda_n)x_n + \lambda_n T_2(x_n))$$

since $(1-\lambda_n)x_n + \lambda_n T_2(x_n) \in Q$. □

5. Numerical experiments

In this section, we investigate through some numerical experiments the effect of the sequence $\{\alpha_n\}$ on the rate convergence of sequences $\{z_n\}$ generated by a particular example of the process (4.6) studied in the previous section. We consider the simple case where:

(1) the Hilbert space $\mathcal{H}$ is $\mathbb{R}^2$ endowed with its natural inner product $\langle x, y\rangle = x_1y_1 + x_2y_2$;
(2) the closed and convex subset Q is given by: $Q = \{x = (x_1, x_2)^t \in \mathbb{R}^2 : x_1, x_2 \geqslant 0\}$;
(3) the contraction mapping $f : Q \to Q$ is defined by $f(x) = \frac{1}{2}(5 + \cos(x_1 + x_2), 6 - \sin(x_1 + x_2))^t$ for all $x = (x_1, x_2)^t \in Q$. Using the mean value theorem, one can easily verify that f is Lipschitz continuous function with Lipschitz constant $\rho \leqslant \frac{\sqrt{2}}{2}$;
(4) the non expansive mapping $S : Q \to Q$ is the identity;
(5) the convex function $\varphi : Q \to \mathbb{R}$ is defined by: $\varphi(x) = \frac{1}{2}\|Bx - b\|^2$, where

$$B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix},\ b = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

a simple calculation yields

$$\nabla\varphi(x) = B^t(Bx - b) = \begin{pmatrix} 5x_1 + 5x_2 - 13 \\ 5x_1 + 5x_2 - 13 \end{pmatrix},\ \forall x = (x_1, x_2)^t \in Q,$$

hence $\nabla\varphi$ is Lipschitz continuous with Lipschitz constant $L = 10$, moreover,

$$\Omega = F_{ix}(S) \cap \arg\min_Q \varphi = \{x = (x_1, x_2)^t \in Q : x_1 + x_2 = 2.6\} = \Delta^2_{2.6},$$

where, for $a > 0$ and $n \in \mathbb{N}$,

$$\Delta^n_a = \{x = (x_1, \dots, x_n)^t \in \mathbb{R}^n : x_1, \dots, x_n \geqslant 0,\ x_1 + \cdots + x_n = a\},$$

let us notice that, by using KKT Theorem, one can easily verify that the projection onto Δ^n_a is given by

$$P_{\Delta^n_a}(x) = (\max(x_1 - \alpha(x), 0), \dots, \max(x_n - \alpha(x), 0))$$

for every $x = (x_1, \dots, x_n)^t \in \mathbb{R}^n$, where $\alpha(x)$ is the unique real solution α of the equation $\sum_{k=1}^n \max(x_k - \alpha, 0) = a$, hence a simple routine on Matlab, using the fact that the unique solution q^* to the variational problem (4.7) is the fixed point of the contraction $P_\Omega \circ f : Q \to Q$, provides a precise numerical approximation of q^* :

$$q^* \simeq (0.9647, 1.6353)^t;$$

(6) the sequence $\{\lambda_k\}$ is constant and equal to $\frac{1}{L} = 0.1$;
(7) the sequence $\{\theta_k\}$ is given by $\theta_k = \frac{1}{k^\theta}$, where θ is a constant which belongs to $(0, 1]$;
(8) the perturbation term $\{e_k\}$ is given by $e_k = \frac{X_k}{k^2}$, where $\{X_k\}$ is a sequence of independent random variables such that every X_k is uniform on the square $[-1, 1] \times [-1, 1]$;
(9) the initial value is $z_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$;
(10) N_{max} the maximal number of iterations k is $N_{max} = 6000$.

We aim to study numerically the relation between θ and the rate of convergence of the sequence $\{z_k\}$ to q^*. We can summarize our numerical results in the following two points.

(A): The convergence of the sequence $\{z_k\}$ to q^* is very slow for small values of the parameter θ as Table 1 shows.

Table 1: Slow convergence of $\{z_k\}$ for small values of θ.

θ	$\min_{k\leqslant N_{max}} \frac{\|z_k-q^*\|}{\|q^*\|}$
0.1	0.4774
0.2	0.1810
0.3	0.0742
0.4	0.0309

(B): The convergence of $\{z_k\}$ to q^* is more clear if θ is close to 1 as it is shown by Tables 2 and 3.

Table 2: Convergence of $\{z_k\}$ for some values of θ closed to 1.

θ	$\min_{k\leqslant N_{max}} \frac{\|z_k-q^*\|}{\|q^*\|}$
0.6	0.0055
0.8	0.0010
0.9	0.0005
1.0	0.0008

Table 3 indicates, for some values of $\varepsilon > 0$ and θ, $N(\varepsilon,\theta)$ the first iteration $k \leqslant N_{max}$ such that $\frac{\|z_k-q^*\|}{\|q^*\|} \leqslant \varepsilon$.

Table 3: $N(\varepsilon,\theta)$.

ε	$\theta = 0.6$	$\theta = 0.8$	$\theta = 0.9$	$\theta = 1.0$
0.5	6	5	4	4
0.10	53	23	14	17
0.05	158	56	36	42
0.01	2200	372	249	314
0.005	ND	854	533	716
0.001	ND	ND	2989	4742

Remark 5.1. $N(\varepsilon,\theta) = ND$ (not defined) means that $\frac{\|z_k-q^*\|}{\|q^*\|} > \varepsilon$ for all the iterations $k \leqslant N_{max}$.

Finally, the schema (Figure 1) shows the convergence of $\{z_k\}$ to q^* for some values of the parameter θ close to 1.

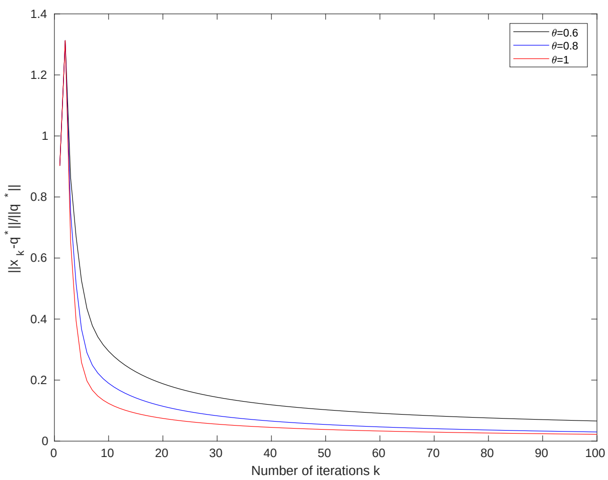

Figure 1: The effect of θ on the speed of the convergence of the algorithm (4.6).

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