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A monotone iterative method for second order nonlinear problems with boundary conditions driven by maximal monotone multivalued operators

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Abstract

In this paper, we study the following second order differential equation: $-(\Phi(u'(t)))' + \phi_p(u(t)) = \varepsilon f(t, u(t))$ a.e. on $\Omega = [0, T]$ under nonlinear multivalued boundary value conditions which incorporate as special cases the classicals boundary value conditions of type Dirichlet, Neumann, and Sturm-Liouville. Using monotone iterative method coupled with lower and upper solutions method, multifunction analysis, theory of monotone operators, and theory of topological degree, we show existence of solution and extremal solutions when the lower and upper solutions are well ordered or not. Since the boundary value conditions do not include the periodic one, we show that our method stay true for the periodic problem.

Keywords: Φ-laplacian, lower and upper solutions, monotone iterative method, Carathéodory function, maximal monotone multivalued operators, Leray-Schauder's degree, extremal solutions.

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1. Introduction

Your text comes here. Separate text sections with this paper is devoted to the study of the following problem:

$$\begin{cases} -\left(\Phi(u'(t))\right)' + \phi_p(u(t)) = \varepsilon f(t, u(t)) \quad \text{a.e. on} \quad \Omega = [0, T], \\ u'(0) \in B_1(u(0)), \quad -u'(T) \in B_2(u(T)). \end{cases} \tag{1.1}$$

where B_1 and B_2 are maximal monotone graphs in $\mathbb{R}^2$, $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is L^p-Caratheodory function, $p > 1$, which is bounded and increasing with respect to the second variable, $\varepsilon \in \{-1, 1\}$, $\Phi_p(z) = |z|^{p-2}z, p > 1$, for all $z \in \mathbb{R}$, is the p-Laplacian operator.

The method of lower and upper-solutions coupled with monotone iterative method is an interesting tool to prove existence of solution and extremal solutions of linear or nonlinear problems in a functional interval formed by a pair of lower and upper solutions. Several authors have used it in this sense. For this purpose, see [1, 3, 4, 7, 10, 11, 13] and references therein. The problems studied in [4, 10] and (1.1) encompasses the classical problems of Dirichlet, Neumann, Neumann-Steklov, Sturm-Liouville, discussed

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in [1, 7, 13]. In [4, 10], the authors assume that the lower and upper-solutions are well-ordered and they use a proof that relies on a fixed point theorem for ordered and reflexive Banach spaces due to HeiKilla-Hu [9]. In this paper, in contrast to [4, 10], we present a version of monotone iterative method which relies on Leray-Schauder topological degree theory, in the study of boundary problems driven by multivalued maximal monotone terms. Moreover, we enrich aforementioned works by taking account the case where the lower and upper solutions are not well ordered.

Thus, the goal of this paper is to establish an existence result of solutions and extremal solutions for the problem (1.1) by using a method which combines the method of lower and upper-solutions, iterative monotone method, the theory of the topological degree to the theory of monotone operators, the analysis of multifunctions when the lower and upper solutions are well ordered or not. Furthermore, since the boundary value conditions in (1.1) do not include the periodic one, we show that our method of proof stay true for the periodic problem.

2. Auxiliary results

Our hypotheses on data of the problem (1.1) are the following:

(H_Φ) $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing continuous map such that:
  (a) $\Phi(0) = 0$;
  (b) there exist $\eta_1, \eta_2, \eta_3 > 0$ such that: $\eta_1 |x|^p \leqslant \Phi(x)x \leqslant \eta_2 + \eta_3 |x|^p$ for all $x \in \mathbb{R}$.

Remark 2.1. Suppose that $\Phi(z) = \Phi_p(z) = |z|^{p-2} z, p \geqslant 2$. Then this function satisfies hypothesis (H_Φ). This function correspond to the one-dimentional operator p-Laplacian. Another interesting case which satisfies hypothesis (H_Φ) is when Φ is defined by $\Phi(z) = a(z)|z|^{p-2} z$ with $a : \mathbb{R} \to]0, +\infty[$ continuous, $a(z) \geqslant k > 0$ for all $z \geqslant 0$ and $z \mapsto a(z)|z|^{p-2} z$ is strictly increasing on $\mathbb{R}$ and $\eta_1 |z|^p \leqslant a(z)|z|^p \leqslant \eta_2 + \eta_3 |z|^p$. In fact, one can write $a(z) = \varphi(|z|)$ with $\varphi :]0, +\infty[\to]0, +\infty[$. For examples, we have:

$$\varphi(|z|) = \frac{\sqrt{1 + \left(1 + |z|^{p-1}\right)^2}}{1 + |z|^{p-1}} \qquad \text{and} \qquad \varphi(|z|) = 1 + \frac{1}{1 + |z|^{p-1}}.$$

It is well-know that under the monotonicity condition and hypotheses (a) and (b), Φ is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Φ^{-1} is strictly monotone and $\left|\Phi^{-1}(z)\right| \to +\infty$ as $|z| \to +\infty$ (see Deimling [8, chap. 3]).

(H_0) The problem (1.1) admits a pair of lower and upper solutions $\{\alpha, \beta\}$.
(H_ε) (a) $\varepsilon = 1$ if $\alpha \leqslant \beta$ on Ω;
  (b) $\varepsilon = -1$ if $\alpha > \beta$ on Ω.
(H_f) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an increasing function with respect to the second variable such that:
  (i) for all $x \in \mathbb{R}, t \longmapsto f(t, x)$ is measurable;
  (ii) for a.e. $t \in \Omega, x \longmapsto f(t, x)$ is continuous;
  (iii) for every $r > 0$, there exists $\gamma_r \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that for a.e. $t \in \Omega$ and for all $x \in \mathbb{R}$ with $|x| \leqslant r$, we have $|f(t, x)| \leqslant \gamma_r(t)$;
(H_B) B_1 and B_2: $\mathbb{R} \longrightarrow P(\mathbb{R})$ are some maximal monotone maps such that $0 \in B_1(0) \cap B_2(0)$.

Remark 2.2. For $i = 1, 2$, $B_i : \mathbb{R} \to P(\mathbb{R})$ being a maximal monotone map, there exists j_i a convex, proper, lower semicontinuous map such that $B_i = \partial j_i$. Let $j_i'(x^-)$ and $j_i(x^+)$ be the left and the right derivative of j_i at x for all $x \in \mathbb{R}$. We have $B_i(x) = \left[j_i'(x^-); j_i(x^+)\right]$ with $|j_i'(x^-)|, |j_i'(x^+)| < +\infty$.

Now, we define our notions of definition of solution, lower and upper solutions of the problem (1.1).

Definition 2.3. A function $u \in C^1(\Omega)$ such that $\Phi(u'(.)) \in W^{1,p}(\Omega)$ is said to be a solution of the problem (1.1), if it verifies (1.1).

Definition 2.4.

(a) A function $\beta \in C^1(\Omega)$ such that $\Phi(\beta'(.)) \in W^{1,p}(\Omega)$ is said to be an upper solution of the problem (1.1), if:

$$\begin{cases} -(\Phi(\beta'(t)))' + \Phi_p(\beta(t)) \geqslant \varepsilon f(t,\beta(t)) \text{ a.e. on } \Omega = [0,T], \\ \beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+, -\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+. \end{cases}$$

(b) A function $\alpha \in C^1(\Omega)$ such that $\Phi(\alpha'(.)) \in W^{1,p}(\Omega)$ is said to be a lower solution of problem (1.1), if:

$$\begin{cases} -(\Phi(\alpha'(t)))' + \Phi_p(\alpha(t)) \leqslant \varepsilon f(t,\alpha(t)) \text{ a.e. on } \Omega = [0,T] \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+, -\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+. \end{cases}$$

Definition 2.5. A lower solution α of (1.1) is said to be strict if all solution u of (1.1) with $u(t) \geqslant \alpha(t), \forall t \in [0,T]$ is such that $u(t) > \alpha(t),\ \forall t \in [0,T]$.

Definition 2.6. A upper solution β of (1.1) is said to be strict if all solution u of (1.1) with $u(t) \leqslant \beta(t), \forall t \in [0,T]$ is such that $u(t) < \beta(t),\ \forall t \in [0,T]$.

Proposition 2.7. *Let α be a lower solution of (1.1) such that:*

(i) *for all $t_0 \in]0,T[$, there exists $\varepsilon_0 > 0$ and Ω_0 an open interval such that $t_0 \in \Omega_0$ and*

$$-\left(\Phi(\alpha'(t))\right)' + \Phi_p(\alpha(t)) \leqslant \varepsilon f(t,x) \ \textit{a.e.}\ \ t \in \Omega_0, \ \ \textit{for all } x \in [\alpha(t), \alpha(t) + \varepsilon_0];$$

(ii) $\alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+^*$;

(iii) $-\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+^*$,

then α is strict lower solution of (1.1).

Proof. Let u be a solution of problem (1) such that $\alpha(t) \leqslant u(t)$ for all $t \in \Omega$. Let us assume that u is not strict, then there exists $\bar{t} \in [0,T]$ such that $\alpha(\bar{t}) = u(\bar{t})$. Whence

$$A = \{t \in [0,T] : \alpha(t) = u(t)\} \neq 0$$

A is closed and bounded. Let $t_0 = \min A$. Then

$$\min_{[0,T]} [u(t) - \alpha(t)] = u(t_0) - \alpha(t_0).$$

(a) If $t_0 \in]0,T[$, then $u'(t_0) - \alpha'(t_0) = 0$ and there exist Ω_0 and $\epsilon_0 > 0$ according to (i). We can choose $t_1 \in \Omega_0$ such that $t_1 < t_0$, $u'(t_1) < \alpha'(t_1)$, and

$$\forall t \in [t_1, t_0], (u(t), u'(t)) \in]\alpha(t), \alpha(t) + \epsilon_0[\times \left]\alpha'(t) - \epsilon_0, \alpha'(t) + \epsilon_0\right[.$$

Therefore, for almost $t \in [t_1, t_0]$,

$$-(\Phi(\alpha'(t)))' - f(t, u(t), u'(t)) - \Xi(u(t)) \leqslant 0.$$

Since Φ is an increasing homeomorphism, we have

$$\Phi(u'(t_0)) - \Phi(\alpha'(t_0)) = 0$$

and

$$\Phi(u'(t_1)) < \Phi(\alpha'(t_1)).$$

Also we have

$$\begin{aligned} \Phi(u'(t_1)) - \Phi(\alpha'(t_1)) &= -\int_{t_1}^{t_0} (\Phi(u'(s)))' - (\Phi(\alpha'(s)))' ds \\ &= -\int_{t_1}^{t_0} -f(s, u(s), u'(s)) - \Xi(u(s)) - (\Phi(\alpha'(s)))' ds \geqslant 0, \end{aligned}$$

which contradicts (2).

(b) We suppose that $t_0 = 0$, then $\alpha'(0) \leqslant u'(0)$ and it follows that

$$\Phi(u'(0)) - \Phi(\alpha'(0)) \geqslant 0.$$

Since $\alpha'(0) \in B_1(\alpha(0)) + e_0, e_0 > 0$, because of the monotonicity of B_1, if $\alpha(0) \geqslant u(0)$, we have $\alpha'(0) > u'(0)$. Then $\Phi(\alpha'(0)) > \Phi(u'(0))$. So $\Phi(u'(0)) - \Phi(\alpha'(0)) < 0$, which contradicts (3).

(c) We suppose that $t_0 = T$, then $\alpha'(T) \geqslant u'(T)$. It follows that:

$$\Phi(u'(T)) - \Phi(\alpha'(T)) \leqslant 0.$$

Since $-\alpha'(T) \in B_2(\alpha(T)) + e_0, e_0 > 0$, because of the monotonicity of B_1, if $\alpha(T) \geqslant u(T)$, we have $\alpha'(T) < u'(T)$. Then $\Phi(\alpha'(T)) < \Phi(u'(T))$. So $\Phi(u'(T)) - \Phi(\alpha'(T)) > 0$, which contradicts (4). Then, t_0 not exist. So, $A = \emptyset$. □

Proposition 2.8. *Let β be an upper solution of (1.1) such that:*

(i) *for all $t_0 \in]0, T[$, there exist $\varepsilon_0 > 0$ and Ω_0 an open interval such that $t_0 \in \Omega_0$ and*

$$-\left(\Phi(\beta'(t))\right)' + \Phi_p(\beta(t)) \geqslant f(t,x) \;\; \textit{a.e.} \;\; t \in \Omega_0, \;\; \textit{for all } x \in [\beta(t) - \varepsilon_0, \beta(t)];$$

(ii) $\beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+^*$;

(iii) $-\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+^*$,

then β is strict upper solution of (1.1).

Proof. The proof is similar to the one of proposition 2.7. □

Let us introduce the following set:

$$D = \left\{u \in C^1(\Omega) : \Phi(u') \in W^{1,q}(0,T), u'(0) \in B_1(u(0)) \text{ and } -u'(T) \in B_2(u(T))\right\}.$$

Lemma 2.9. *If the hypotheses (H_Φ) and (H_B) hold, then for all $h \in L^q(\Omega)$, the problem*

$$(P_h) \begin{cases} -(\Phi(u'(t)))' + \phi_p(u(t)) = h(t) \textit{ a.e. on } \Omega = [0,T], \\ u'(0) \in B_1(u(0)), \; -u'(T) \in B_2(u(T)). \end{cases} \tag{2.1}$$

has an unique solution u_h in $C^1(\Omega)$.

Proof. Let $l, m \in \mathbb{R}$. We consider the following two-point boundary value problem:

$$\begin{cases} -(\Phi(u'(t)))' + \Phi_p(u(t)) = h(t) \text{ a.e. on } \Omega = [0,T], \\ u(0) = l, \quad u(T) = m. \end{cases} \tag{2.2}$$

Let us set $\gamma(t) = \left(1 - \frac{t}{T}\right) l + \frac{t}{T} m$. Then $\gamma(0) = l$ and $\gamma(T) = m$. We consider the function y defined by $y(t) = u(t) - \gamma(t)$ and rewrite (2.2) in the terms of the function y:

$$\begin{cases} -(\Phi(y'(t) + \gamma'(t)))' + \Phi_p(y(t) + \gamma(t)) = h(t) \text{ a.e. on } \Omega = [0,T], \\ y(0) = y(T) = 0. \end{cases} \tag{2.3}$$

This is a homogeneous Dirichlet problem for (2.2). To solve (2.3), we argue as follows. Let $\pi : W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega)$ be nonlinear operator defined by:

$$\langle \pi(y), z \rangle_0 = \int_0^T \Phi(y'(t) + \gamma'(t)) z'(t) dt + \int_0^T \Phi_p(y(t) + \gamma(t)) z(t) dt, \forall y, z \in W_0^{1,p}(\Omega),$$

where $\langle\rangle_0$ denotes the duality brackets for the pair $\left(W^{-1,q}(\Omega), W_0^{1,p}(\Omega)\right)$. We show that π is strictly monotone, demicontinuous, and coercive (see the proof of proposition 3.10 of Béhi-Adjé-Goli [4]) . So, π is surjective. Moreover, since π is strictly monotone, we infer that there exists an unique $y \in W_0^{1,p}((0,T))$ such that $\pi(y) = h$. For any test function ϕ, we have

$$\langle \pi(y), \phi\rangle_0 = \langle h, \phi\rangle_0 \Leftrightarrow \int_0^T \Phi(y'(t)+\gamma'(t))\phi'(t)dt = \int_0^T (h(t) - \Phi_p(y(t)+\gamma(t)))\,\phi(t)dt.$$

From the definition of the distributional derivative, it follows that

$$-\left(\Phi(y'(t)+\gamma'(t))\right)' = h(t) - \Phi_p(y(t)+\gamma(t)) \text{ a.e. on } \Omega.$$

Whence y is the unique solution of problem (2.3). Then $u = y+\gamma \in C^1(\Omega)$ is the unique solution of the problem (2.2). We can define the solution map $\sigma : \mathbb{R}\times\mathbb{R} \longrightarrow C^1(\Omega)$, which assigns to each pair (l, m) the unique solution of the problem (2.2). Let $Q : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$ be defined by:

$$Q(l, m) = \left(-\Phi(\sigma(l, m)'(0)), \Phi(\sigma(l, m)'(T))\right).$$

We show that Q is monotone, continuous, and coercive (see the proof of proposition 3.10 of Béhi-Adjé [4]). We infer that Q is maximal monotone (being continuous, monotone) and coercive. Thus Q is surjective. Now, let $B : \mathbb{R}\times\mathbb{R} \longrightarrow P(\mathbb{R}\times\mathbb{R})$ be defined by

$$B(l, m) = (\Phi\circ B_1(l), \Phi\circ B_2(m)) \text{ for all } (l, m) \in \mathbb{R}\times\mathbb{R}.$$

We have B is maximal monotone (see Claim 4 in the proof of Proposition 3.8 in Bader-Papageorgiou [2])). Next, let $\theta : \mathbb{R}\times\mathbb{R} \longrightarrow P(\mathbb{R}\times\mathbb{R})$ be defined by

$$\theta(l, m) = Q(l, m) + B(l, m) \text{ for all } (l, m) \in \mathbb{R}\times\mathbb{R}.$$

Then θ is maximal monotone (see Brezis [5], Corollary 2.7, p. 36 or Zeidler [12], Theorem 32.I, p. 897). Moreover, since Q is coercive, B is maximal monotone and $(0, 0) \in B(0, 0)$, it follows that θ is coercive. Thus θ is surjective. We infer that we can find $(l, m) \in \mathbb{R}\times\mathbb{R}$ such that $(0, 0) \in \theta(l, m)$. So $\Phi(u(0)) \in \Phi\circ B_1(l)$ and $-\Phi(u(T)) \in \Phi\circ B_2(m)$. Whence, by acting with Φ^{-1}, we obtain $(u'(0), -u'(T)) \in (B_1(l), B_2(m))$. Therefore $x_0 = \sigma(l, m)$ is the unique solution of the problem (2.1). □

Lemma 2.10. *Suppose that hypotheses* (H_B) *and* (H_f) *hold and* α *and* β *are lower and upper solutions of* (1.1). *If* u *is solution of* (1.1), *then* $\|u'\|_\infty < M$ *with*

$$M = \Phi^{-1}\left[\Phi\left(\frac{\max\{|\alpha(T)-\beta(0)|, |\beta(T)-\alpha(0)|\}}{T}\right) + T\left(\max\left\{\|\alpha\|_\infty^{p-1}, \|\beta\|_\infty^{p-1}\right\} + \|\gamma_r\|_q\right)\right]$$

and

$$r > \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

Proof. Since $u \in C^1(\Omega)$, by the mean value theorem, we can find $t_0 \in (0, T)$ such that $u(T) - u(0) = Tu'(t_0)$. It follows that $u'(t_0) = \dfrac{u(T)-u(0)}{T}$. Then, from (1.1), we have

$$|u'(t_0)| \leqslant \frac{\max\{|\alpha(T)-\beta(0)|, |\beta(T)-\alpha(0)|\}}{T}.$$

Then,

$$\Phi(|u'(t_0)|) \leqslant \Phi\left(\frac{\max\{|\alpha(T)-\beta(0)|, |\beta(T)-\alpha(0)|\}}{T}\right). \tag{2.4}$$

Therefore,

$$|\Phi_p(u(t))| \leqslant \max\left\{\|\alpha\|_\infty^{p-1}, \|\beta\|_\infty^{p-1}\right\}. \tag{2.5}$$

By integration of (1.1) on (t_0, t), $\forall\, t \in (t_0, T)$ (similarly, we can integrate (1.1) on $(t, t_0), \forall\, t \in (0, t_0)$), we have

$$\Phi(u'(t)) = \Phi(u'(t_0)) + \int_{t_0}^{t} (\Phi_p(u(s)) - \varepsilon f(s, u(s)))\, ds \text{ on } \Omega.$$

Then, we have

$$u'(t) = \Phi^{-1}\left[\Phi(u'(t_0)) + \int_{t_0}^{t} (\Phi_p(u(s)) - \varepsilon f(s, u(s)))\, ds\right] \text{ on } \Omega.$$

It follows that

$$|u'(t)| \leqslant \Phi^{-1}\left[\Phi(|u'(t_0)|) + \int_{t_0}^{t} (|\Phi_p(u(s))| + |f(s, u(s))|)\, ds\right] \text{ on } \Omega. \tag{2.6}$$

Thus, using hypotheses (2.4), (2.5), and (2.6), we obtain

$$\|u'\|_\infty \leqslant \Phi^{-1}\left[\Phi\left(\frac{\max\{|\alpha(T) - \beta(0)|, |\beta(T) - \alpha(0)|\}}{T}\right) + T\left(\max\left\{\|\alpha\|_\infty^{p-1}, \|\beta\|_\infty^{p-1}\right\} + \|\gamma_r\|_q\right)\right].$$

□

We introduce the following operator $G : C^1(\Omega) \to D \subseteq C^1(\Omega)$ defined by:

$$G(u)(t) = u(0) + \int_0^t \Phi^{-1} \circ \left(\Phi(u'(0)) - \int_0^y (\varepsilon f(s, u(s)) - \Phi_p(u(s)))\, ds\right) dy.$$

Lemma 2.11. G *is completely continuous.*

Proof. Let $u_n \longrightarrow u$ in $C^1(\Omega)$. To establish the continuity of the operator G, we will show that $G(u_n) \longrightarrow G(u)$ in $C^1(\Omega)$. That's mean $G(u_n) \longrightarrow G(u)$ in $C(\Omega)$ and $(G(u_n))' \longrightarrow (G(u))'$ in $C(\Omega)$. For $n \geqslant 1$, we have:

$$\begin{aligned}
&-(\Phi(u_n'(t)))' + \Phi_p(u_n(t)) = \varepsilon f(t, u_n(t)) \text{ a.e. } t \in [0, T] \\
&\qquad \Leftrightarrow u_n'(t) = \Phi^{-1} \circ (\Phi(u_n'(0)) - \int_0^t \varepsilon f(s, u_n(s)) - \Phi_p((u_n(s))ds) \text{ a.e. } t \in [0, T] \\
&\qquad \Leftrightarrow u_n'(t) = (G(u_n))'(t) \text{ a.e. } t \in [0, T].
\end{aligned}$$

Since Φ, Φ_p and N are continuous, respectively, in $L^q(\Omega)$ and $C(\Omega)$, we have

$$\lim_{n \to +\infty} N(u_n)(.) = f(., u(.)) \text{ in } L^q(\Omega), \lim_{n \to +\infty} \Phi(u_n'(0)) = \Phi(u'(0))$$

and

$$\lim_{n \to +\infty} \Phi_p(u_n(t)) = \Phi_p(u(t)),$$

where N is the Nemytski's operator associated to f. Using the previous arguments and the dominated convergence theorem, we have

$$\lim_{n \to +\infty} \int_0^t \varepsilon f(t, u_n(s)) - \Phi_p(u_n(t))dt = \int_0^t \varepsilon f(t, u(s)) - \Phi_p(u(t))dt.$$

It follows that

$$\lim_{n \to +\infty} \Phi(u_n'(0)) - \int_0^t \varepsilon f(t, u_n(s)) - \Phi_p(u_n(t))dt = \Phi(u'(0)) - \int_0^t \varepsilon f(t, u(s)) - \Phi_p(u(t))dt.$$

Since Φ is an homeomorphism, Φ^{-1} exists and is continuous. Finally, we have

$$\lim_{n\to+\infty} \Phi^{-1}\circ\left(\Phi(u_n'(0))-\int_0^t f(t,u_n(s))-\Phi_p(u_n(t))dt\right)$$
$$=\Phi^{-1}\circ\left(\Phi(u'(0))-\int_0^t \varepsilon f(t,u(s))-\Phi_p(u(t))dt\right)\Leftrightarrow \lim_{n\to+\infty}(G(u_n))'=(G(u))'\in C(\Omega).$$

By integration, we obtain: $\lim_{n\to+\infty} G(u_n)=G(u)$ in $C(\Omega)$. Therefore, G is continuous. Let us show that G is relatively compact. If $u_n \longrightarrow u$ in $C^1(\Omega)$, then there exists $R>0$ such that $u_n, u\in B_{C^1}(R)$. Let Π be a bounded set of $C^1(\Omega)$. We set $\Delta=\{G(u) : u\in\Pi\}$. Since Π is bounded, there exists $R>0$ such that:

$$\|G(u)\|_\infty < R+MT.$$

It follows that:

$$\|G(u)\|_{C^1} < R+(M+1)T.$$

Therefore, there exist $R_1>0$ such that $G(u)\in\overline{B_{C^1}(R_1)}$. For $u\in\Pi$ and $s_1,s_2\in\Omega$,

$$\left|\Phi(G(u))'(s_1)-\Phi(G(u))'(s_2)\right| = \left|\int_{s_2}^{s_1} N_f(u)(t)-\Phi_p(u(t))dt\right| \leqslant |s_2-s_1|^{\frac{1}{p}}\left(\|\gamma_r\|_q+\|R+MT\|_q^{p-1}\right).$$

We infer that for all $\epsilon>0$, there exist $\delta>0$ such that

$$|s_1-s_2|<\delta \Rightarrow \left|\Phi(G(u))'(s_1)-\Phi(G(u))'(s_2)\right|<\epsilon.$$

It suffice to take $\delta=\left(\dfrac{\epsilon}{\|\gamma_r\|_q+\|R+MT\|_q^{p-1}}\right)^p$. Therefore, Φ being an homeomorphism, for all $\epsilon>0$, it exists $\delta>0$, such that for all $u\in\Pi$, $s_1,s_2\in\Omega$, if $|s_1-s_2|<\delta$, then

$$\left|(G(u))'(s_1)-(G(u))'(s_2)\right| = \left|\Phi^{-1}\circ\Phi((G(u))'(s_1))-\Phi^{-1}\circ\Phi((G(u))'(s_2))\right|<\epsilon.$$

Δ is uniformly equicontinuous and bounded on $C^1(\Omega)$. By Ascoli-Arzela 's theorem, $\Delta=G(\Pi)$ is relatively compact in $C^1(\Omega)$. Since G is continuous and $G(\Pi)$ is relatively compact in $C^1(\Omega)$ for every bounded subset Π of $C^1(\Omega)$, G is completely continuous. □

Proposition 2.12. *If hypotheses* (H_0), (H_Φ), *and* (H_f) *hold, then the problem* (1.1) *admits a solution* u *if* u *is a fixed point of the operator* G.

Proof. If u is solution of (1.1), then $(\Phi(u'))'\in L^q(\Omega)$ because of hypothesis $(H_f)(iv)$. By integration, we have $\Phi(u')\in L^q(\Omega)$. So, $\Phi(u')\in W^{1,q}(\Omega)$. We also have $u'(0)\in B_1(u(0))$ and $-u'(T)\in B_2(u(T))$. Therefore, $u\in D$. Furthermore, we have

$$-\left(\Phi(u'(t))\right)'=f(t,u(t))-\Phi_p(u(t)) \text{ ae on } \Omega.$$

That's mean

$$u(t)=G(u)(t) \text{ ae on } \Omega.$$

As consequence, u is a fixed point of G. Reciprocally, if u is a fixed point of G, we have $u\in D\subseteq C^1(\Omega)$, and $G(u)(t)=u(t)$ ae on Ω. Then, $-(\Phi(u'(t)))'=f(t,u(t))-\Phi_p(u(t))$ ae on Ω and $u'(0)\in B_1(u(0))$ and $-u'(T)\in B_2(u(T))$. So, u is solution of (1.1). Finally, by Lemma 2.10, we have $\forall u\in C^1(\Omega), \|(G(u))'\|_\infty < M$. □

3. Existence of solutions and extremal solutions with well-ordered lower and upper solutions

3.1. Existence results

We introduce the functional interval:

$$U = [\alpha, \beta] = \left\{u \in W^{1,p}\left((0,T)\right) : \ \alpha(t) \leqslant u(t) \leqslant \beta(t) \text{ for all } \ t \in \Omega\right\},$$

and the operator $\tau : W^{1,p}\left((0,T)\right) \longrightarrow W^{1,p}\left((0,T)\right)$ defined by

$$\tau(u)(t) = \max\{\alpha(t), \min\{u(t), \beta(t)\}\} = \begin{cases} \alpha(t), & \text{if } u(t) < \alpha(t), \\ u(t), & \text{if } \alpha(t) \leqslant u(t) \leqslant \beta(t), \\ \beta(t), & \text{if } u(t) > \beta(t). \end{cases}$$

For all $(t, x) \in \Omega \times W^{1,p}(\Omega)$, let us set $N_1(x)(t) = f_1(t, x) = f(t, \tau(x)(t))$. For all $x \in [\alpha(t), \beta(t)]$, we have $f_1(t, x) = f(t, x)$. Moreover, a.e. $t \in \Omega$ and all $x \in \mathbb{R}$, we have $|f_1(t, x)| \leqslant \gamma_r(t)$ with $r = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} + MT$. We consider the following auxilary problem

$$\begin{cases} -(\Phi(u'(t)))' + \phi_p(u(t)) = f_1(t, u(t)) \text{ a.e. on } \Omega = [0, T], \\ u'(0) \in B_1(u(0)), \ -u'(T) \in B_2(u(T)). \end{cases} \tag{3.1}$$

The problem (3.1) is equivalent to fixed point problem of the operator $\overline{G} : C^1(\Omega) \to D \subseteq C^1(\Omega)$ defined by

$$\overline{G}(u)(t) = u(0) + \int_0^t \Phi^{-1} \circ \left(\Phi(u'(0)) - \int_0^y \left(f_1(s, u(s)) - \Phi_p(u(t))\right) ds\right) dy,$$

we have

$$\left\|\overline{G}(u)\right\|_\infty < \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} + MT.$$

Theorem 3.1. *Suppose that there exist a lower solution α and a upper solution β such that $\forall t \in [0, T]$, $\alpha(t) \leqslant \beta(t)$. Then the problem (1.1) admits at least one solution u such that*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad \forall t \in [0, T].$$

Moreover, if α and β are strict, then

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad \forall t \in [0, T] \quad \text{and} \quad d_{LS}[Id_{C^1} - G, \Pi_{\alpha,\beta}, 0] = 1,$$

where $\Pi_{\alpha,\beta} = \left\{u \in C^1(\Omega) : \forall t \in [0, T], \alpha(t) \leqslant u(t) \leqslant \beta(t)\right\}$ and G is the fixed point operator of problem (1.1).

Proof. The proof will be established in many steps.

Claim 3.2. Every solution of (3.1) is such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $\forall t \in \Omega$.

Proof. $\alpha \in C^1(\Omega)$ is a lower solution of the problem (1.1), then

$$\begin{cases} -(\Phi(\alpha'(t)))' + \Phi_p(\alpha(t)) \leqslant f(t, \alpha(t)) \text{ a.e. on } \Omega = [0, T], \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+, -\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+. \end{cases} \tag{3.2}$$

Substraction of (3.2) from (3.1) yields

$$(\Phi(\alpha'(t)))' - (\Phi(u'(t))' + \Phi_p(u(t)) - \Phi_p(\alpha(t)) \geqslant f_1(t, u(t)) - f(t, \alpha(t)). \tag{3.3}$$

We multiply (3.3) by $(\alpha - u)^+ \in W^{1,p}\left((0,T)\right)$ and then integrate on Ω. We obtain

$$\begin{aligned} &\int_0^T \left[(\Phi(\alpha'(t)))' - (\Phi(u'(t)))'\right] (\alpha - u)^+ (t) dt \\ &\quad - \int_0^T \left(\Phi_p(\alpha(t)) - \Phi_p(u(t))\right) (\alpha - u)^+ (t) dt \geqslant \int_0^T [f_1(t, u(t)) - f(t, \alpha(t))] (\alpha - u)^+ (t) dt. \end{aligned} \tag{3.4}$$

The integration by parts of the left-hand side in inequality, yields

$$\begin{aligned}
&\int_0^T \left[(\Phi(\alpha'(t))' - (\Phi(u'(t))'\right](\alpha-u)^+(t)dt - \int_0^T (\Phi_p(\alpha(t)) - \Phi_p(u(t)))(\alpha-u)^+(t)dt \\
&= \big(\Phi(\alpha'(T)) - \Phi(u'(T))\big)(\alpha-u)^+(T) - \big(\Phi(\alpha'(0)) - \Phi(u'(0))\big)(\alpha-u)^+(0) \\
&\quad - \int_0^T \left[\Phi(\alpha'(t)) - \Phi(u'(t))\right](\alpha-u)'^+(t)dt - \int_0^T (\Phi_p(\alpha(t)) - \Phi_p(u(t)))(\alpha-u)^+(t)dt \\
&\geqslant \int_0^T [f_1(t,u(t)) - f(t,\alpha(t))](\alpha-u)^+(t)dt.
\end{aligned}$$

We set

$$\left[(\alpha-u)^+\right]'(t) = \begin{cases} (\alpha-u)'(t), & \text{if } \alpha(t) > u(t), \\ 0, & \text{if } \alpha(t) \leqslant u(t). \end{cases}$$

Also, from the boundary conditions in (3.1) and (3.2), we have:

$$-u'(T) \in B_1(u(T)) \text{ and } -\alpha'(T) \in B_1(\alpha(T)) + e_T \ \text{ with } \ e_T \geqslant 0.$$

If $\alpha(T) \geqslant u(T)$, then from the monotony of B_2 (see hypothesis (H_B)), we have $\alpha'(T) \leqslant u'(T)$. Since Φ is increasing, we have $\Phi(\alpha'(T)) \leqslant \Phi(u'(T))$. So, it follows that

$$(\Phi(\alpha'(T)) - \Phi(u'(T)))(\alpha(T) - u(T)) \leqslant 0. \tag{3.5}$$

In a similar fashion, using the boundary conditions $u'(0) \in B_1(u(0))$ and $\alpha'(0) \in B_1(\alpha(0)) + e_0$ with $e_0 \geqslant 0$, if $\alpha(0) \geqslant u(0)$, we have $\alpha'(0) \geqslant u'(0)$. We infer that $\Phi(\alpha'(0)) \geqslant \Phi(u'(0))$. It follows that

$$(\Phi(\alpha'(0)) - \Phi(u'(0)))(\alpha(0) - u(0)) \geqslant 0. \tag{3.6}$$

Also, since Φ is an increasing homeomorphism, we have

$$\int_0^T \big(\Phi(\alpha'(t)) - \Phi(u'(t))\big)(\alpha-u)'^+(t)dt = \int_{\{\alpha>u\}} \big(\Phi(\alpha'(t)) - \Phi(u'(t))\big)(\alpha-u)'(t)dt \geqslant 0, \tag{3.7}$$

where $\{\alpha > u\} = \{t \in [0,T] : \alpha(t) > u(t)\}$. Moreover, since Φ_p is strictly increasing, we have

$$\int_0^T \big(\Phi_p(\alpha'(t)) - \Phi_p(u'(t))\big)(\alpha-u)'(t)dt = \int_{\{\alpha>u\}} \big(\Phi_p(\alpha'(t)) - \Phi_p(u'(t))\big)(\alpha-u)'(t)dt > 0. \tag{3.8}$$

Using the inequalities (3.5)), (3.6), (3.7), and (3.8) in the first member of (3.4), we obtain

$$\int_0^T \left[(\Phi(u'(t))' - (\Phi(\alpha'(t))'\right](\alpha-u)^+(t)dt - \int_0^T (\Phi_p(\alpha(t)) - \Phi_p(u(t)))(\alpha-u)^+(t)dt < 0. \tag{3.9}$$

Furthermore

$$\begin{aligned}
f_1(t,u(t)) - f(t,\alpha(t))) &= f(t,\alpha(t)) - f(t,\alpha(t)) = 0 \text{ a.e. on } \{\alpha > u\} \\
&\Rightarrow \int_0^T (f_1(t,u(t)) - f(t,\alpha(t)))(\alpha-u)^+(t)dt = 0.
\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10) in (3.4), we have a contradiction when $|\{\alpha > u\}| > 0$. Therefore , for all $t \in \Omega$, $\alpha(t) \leqslant u(t)$. In a similar fashion we show that $u(t) \leqslant \beta(t)$ for all $t \in \Omega$; thus $u \in U$. □

Claim 3.3. The operator $\overline{G}$ is continuous and completely continuous.

Proof. The proof is similar to the one of Lemma 2.11. □

Claim 3.4. The problem (1.1) admits at least one solution such that

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad \forall t \in [0,T].$$

Proof. $\overline{G}(C^1(\Omega)) \subset B_{C^1}(R)$ for all $R > \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} + (M+1)T$. Then, by Leray-Schauder's theorem, we can say that the operator $\overline{G}$ has a fixed point u in the ball $B_{C^1}(R)$, which is solution of problem (3.1). Therefore, by Claim 3.2, u is also solution of (1.1). □

Claim 3.5. If α and β are strict, then

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad \forall t \in [0,T] \quad \text{and} \quad d_{LS}[Id_{C^1} - G, \Pi_{\alpha,\beta}, 0] = 1.$$

Proof. Suppose that α is a strict lower solution and β is a strict upper solution of (1.1). Let

$$R > \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} + M(T+1),$$

be enough small, such that $\overline{G}u \neq u \quad \forall\, u \in \partial B_{C^1}(R)$. Because of complete continuity of $\overline{G}$, we can compute the degree of $Id_{C^1} - \overline{G}$. The function H defined by $H(t,u) = t\overline{G}(u)$ is compact on $[0,1] \times B_{C^1}(R)$. Suppose that there exist $t \in [0,1]$ and $v \in \partial B_{C^1}(R)$ such that $v - H(t,v) = 0$, then $v = t\overline{G}(v)$. However $\|v\|_{C^1} = R$, so $t\left\|\overline{G}(v)\right\|_{C^1} = R$, which contradict the fact that $\left\|\overline{G}(v)\right\|_{C^1} < R$. So, we can apply homotopy invariance property and those of normalisation of Leray-Schauder 's degree to obtain:

$$d_{LS}\left[Id_{C^1} - \overline{G}, B_{C^1}(R), 0\right] = d_{LS}\left[Id_{C^1}, B_{C^1}(R), 0\right] = 1.$$

Since G is a completely continuous operator associated to (1.1), we have $\overline{G} = G$ on $\Pi_{\alpha,\beta}$. Therefore

$$d_{LS}\left[Id_{C^1} - G, B_{C^1}(R), 0\right] = 1.$$

□

Finally, by Claims 3.4 and 3.5, we obtain the proof of Theorem 3.1 □

3.2. *Existence of extremal solutions*

Theorem 3.6. *If the hypotheses* (H_ε), (H_f), (H_Φ), *and* (H_B) *hold, then there exist two monotones sequences* $(\alpha_k)_{k\geqslant 1}$ *and* $(\beta_k)_{k\geqslant 1}$ *in* $C^1(\Omega)$ *such that* $\phi(\alpha'_k) \in W^{1,p}(\Omega)$ *and* $\phi(\beta'_k) \in W^{1,p}(\Omega)$ *that converge uniformly in* $C^1(\Omega)$ *to* η *and* w *solutions of problem* (1.1) *and verify inequalities*

$$\alpha(t) \leqslant \alpha_1(t) \leqslant \alpha_2(t) \leqslant \cdots \leqslant \alpha_k(t) \leqslant \eta \leqslant w \leqslant \beta_k(t) \leqslant \cdots \leqslant \beta_2(t) \leqslant \beta_1(t) \leqslant \beta(t). \tag{3.11}$$

Proof. For all $v \in U$ fixed, we consider the following problem

$$(P_v)\begin{cases} -(\Phi(u'(t)))' + \phi_p(u(t)) = f(t, v(t)) \text{ a.e. on } \Omega = [0,T], \\ u'(0) \in B_1(u(0)),\ -u'(T) \in B_2(u(T)). \end{cases} \tag{3.12}$$

Then, by Lemma 2.9, the problem (3.12) admits an unique solution u_v. Let $L : U \to U$ be the operator defined by $\forall v \in U,\ Lv = u_v$, where u_v is the unique solution of problem (P_v). Let us show that

(i) $\alpha \leqslant L\alpha$;
(ii) $\beta \geqslant L\beta$;
(iii) L is nondecreasing.

In order to prove (i), we set $L\alpha = \alpha_1$. We have

$$\begin{cases} -(\Phi(\alpha'(t)))' + \phi_p(\alpha(t)) \leqslant f(t,\alpha(t)) \text{a.e. on } \Omega = [0,T], \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+,\ -\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+, \end{cases}$$
$$\begin{cases} -(\Phi(\beta'(t)))' + \phi_p(\beta(t)) \geqslant f(t,\beta(t)) \geqslant f(t,\alpha(t)) \text{ a.e. on } \Omega = [0,T] \\ \beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+,\ -\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+. \end{cases}$$

So α and β are lower and upper solutions of (P_α) such that $\alpha(t) \leqslant \beta(t), \forall t \in \Omega$. Then by the Lemma 2.9, α_1 is the unique solution of (P_α) such that $\alpha_1 \in U$. In order to prove (ii), we set $L\beta = \beta_1$. We have

$$\begin{cases} -(\Phi(\alpha'(t)))' + \phi_p(\alpha(t)) \leqslant f(t,\alpha(t)) \leqslant f(t,\beta(t)) \text{ a.e. on } \Omega = [0,T], \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+,\ -\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+, \end{cases}$$
$$\begin{cases} -(\Phi(\beta'(t)))' + \phi_p(\beta(t)) \geqslant f(t,\beta(t)) \text{ a.e. on } \Omega = [0,T], \\ \beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+,\ -\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+. \end{cases}$$

So α and β are lower and upper solutions (P_β) such that $\alpha(t) \leqslant \beta(t), \forall t \in \Omega$. Then by Lemma 2.9, β_1 is the unique the solution of (P_β) such that $\beta_1 \in U$.

Let us show (iii). Let $\pi, \nu_2 \in U$ such that $\nu_1 \leqslant \nu_2$. We set $L\nu_i = u_i, \forall i \in \{1;2\}$. We have

$$\begin{cases} -\left(\Phi(u_1'(t))\right)' + \phi_p(u_1(t)) \leqslant f(t,\nu_1(t)) \leqslant f(t,\nu_2(t)) \text{ a.e. on } \Omega = [0,T], \\ u_1'(0) \in B_1(u_1(0)) + \mathbb{R}_+,\ -u_1'(T) \in B_2(u_1(T)) + \mathbb{R}_+, \end{cases}$$
$$\begin{cases} -(\Phi(\beta'(t)))' + \phi_p(\beta(t)) \geqslant f(t,\beta(t)) \geqslant f(t,\nu_2(t)) \text{ a.e. on } \Omega = [0,T], \\ \beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+,\ -\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+. \end{cases}$$

So u_1 and β are lower and upper solutions of the problem (P_{ν_2}) such that $u_1(t) \leqslant \beta(t), \forall t \in \Omega$. By Lemma 2.9, u_2 the unique solution of (P_{ν_2}) is such that $u_2 \in [u_1, \beta]$. Thus L is non decreasing on U. We define the sequences $(\alpha_k)_{k\geqslant 1}$ and $(\beta_k)_{k\geqslant 1}$ as

$$\alpha_0 = \alpha, \alpha_1 = L\alpha = L\alpha_0, \dots, \alpha_{k+1} = L\alpha_k, k = 1,2,\dots,$$
$$\beta_0 = \beta, \beta_1 = L\beta = L\beta_0, \dots, \beta_{k+1} = L\beta_k, k = 1,2,\dots.$$

By induction, $\alpha_k(t) \leqslant \beta_k(t)$ for all $k \in \mathbb{N}$ and for all $t \in \Omega$. Then

$$\alpha \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k \leqslant \beta_k \leqslant \dots \beta_2 \leqslant \beta_1 \leqslant \beta$$

in $[0,T]$. We have, for all k, $u_k \in \{\alpha_k, \beta_k\}$.

Let us show that $\{u_k\}_{k\geqslant 1}$ is bounded in $W^{1,p}(\Omega)$. Let us consider the following sequence of problems

$$-(\Phi(u_n'(t)))' + \Phi_p(u_n(t)) = h(t) \text{ a.e. on } \Omega = [0,T]. \tag{3.13}$$

By integration by parts, we obtain

$$-\Phi(u_n'(T))u_n(T) + \Phi(u_n'(0))u_n(0) + \int_0^T \Phi(u_n'(t))u_n'(t)dt + \int_0^T \Phi_p(u_n(t))u_n(t)dt = \langle h(t), u_n \rangle_p. \tag{3.14}$$

Since $u_n \in D$, we have $u_n'(0) \in B_1(u_n(0))$ and $-u_n'(T) \in B_1(u_n(T))$ for all $n \geqslant 1$. We recall that $(0,0) \in GrB_i, i = 1,2$, then

$$u_n'(0)u_n(0) \geqslant 0 \text{ and } u_n'(T)u_n(T) \leqslant 0. \tag{3.15}$$

Moreover, the map Φ being increasing, we have

$$\Phi(u_n'(0))u_n'(0) \geqslant 0 \text{ and } \Phi(u_n'(T))u_n'(T) \geqslant 0. \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\Phi(u_n'(0))u_n(0) \geqslant 0 \text{ and } \Phi(u_n'(T))u_n(T) \leqslant 0. \tag{3.17}$$

From (3.14) and (3.17), we infer that :

$$\langle h(t), u_n\rangle_2 \geqslant \int_0^T \Phi(u_n'(t))u_n'(t)dt + \int_0^T \Phi_p(u_n(t))u_n(t)dt. \tag{3.18}$$

By hypothesis (b) on Φ, we have

$$\int_0^T \Phi(u_n'(t))u_n'(t)dt + \int_0^T \Phi_p(u_n(t))u_n(t)dt \geqslant \int_0^T \left(\eta_1 \left|u_n'(t)\right|^p + |u_n(t)|^p\right) dt. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\langle h(t), u_n\rangle_p \geqslant \int_0^T \left(\eta_1 \left|u_n'(t)\right|^p + |u_n(t)|^p\right) dt.$$

Whence

$$\|u_n\|^{p-1} \leqslant \eta_7 \quad \text{for some} \quad \eta_7 > 0.$$

Therefore the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}((0,T))$ is bounded. Passing to the subsequence, if necessary, it converges weakly in $W^{1,p}(\Omega)$. Due to the compact embedding of $W^{1,p}(\Omega)$ in $C(\Omega)$, the sequence $\{u_k\}_{k\geqslant 1}$ converges strongly in $C(\Omega)$. Furthermore, it follows directly from (3.13) that the sequence $\left\{(\phi(u_k'))'\right\}_{k\geqslant 1}$ is bounded in $L^q(\Omega)$. Then, by integration, the sequence $\left\{\phi(u_k')\right\}$ is bounded in $L^q(\Omega)$. Therefore, the sequence $\left\{\phi(u_k')\right\}_{k\geqslant 1}$ is bounded in $W^{1,q}(\Omega)$. Then, the sequence $\left\{\phi(u_k')\right\}_{k\geqslant 1}$ admits a subsequence which converges weakly in $W^{1,q}(\Omega)$. Due to the compact embedding of $W^{1,q}(\Omega)$ in $C(\Omega)$, this subsequence $\left\{\phi(u_k')\right\}_{k\geqslant 1}$ converges strongly to $\phi(u')$ in $C(\Omega)$. Since ϕ is a homeomorphism, ϕ^{-1} exists. Then, acting by ϕ^{-1}, we have that the sequence $\left\{u_k'\right\}_{k\geqslant 1}$ converges strongly to u' in $C(\Omega)$. Furthermore, the sequence $\{\phi_p(u_k)\}_{k\geqslant 1}$ converges strongly to $\phi_p(u)$ in $C(\Omega)$.

Let us set $v = \lim_{k\to\infty} \alpha_k$ and $w = \lim_{k\to\infty} \beta_k$. Then, $v, w \in C(\Omega)$ and $v' = \lim_{n\to\infty} \alpha_k'$, $w' = \lim_{n\to\infty} \beta_k'$. For all $k \in \mathbb{N}$ and for all $u_k \in \{\alpha_k, \beta_k\}$, we consider the sequence of problems

$$\begin{cases} -\left(\Phi(u_k'(t))\right)' + \phi_p(u_k(t)) = f(t, u_k(t)) \text{ a.e. on } \Omega = [0,T], \\ u_k'(0) \in B_1(u_k(0)),\ -u_k'(T) \in B_2(u_k(T)). \end{cases}$$

Passing to the limit as $k \to +\infty$, we see that v and w are some solutions of the problem (1.1). Also we have, for $k \in \mathbb{N}^*$,

$$\alpha \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k \leqslant v \leqslant w \leqslant \beta_k \leqslant \cdots \beta_2 \leqslant \beta_1 \leqslant \beta \text{ on } \Omega.$$

□

4. Existence of solutions and extremal solutions with non well-ordered lower and upper solutions

4.1. Existence results

Theorem 4.1. *Suppose that there exist a lower solution α and an upper solution β of (1.1) such that*

$$\exists \bar{t} \in [0,T] \text{ such that } \alpha(\bar{t}) > \beta(\bar{t}). \tag{4.1}$$

Then the problem (1.1) admits at least one solution u, such that

$$\min\{\alpha(t_u), \beta(t_u)\} \leqslant u(t_u) \leqslant \max\{\alpha(t_u), \beta(t_u)\} \text{ for some } t_u \in [0,T] \tag{4.2}$$

and

$$\|u\|_\infty \leqslant \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} + MT. \tag{4.3}$$

Proof. We set $\lambda = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. We consider the function $f^* : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, for $i = 1, 2$, and multifunctions $B_i^* : \mathbb{R} \to P(\mathbb{R})$ defined, respectively, by

$$f^*(t,u) = \begin{cases} 2, & \text{if } u > \lambda + 1, \\ (1+\lambda-u)\,[f(t,u)] + 2(u-\lambda), & \text{if } \lambda < u \leqslant \lambda + 1, \\ f(t,u), & \text{if } -\lambda \leqslant u \leqslant \lambda, \\ (1+\lambda+u)\,[f(t,u)] + 2(u+\lambda), & \text{if } -\lambda - 1 \leqslant u < -\lambda, \\ -2, & \text{if } u < -\lambda - 1, \end{cases}$$

and

$$B_i^* = B_i + A_i,$$

where

$$A_i(u) = \begin{cases} \bar{c}_i^*(u) + 1, & \text{if } u > \lambda + 1, \\ (\bar{c}_i^*(u)+1)(u-\lambda), & \text{if } \lambda < u \leqslant \lambda + 1, \\ 0, & \text{if } -\lambda \leqslant u \leqslant \lambda, \\ (\bar{c}_i^*(u)+1)(u+\lambda), & \text{if } -\lambda - 1 \leqslant u < -\lambda, \\ -(\bar{c}_i^*(u)+1), & \text{if } u < -\lambda - 1, \end{cases}$$

$B_i(u) = [j'(u^-); j'(u^+)]$, and $\bar{c}_i^*(u) = \max\{|j'(u^-)|, |j'(u^+)|\}$ a positive real number which depends on u. For $i = 1, 2$, A_i is maximal monotone and $\operatorname{int}(A_i) \cap B_i = \emptyset$. Then, the multifunction B_i^* is maximal monotone. We also have $0 \in B_1^*(0) \cap B_2^*(0)$. Furthermore, f^* is Caratheodory function. We consider the following modified problem

$$\begin{cases} -(\Phi(u'(t)))' + \Phi_p(u(t)) = \varepsilon f^*(t, u(t)) \text{ a.e. on } [0,T], \\ u'(0) \in B_1^*(u(0)), -u'(T) \in B_2^*(u(T)). \end{cases} \tag{4.4}$$

We can verified that α is a lower solution and β is a upper solution of (4.4). Let $\widetilde{\beta}, \widetilde{\alpha} : \mathbb{R} \to \mathbb{R}$ defined by $\widetilde{\beta}(t) = \lambda + 2$ and $\widetilde{\alpha}(t) = -\lambda - 2, \forall t \in \mathbb{R}$. We have

$$-\left(\Phi(\widetilde{\beta}'(t))\right)' + \Phi_p(\widetilde{\beta}(t)) = (\lambda+2)^2 > 2 = f^*(t, \widetilde{\beta}(t)) \text{ a.e. on } [0,T], \quad \widetilde{\beta}'(0) = 0, \quad \text{and} \quad -\widetilde{\beta}'(T) = 0.$$

Since $B_1^*(\widetilde{\beta}(0))$ and $B_2^*(\widetilde{\beta}(T))$ are some closed interval of $\mathbb{R}_+^*$, we can find $e_i \in \mathbb{R}_+^*$, for $i = 1, 2$, such that

$$0 = \widetilde{\beta}'(0) \in B_2^*(\widetilde{\beta}(0)) - e_0 \quad \text{and} \quad 0 = -\widetilde{\beta}'(T) \in B_2^*(\widetilde{\beta}(T)) - e_T.$$

Therefore, $\widetilde{\beta}$ is a upper solution of (4.4),

$$-\left(\Phi(\widetilde{\alpha}'(t))\right)' + \Phi_p(\widetilde{\alpha}(t)) = -(\lambda+2)^2 < -2 = f^*(t, \widetilde{\alpha}(t)) \text{ a.e. on } [0,T], \quad \widetilde{\alpha}'(0) = 0, \quad \text{and} \quad -\widetilde{\alpha}'(T) = 0.$$

Since $B_1^*(\widetilde{\alpha}(0))$ and $B_2^*(\widetilde{\alpha}(T))$ are some closed interval of $\mathbb{R}_+^*$, we can find $l_i \in \mathbb{R}_+^*$, for $i = 1, 2$, such that

$$0 = \widetilde{\alpha}'(0) \in B_2^*(\widetilde{\alpha}(0)) + l_0 \quad \text{and} \quad 0 = -\widetilde{\alpha}'(T) \in B_2^*(\widetilde{\alpha}(T)) + l_T.$$

Therefore, $\widetilde{\alpha}$ is a lower solution of (4.4). Furthermore,

$$\forall t \in [0,T], \quad \widetilde{\alpha}(t) \leqslant \min\{\alpha(t), \beta(t)\} \leqslant \max\{\alpha(t), \beta(t)\} \leqslant \widetilde{\beta}(t).$$

Let us introduce the sets

$$\Pi_{\widetilde{\alpha},\beta} = \left\{u \in C^1 : \forall t \in [0,T], \widetilde{\alpha}(t) < u(t) < \beta(t)\right\}, \quad \Pi_{\alpha,\widetilde{\beta}} = \left\{u \in C^1 : \forall t \in [0,T], \alpha(t) < u(t) < \widetilde{\beta}(t)\right\},$$

and

$$\Pi_{\widetilde{\alpha},\widetilde{\beta}} = \left\{u \in C^1 : \forall t \in [0,T], \widetilde{\alpha}(t) < u(t) < \widetilde{\beta}(t)\right\}.$$

By using the definition (4.1), we obtain

$$\Pi_{\widetilde{\alpha},\beta} \cap \Pi_{\alpha,\widetilde{\beta}} = \emptyset.$$

Also we have

$$\Pi_{\widetilde{\alpha},\beta} \cup \Pi_{\alpha,\widetilde{\beta}} \subset \Pi_{\widetilde{\alpha},\widetilde{\beta}}.$$

Let us consider

$$\Pi = \Pi_{\widetilde{\alpha},\widetilde{\beta}} \setminus \left(\overline{\Pi}_{\widetilde{\alpha},\beta} \cup \overline{\Pi}_{\alpha,\widetilde{\beta}} \right).$$

Then

$$\Pi = \left\{ u \in \Pi_{\widetilde{\alpha},\widetilde{\beta}} : \exists (t_1,t_2) \in [0,T]^2 \text{ such that } \beta(t_1) < u(t_1) \text{ and } u(t_2) < \alpha(t_2) \right\}$$

and

$$\partial \Pi_{\widetilde{\alpha},\widetilde{\beta}} = \partial \Pi_{\alpha,\widetilde{\beta}} \cup \partial \Pi_{\alpha,\widetilde{\beta}} \cup \partial \Pi.$$

Since all constant functions between $\widetilde{\beta(t)}$ and $\widetilde{\alpha(t)}$ are into Π, Π is non-empty. Let G^* be the fixed point operator associated with problem (4.4). Suppose that there exists $u \in \partial\Pi$ such that $G^*(u) = u$ and $\|u\|_\infty = \lambda + 2$. There exists $t_0 \in [0,T]$ such that $u(t_0) = \max_{[0,T]} u = \lambda + 2$ or $u(t_0) = \min_{[0,T]} u = -\lambda - 2$. Let us consider the case $u(t_0) = \max_{[0,T]} u = \lambda + 2$. If $t_0 \in]0,T[$, then $u'(t_0) = 0$ and there exists $\epsilon > 0$ such that $u(t) > \lambda + 1$ for all $t \in [t_0, t_0 + \epsilon]$. Moreover $-(\Phi(u'(t)))' + \Phi_p(u(t)) = -2$. Then $(\Phi(u'(t)))' = 2 + |u(t)|^p$. Whence, $\Phi(u'(t)) = \int_{t_0}^t (\Phi(u'(t)))' \, dt > 0$, for all $t \in [t_0, t_0 + \epsilon]$. It follows that u is increasing on $[t_0, t_0 + \epsilon]$. That's contradict the existence of t_0. If $t = 0, u'(0) = 0$, and we obtain the contradiction $0 \in B_1^*(u(0)) \subset \mathbb{R}_+^*$. If $t_0 = T, u'(T) = 0$, and we obtain the contradiction $0 \in B_2^*(u(T)) \subset \mathbb{R}_+^*$. In the similar fashion, we obtain contradiction with the case $u(t_0) = \min_{[0,T]} u = -\lambda - 2$. Therefore

$$[u \in \partial\Pi, G^*(u) = u] \Rightarrow \|u\|_\infty < \lambda + 2. \tag{4.5}$$

Let $u \in \partial\Pi$ such that $G^*(u) = u$. It becomes from (4.5) that $\|u\|_\infty < \lambda + 2$, and $u \in \partial\Pi_{\widetilde{\alpha},\beta} \cup \partial\Pi_{\alpha,\widetilde{\beta}}$. It follows, there exists $t_0 \in [0,T]$ such that $u(t_0) = \alpha(t_0)$ or $u(t_0) = \beta(t_0)$, that implies

$$|u(t_0)| < \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

We infer that

$$[u \in \partial\Pi, G^*(u) = u \Rightarrow \|u\|_\infty < \lambda]. \tag{4.6}$$

We have two cases.

1st **case:** We assume there exists $u \in \partial\Pi$ such that $G^*(u) = u$. From (4.6), we infer that $\|u\|_\infty < \lambda$, that implies that u is a solution of (1.1), and (4.2) and (4.3) are satisfied. Then, there exists $\sigma \in [0,T]$ such that $u(\sigma) = \alpha(\sigma)$ or $u(\sigma) = \beta(\sigma)$.

2nd **case:** We assume that $G^*(u) \neq u$ for all $u \in \partial\Pi$. Then, as in the proof of Theorem 3.1, for $R > \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$, we have:

$$d_{LS}\left(Id_{C^1} - G^*, \Pi_{\widetilde{\alpha},\widetilde{\beta}}, 0\right) = d_{LS}\left(Id_{C^1} - G^*, B_{C^1}(R), 0\right) = d_{LS}\left(Id_{C^1}, B_{C^1}(R), 0\right) = 1.$$

By additivity property of Leray Schauder's degree, we infer that:

$$\begin{aligned} &d_{LS}\left(Id_{C^1} - G^*, \Pi, 0\right) \\ &\quad = d_{LS}\left(Id_{C^1} - G^*, \Pi_{\widetilde{\alpha},\widetilde{\beta}}, 0\right) - d_{LS}\left(Id_{C^1} - G^*, \Pi_{\alpha,\widetilde{\beta}}, 0\right) - d_{LS}\left(Id_{C^1} - G^*, \Pi_{\widetilde{\alpha},\beta}, 0\right) = -1. \end{aligned}$$

So, there exists $u \in \Pi$ such that $G^*(u) = u$ and $\|u\|_\infty < \lambda$. Therefore u is solution of (1.1), and (4.2) and (4.3) are satisfied.

□

4.2. Existence of extremal solutions

Theorem 4.2. *If the hypotheses* (H_f), (H_Φ), *and* (H_B) *hold, then there exists a sequence of lower-solutions* $(\alpha_k)_{k\geqslant 1}$ *and a sequence of upper-solutions* $(\beta_k)_{k\geqslant 1}$ *in* $C^1(\Omega)$ *of* (1.1) *such that*

$$\exists \bar{t} \in [0,T] \text{ such that } \alpha_k(\bar{t}) > \beta_k(\bar{t}).$$

Then there exist two monotones sequences $(\min\{\alpha_k,\beta_k\})_{k\geqslant 1}$ *and* $(\max\{\alpha_k,\beta_k\})_{k\geqslant 1}$ *in* $C^1(\Omega)$ *such that* $\Phi(\alpha_k')$, $\Phi(\beta_k') \in W^{1,p}(\Omega)$, *which converge uniformly in* $C^1(\Omega)$ *to* $\overline{\eta}$ *and* $\overline{w}$ *solutions of problem* (1.1) *and verify inequlities*

$$\begin{aligned} \min\{\alpha,\beta\} \leqslant \min\{\alpha_1,\beta_1\} \leqslant \cdots \leqslant \min\{\alpha_k,\beta_k\} < \overline{\eta} \\ \leqslant \overline{w} < \max\{\alpha_k,\beta_k\} \leqslant \cdots \leqslant \max\{\alpha_2,\beta_2\} \leqslant \max\{\alpha_1,\beta_1\} \leqslant \max\{\alpha,\beta\}. \end{aligned}$$

Proof. Let us introduce the following set

$$U^* = [\min\{\alpha,\beta\},\max\{\alpha,\beta\}] = \left\{u \in W^{1,p}(\Omega) : \min\{\alpha(t),\beta(t)\} \leqslant u(t) \leqslant \max\{\alpha(t),\beta\} \text{ on } \Omega\right\}.$$

For all $\nu \in U^*$ fixed, we consider the following problem

$$(P_\nu) \begin{cases} -(\Phi(u'(t)))' + \phi_p(u(t)) = \varepsilon f(t,\nu(t)) \text{ a.e. on } \Omega = [0,T], \\ u'(0) \in B_1(u(0)),\ -u'(T) \in B_2(u(T)). \end{cases} \tag{4.7}$$

Then, by Lemma 2.9, the problem (4.7) admits an unique solution u_ν. Let $L^* : U^* \to U^*$ be the operator defined by $\forall \nu \in U^*$, $L^*\nu = u_\nu$, where u_ν is the unique solution of problem (P_ν). When α and β are well ordered, for some $t \in \Omega$, L^* coincides with L. Then by Theorems 3.1 and 3.6, the problem admits a solution and some extremal solutions.

Now suppose that α and β are in reversed order for some $t \in \Omega$. Let us introduce the following set

$$\Lambda = \{t \in \Omega : \beta(t) < u(t) < \alpha(t)\}.$$

Let us show that

(i) $\alpha \geqslant L^*\alpha$;
(ii) $\beta \leqslant L^*\beta$;
(iii) L^* is decreasing.

In order to prove (i), we set $L^*\alpha = \alpha_1$. We have

$$\begin{cases} -(\Phi(\beta'(t)))' + \phi_p(\beta(t)) \geqslant -f(t,\beta(t)) > -f(t,\alpha(t)) \text{ a.e. on } \Lambda, \\ \beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+,\ -\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+, \end{cases}$$
$$\begin{cases} -(\Phi(\alpha'(t)))' + \phi_p(\alpha(t)) \leqslant -f(t,\alpha(t)) \text{ a.e. on } \Lambda, \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+,\ -\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+. \end{cases}$$

So α and β are lower and upper solution of (P_α) such that $\alpha(t) > \beta(t)$, on Λ. Then by Lemma 2.9, α_1 is the unique solution of (P_α) such that $\alpha_1 \in U^*$. In order to prove (ii), we set $L^*\beta = \beta_1$. We have

$$\begin{cases} -(\Phi(\alpha'(t)))' + \phi_p(\alpha(t)) \leqslant -f(t,\alpha(t)) < -f(t,\beta(t)) \text{ a.e. on } \Lambda, \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+,\ -\alpha'(T) \in B_2(\alpha(T) + \mathbb{R}_+, \end{cases}$$
$$\begin{cases} -(\Phi(\beta'(t)))' + \phi_p(\beta(t)) \geqslant -f(t,\beta(t)) \text{ a.e. on } A, \\ \beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+,\ -\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+. \end{cases}$$

So α and β are lower and upper solutions (P_β) such that $\alpha(t) > \beta(t)$, on Λ. Then by Lemma 2.9, β_1 is the unique solution of (P_β) such that $\beta_1 \in U^*$.

Let us show (iii). Let $v_1, v_2 \in U^*$ such that $v_1 \leqslant v_2$. We set $Lv_i = u_i, \forall i \in \{1;2\}$. We have

$$\begin{cases} -(\Phi(\alpha'(t)))' + \phi_p(\alpha(t)) \leqslant -f(t, v_2(t)) \text{ a.e. on } \Lambda, \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+, \ -\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+, \end{cases}$$
$$\begin{cases} -\left(\Phi(u_1'(t))\right)' + \phi_p(u_1(t)) \geqslant -f(t, v_1(t)) \geqslant -f(t, v_2(t)) \text{ a.e. on } \Lambda, \\ u_1'(0) \in B_1(u_1(0)) + \mathbb{R}_+, \ -u_1'(T) \in B_2(u_1(T)) + \mathbb{R}_+. \end{cases}$$

So α and u_1 are lower and upper solutions of the problem (P_{v_2}) such that $u_1(t) > \beta(t), \forall t \in \Omega$. By the Lemma 2.9, u_2 the unique solution of (P_{v_2}) is such that $u_2 \in]\beta, u_1]$. Thus L^* is decreasing on U^*. Then, arguing as in the proof of Theorem 3.6, we find two sequences $(\alpha_k)_{k\geqslant 1}$ and $(\beta_k)_{k\geqslant 1}$, which converge uniformely in $C^1(\Omega)$ to η^* and w^* solutions of (1.1) such that, for $k \in \mathbb{N}^*$,

$$\beta \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k < \eta^* \leqslant w^* < \alpha_k \leqslant \cdots \alpha_2 \leqslant \alpha_1 \leqslant \alpha \text{ on } \Lambda. \tag{4.8}$$

Let us consider the two sequences $(\min\{\alpha_k, \beta_k\})_{k\geqslant 1}$ and $(\max\{\alpha_k, \beta_k\})_{k\geqslant 1}$. From (3.11) and (4.8), we deduce

$$\begin{aligned} \min\{\alpha, \beta\} &\leqslant \min\{\alpha_1, \beta_1\} \leqslant \cdots \leqslant \min\{\alpha_k, \beta_k\} \\ &\leqslant \max\{\alpha_k, \beta_k\} \leqslant \cdots \leqslant \max\{\alpha_2, \beta_2\} \leqslant \max\{\alpha_1, \beta_1\} \leqslant \max\{\alpha, \beta\} \text{ on } \Lambda. \end{aligned}$$

We have, for all $k \in \mathbb{N}^*, \overline{u}_k \in \{\min\{\alpha_k, \beta_k\}, \max\{\alpha_k, \beta_k\}\}$. Arguing as in the proof of Theorem 3.6, it follows that the sequence $(\overline{u}_k)_{k\geqslant 1}$ is bounded. So it admits a subsequence, which converge to a solution $\overline{u}$ of problem (1.1). Therefore, the sequences $(\min\{\alpha_k, \beta_k\})_{k\geqslant 1}$ and $(\max\{\alpha_k, \beta_k\})_{k\geqslant 1}$ admit some subsequences, which converge to $\overline{\eta} = \min\{\eta, \eta^*\}$ and $\overline{w} = \max\{w, w^*\}$, respectively. Thus, for $k \in \mathbb{N}^*$

$$\begin{aligned} \min\{\alpha, \beta\} &\leqslant \min\{\alpha_1, \beta_1\} \leqslant \cdots \leqslant \min\{\alpha_k, \beta_k\} < \overline{\eta} \leqslant < \overline{w} \\ &< \max\{\alpha_k, \beta_k\} \leqslant \cdots \leqslant \max\{\alpha_2, \beta_2\} \leqslant \max\{\alpha_1, \beta_1\} \leqslant \max\{\alpha, \beta\} \text{ on } \Omega. \end{aligned}$$

□

5. Special cases

For $i = 1, 2$, let I_i be a real closed interval containing 0. We consider the following indicator function

$$E_{I_i}(x) = \begin{cases} 0, & \text{if } x \in I_i, \\ +\infty, & \text{if otherwise.} \end{cases}$$

We set $B_i = \partial E_{I_i}$, the subdifferential in the sense of convex analysis of E_{I_i}, for all $i = 1, 2$.

1. If $I_i = \{0\}$, then $B_i(x) = \mathbb{R}$ for all $x \in \mathbb{R}$ and for all $i = 1, 2$. Hence, the problem (1.1) becomes the classical problem of Dirichlet.
2. If $I_i = \mathbb{R}$, then $B_i(x) = \{0\}$ for all $x \in \mathbb{R}$ and for all $i = 1, 2$. Hence, the problem (1.1) becomes the classical problem of Neumann.
3. If $B_i(x) = \frac{1}{a_i}x, a_i \neq 0$, for all $i = 1, 2$, then the problem (1.1) becomes the classical problem of Sturm-Liouville.

6. Example

Let us consider the following problem

$$\begin{cases} -\left(\left(1 + \dfrac{1}{1 + (u'(t))^3}\right)(u'(t))^3\right)' + (u(t))^3 = \varepsilon\left(\dfrac{(u(t))^3}{e^{3t} + 1} + 1\right) \text{ a.e. on } \Omega = [0, T], \\ u'(0) \in B_1(u(0)), \ -u'(T) \in B_2(u(T)). \end{cases} \tag{6.1}$$

Here

$$\Phi(x)=\left(1+\frac{1}{1+x^3}\right)x^3,\quad \Phi_p(x)=x^3,\quad \text{with } p=4;\qquad f(t,x)=\frac{x^3}{e^{3t}+1}+1.$$

It easy to check that the hypotheses (H_Φ) and (H_f) are satisfied. Suppose that, for $i=1,2$,

$$B_i(x)=\begin{cases}\emptyset, & \text{if } x>e^T+1,\\ \mathbb{R}_+, & \text{if } x=e^T+1,\\ \{0\}, & \text{if } x\in\left]-e^T-1,e^T+1\right[,\\ \mathbb{R}_-, & \text{if } x=-e^T-1,\\ \emptyset, & \text{if } x<-e^T-1.\end{cases}\qquad \text{or}\qquad B_i(x)=\partial E_{I_i}(x).$$

Then hypothesis (H_B) is satisfied. Let us set $\alpha(t)=1$ and $\beta(t)=2$. We check that α and β are, respectively, well ordered lower and upper solutions of (6.1). Then, by Theorem 3.1, the problem (6.1) admits a solution in the functional interval $[\alpha,\beta]$. Let us set $\alpha(t)=e^t$ and $\beta(t)=t+5$. We check that α and β are, respectively, non well ordered lower and upper solutions of (6.1). Then, by Theorems 4.1 and 4.2, problem (6.1) admits a solution and extremal solutions in the functional interval $[\min\{\alpha,\beta\},\max\{\alpha,\beta\}]$.

7. Periodic problem

We recall that the problem (1.1) does not contain the periodic problem. However our method can be used to deal with it. Let us consider the following periodic problem

$$\begin{cases}-(\Phi(u'(t)))'+\Phi_p(u(t))=f(t,(u(t))\ \text{a.e. on } \Omega=[0,T],\\ u(0)=u(T),\ u'(0)=(u'(T),\end{cases}\tag{7.1}$$

when the lower and upper solutions are well ordered. We define the set D as

$$D=\left\{u\in C^1(\Omega):\Phi(u')\in W^{1,q}(0,T),u(0)=u(T)\text{ and } u'(0)=u'(T)\right\}.$$

Arguing as in the proof of Lemma 2.9, we prove that the following problem has an unique solution

$$\begin{cases}-(\Phi(u'(t)))'+\Phi_p(u(t))=h(t)\ \text{a.e. on } \Omega=[0,T],\\ u(0)=u(T),\ u'(0)=(u'(T).\end{cases}$$

We replace in this case the auxiliary problem (2.2) by the following nonhomogeneous Dirichlet problem

$$\begin{cases}-(\Phi(u'(t)))'+\Phi_p(u(t))=h(t)\ \text{a.e. on } \Omega=[0,T],\\ u(0)=u(T)=a.\end{cases}$$

It follows that $G:C^1(\Omega)\to D\subseteq C^1(\Omega)$ defined by

$$G(u)(t)=u(0)+\int_0^t\Phi^{-1}\circ\left(\Phi(u'(0))-\int_0^y(\varepsilon f(s,u(s))-\Phi_p(u(s)))\,ds\right)dy,$$

is completely continuous and so as in the proof of Theorem 3.1, we end up solving the abstract fixed point problem

$$u=G(u).\tag{7.2}$$

Using Leray-Schauder topological degree, we solve (7.2) and show that $u\in C^1(\Omega)$ is a solution of (1.1) in the interval U. Finally, arguing as in the proof of Theorem 3.6, we show existence of monotones sequences $(\alpha_k)_{k\geqslant1}$ and $(\beta_k)_{k\geqslant1}$ in $C^1(\Omega)$ such that $\phi(\alpha_k')\in W^{1,p}(\Omega)$ and $\phi(\beta_k')\in W^{1,p}(\Omega)$, which converge uniformly in $C^1(\Omega)$ to η and w solutions of problem (7.1) and verify inequalities

$$\alpha(t)\leqslant\alpha_1(t)\leqslant\alpha_2(t)\leqslant\cdots\leqslant\alpha_k(t)\leqslant\eta\leqslant w\leqslant\beta_k(t)\leqslant\cdots\leqslant\beta_2(t)\leqslant\beta_1(t)\leqslant\beta(t).$$

Now, we consider the following periodic problem

$$\begin{cases} -(\Phi(u'(t)))' + \Phi_p(u(t)) = -f(t,(u(t)) \text{ a.e. on } \Omega = [0,T], \\ u(0) = u(T),\ u'(0) = u'(T), \end{cases} \tag{7.3}$$

when the lower and upper solutions are non well ordered. Then, we consider the following modified problem:

$$\begin{cases} -(\Phi(u'(t)))' + \Phi_p(u(t)) = f^*(t,u(t)) \text{ ae on } \Omega = [0,T], \\ u(0) = u(T), \\ u'(0) = K^*(u(T),u'(T)), \end{cases}$$

where f^* is defined as in the proof of Theorem 4.1 and $K^* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by:

$$K^*(u,y) = \begin{cases} 3, & \text{if } u > \lambda + 1, \\ (1+\lambda-u)y + 3(u-\lambda), & \text{if } \lambda < u \leqslant \lambda + 1, \\ y, & \text{if } -\lambda \leqslant u \leqslant \lambda, \\ (1+\lambda+u)y + 3(u+\lambda), & \text{if } -\lambda - 1 \leqslant u < -\lambda, \\ -3 & \text{if } u < -\lambda - 1. \end{cases}$$

Finally, arguing as in the proof of Theorems 4.1 and 4.2, we obtain the analogous results for the problem (7.3).

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