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Some observations on statistical convergence of uncertain double sequences of fuzzy numbers

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Abstract

In this paper, we expand upon different concepts of convergence, including convergence in distribution, in the mean, and in measure, while also exploring uniform almost sure convergence of uncertain double sequences of fuzzy numbers in statistical context. Additionally, we present intriguing findings concerning these convergence concepts.

Keywords: Uncertain sequence, fuzzy number, statistical convergence.

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1. Introduction and background

The uncertain theory, initially introduced by Liu [13] within the framework of measure theory, has gained widespread recognition due to its versatile applications in various domains, including probability theory, statistics, fuzzy set theory, measure theory, summability theory, and more. This theory has found utility in risk assessment and uncertain reliability analysis (Liu [10]), modeling human language using uncertainty logic (Liu [11]), continuous uncertain measure theory (Gao [6]), and various pure and theoretical mathematical fields.

Furthermore, the uncertain theory has extended its influence into practical applications such as uncertain finance (Liu [13]), uncertain optimization theories (Liu [12]), and others. For an in-depth exploration of uncertain complex sequences, their convergence, and practical applications, interested readers can refer to the works of Baliarsingh et al. [1], Chen et al. [2], Tripathy and Nath [21], Yuan et al. [22] and the references provided therein. In the following sections, we draw from these aforementioned papers to present the definitions of uncertain measure and related concepts.

In 1965, Zadeh [23] introduced the concept of fuzzy numbers and conducted an in-depth exploration of this notion, focusing on its arithmetic operations. Subsequently, this concept found applications in various practical and scientific domains, including fuzzy programming, fuzzy optimization, fuzzy measures for fuzzy events, and more. The idea was also extended to sequence spaces and summability theory by

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Matloka [14] in a broader context. Nanda [15] integrated the concept into topology and vector spaces by introducing the idea of a fuzzy metric. Furthermore, applications of this concept have been identified in the theory of double sequence spaces by Savas and Mursaleen [19] sequence spaces with fuzzy mappings by Tripathy and Nanda [20] sequence spaces with ideal convergence using Orlicz functions by Hazarika and Savas [7].

In the classical sense, sequence convergence typically involves most terms of the sequence being in a small neighborhood of the limit. However, applying this concept to all types of sequences can be challenging. Statistical convergence of sequences, introduced by Fast [4] and further developed by Fridy [5], offers a broader perspective on convergence. Notable research works in this area include those by Demirci and Gürdal [3], Huban and Gürdal [8, 9], Savaş and Gürdal [17, 18].

Our study's main goal is to examine the practical application of uncertainty theory within the domains of optimization and fuzzy set theory. This investigation enables us to establish correlations between this theoretical framework and a diverse range of applications in various computational and applied fields.

Recently, summability theory has assumed a crucial role in forging connections among various mathematical domains, including sequence spaces, operator theory, fuzzy set theory, and uncertainty theory. This research, employing statistical convergence concepts, establishes a bridge between uncertainty theory and fuzzy set theory.

2. Preliminaries

In this section, we give significant existing conceptions and results which are crucial for our findings. Next, we present the definitions of uncertain measure and its associated concepts. Using $\mathcal{L}$, we represent a σ-algebra on a non-empty set Γ. An uncertain measure, denoted as $\mathcal{M}$, is characterized by four key axioms.

(i) Normality, which states $\mathcal{M}\{\Gamma\} = 1$.

(ii) Duality, represented as $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$.

(iii) Subadditivity, meaning that for every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$,
$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leqslant \sum_{j=1}^{\infty} \mathcal{M}\left\{\Lambda_j\right\}.$$

(iv) The product axiom, known as
$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\},$$
applies when Λ_k represents arbitrarily selected events from $\mathcal{L}_k$ for $k = 1, 2, 3, \ldots$.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ forms an uncertainty space, where Λ, an element in $\mathcal{L}$, represents an event. A measurable function ϕ mapping from $(\Gamma, \mathcal{L}, \mathcal{M})$ to $\mathbb{R}$ is known as an uncertain variable. In other words, for a Borel set $\mathcal{B}$ (a subset of real numbers), the set
$$\{\phi \in \mathcal{B}\} = \{\gamma \in \Gamma : \phi(\gamma) \in \mathcal{B}\},$$
constitutes an event. If ϕ is an uncertain variable, its uncertainty distribution ψ is defined as:
$$\psi(x) = \mathcal{M}\{\phi \leqslant x\}, \quad \forall x \in \mathbb{R}.$$
The expected value of an uncertain variable ρ is given by
$$E[\phi] = -\int_{-\infty}^{0} \mathcal{M}\{\phi \leqslant \mu\} d\mu + \int_{0}^{\infty} \mathcal{M}\{\phi \geqslant \mu\} d\mu,$$

if at least one of the above two integrals exists and is finite. The uncertain sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is called to converge almost surely to ϕ if there is an event Λ with $\mathcal{M}\{\Lambda\}=1$ and for all $\gamma\in\Lambda$,

$$\lim_{n\to\infty}|\phi_n(\gamma)-\phi(\gamma)|=0.$$

In this situation, we denote it as $\phi_n\to\phi$, almost surely. It is called to be convergent in measure to ϕ if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\mathcal{M}\{|\phi_n-\phi|\geqslant\varepsilon\}=0,$$

and in mean to ϕ if

$$\lim_{n\to\infty}E\left[|\phi_n-\phi|\right]=0.$$

Consider the uncertainty distributions ψ and ψ_n for uncertain variables ϕ and ϕ_n, respectively. The sequence $\{\phi_n\}$ converges in distribution to ϕ if, for all x where $\psi(x)$ is continuous,

$$\lim_{n\to\infty}\psi_n(x)=\psi(x).$$

An uncertain sequence $\{\phi_n\}$ is considered to be convergent uniformly almost surely to ϕ if there is a sequence of events $\{E_k\}$ with $\mathcal{M}\{E_k\}\to 0$ such that it uniformly converges to ϕ in the set $\Gamma-E_k$ for fixed $k\in\mathbb{N}_0$.

A function $Q:\mathbb{R}\to[0,1]$ is designated as a fuzzy number under the following conditions: (i) Q is a normal function, meaning that there exists a value $u\in\mathbb{R}$ for which $Q(u)=1$; (ii) Q is fuzzy convex, i.e., $Q(u)\geqslant\min\{Q(p),Q(q)\}$, where $p<u<q$; (iii) Q is upper semi-continuous; (iv) the closure of $Q^0=\{u\in\mathbb{R}:Q(u)>0\}$ is compact.

We denote the collection of all fuzzy numbers on the real numbers set $\mathbb{R}$ as $L(\mathbb{R})$. In essence, we can establish a connection between the set $\mathbb{R}$ as $L(\mathbb{R})$ as follows: for any $r\in\mathbb{R}$, we can define a function $\bar{s}(u)\in L(\mathbb{R})$ using the rule

$$\bar{s}(u)=\begin{cases}1, & \text{if } u=r\\ 0, & \text{if } u\neq r.\end{cases}$$

Take $\alpha\in(0,1]$. We define the α-level cut of the fuzzy number Q as:

$$[Q]_\alpha=\{u\in\mathbb{R}:Q(u)\geqslant\alpha\}.$$

Additionally, the distance between two fuzzy numbers Q and R can be expressed as:

$$d(Q,R)=\sup_{\alpha\in[0,1]}d_H\left([Q]_\alpha,[R]_\alpha\right).$$

Notably, the symbol $d_H(.,.)$ denotes the Hausdorff metric, defined as:

$$d_H\left([Q]_\alpha,[R]_\alpha\right)=\max\left\{\left|[Q]_\alpha^- - [R]_\alpha^-\right|,\left|[Q]_\alpha^+ - [R]_\alpha^+\right|\right\},$$

where $[Q]_\alpha^+$and $[Q]_\alpha^-$, respectively, represent the upper and lower bounds of the α-level cut of the fuzzy number Q. The set $L(\mathbb{R})$ forms a complete metric space using the metric d. A sequence $\{\xi_n\}$ is said to be statistically convergent to ξ provided that for every $\varepsilon>0$ the set $K(\varepsilon)=\{k\in\mathbb{N}:|\xi_k-\xi|\geqslant\varepsilon\}$ has the natural density zero (see, Fast [4]), i.e.,

$$\delta(K(\varepsilon))=\lim_{n\to\infty}\frac{1}{n}\left|\{k\leqslant n:|\xi_k-\xi|\geqslant\varepsilon\}\right|=0,$$

where vertical line represents the cardinality of the set enclosed. Following the primary definition of statistically convergent of sequence, the fuzzy analog statistically convergence has been given by Nuray and Savas [16] as follows. A sequence $\{Q_k\}_{\in\mathbb{N}}$ of fuzzy numbers is said to be statistically convergent to Q_0 provided that for every $\varepsilon>0$,

$$\delta\left(k:d\left(Q_k,Q_0\right)\geqslant\varepsilon\right)=\lim_{n\to\infty}\frac{1}{n}\left|\{k\leqslant n:d\left(Q_k,Q_0\right)\geqslant\varepsilon\}\right|=0.$$

Extending the application of fuzzy numbers, we delve into the theory of statistical convergence for uncertain fuzzy double sequences.

3. Main results

In this section, we expand upon the definitions provided in the preliminaries section of uncertain theory in fuzzy statistical context.

Definition 3.1. The uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is called to be statistical convergent to Q_0 if for each $\varepsilon > 0$, there exists an event Λ and for all $\gamma \in \Lambda$ such that

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\right) = 0.$$

In such cases, we denote this as st-lim $Q_{kl}(\gamma) = Q_0(\gamma)$.

Definition 3.2. The uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is considered to be statistically bounded, if for every $\gamma \in \Lambda$ there exists a real number T such that

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}(\gamma), \overline{0}\right) > T\right) = 0.$$

Definition 3.3. The uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is called to be statistically convergent almost surely to Q_0 if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\right) = 0,$$

for every $\gamma \in \Lambda$. In this case we write $Q_{kl} \to Q_0$, statistical convergent almost surely.

Definition 3.4. The uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is called to be statistically convergent in measure to Q_0 if

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\{\gamma \in \Lambda : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\}\right) \geqslant \delta^*\right) = 0,$$

for every $\varepsilon, \delta^* > 0$.

Definition 3.5. The uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is said to be statistically convergent in mean to Q_0 if

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : E\left[d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\right] \geqslant \varepsilon\right) = 0,$$

for every $\varepsilon > 0$ and $\gamma \in \Lambda$.

Definition 3.6. Let ψ, ψ_{kl} be the uncertainty distributions of uncertain variables Q, Q_{kl}, respectively. Then, the uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is statistically convergent in distribution to Q_0 if for every $\varepsilon > 0$,

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : d\left(\psi_{kl}(x), \psi(x)\right) \geqslant \varepsilon\right) = 0,$$

for all x at which $\psi(x)$ is continuous.

Definition 3.7. The uncertain double sequence $\{Q_{kl}\}$ of fuzzy number is called to be statistically convergent uniformly almost surely to Q_0 if for every $\varepsilon > 0$, there exist $\{E_k\}$ with $\mathcal{M}\left(E_k\right) \to 0$ such that

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\right) = 0,$$

for every $\gamma \in \Gamma - \{E_k\}$.

Example 3.8. Consider the event $\gamma \in \Lambda$ and associated fuzzy uncertain double sequence $\{Q_{kl}\}$, defined by

$$Q_{kl}(\gamma) = \begin{cases} \frac{1}{2}, & \text{if } k = m^2, l = n^2, \\ p(\gamma), & \text{if not,} \end{cases} \qquad (m, n \in \mathbb{N}).$$

For given $\varepsilon > 0$, it is evident that

$$\begin{aligned} \delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}(\gamma), p(\gamma)\right) \geqslant \varepsilon\right) &= \lim_{r,s\to\infty} \frac{1}{rs} \left|\{0 < k \leqslant r, 0 < l \leqslant s : d\left(Q_{kl}(\gamma), p(\gamma)\right) \geqslant \varepsilon\}\right| \\ &\leqslant \lim_{r,s\to\infty} \frac{\sqrt{rs}}{rs} = 0. \end{aligned}$$

We say that st-lim $Q_{kl} = p$.

Example 3.9. Consider the event $\gamma \in \Lambda$, a fixed fuzzy number p and associated fuzzy uncertain sequence $\{Q_{kl}\}$, defined by

$$Q_{kl}(\gamma) = \begin{cases} p(\gamma), & \text{if } k = m^2, l = n^2,\ m, n \neq 1, \\ \frac{1}{4}, & \text{if } k = m^3, l = n^3, \\ 0, & \text{if not,} \end{cases} \qquad (m, n \in \mathbb{N}).$$

Now, we can calculate

$$d\left(Q_{kl}(\gamma), 0\right) = \begin{cases} \sup_{\alpha \in [0,1]} \max\{|[p(\gamma)]_\alpha^-|, |[p(\gamma)]_\alpha^+|\}, & \text{if } k = m^2, l = n^2,\ m, n \neq 1, \\ \frac{1}{4}, & \text{if } k = m^3, l = n^3, \\ 0, & \text{if not,} \end{cases} \qquad (m, n \in \mathbb{N}).$$

For every $\gamma \in \Lambda$ and given $\varepsilon > 0$, we have

$$\begin{aligned} \delta\left((k, l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}(\gamma), 0\right) \geqslant \varepsilon\right) &= \lim_{r,s \to \infty} \frac{1}{rs} \left|\{0 < k \leqslant r, 0 < l \leqslant s : d\left(Q_{kl}(\gamma), 0\right) \geqslant \varepsilon\}\right| \\ &\leqslant \lim_{r,s \to \infty} \frac{\sqrt{rs} + \sqrt[3]{rs}}{rs} = 0. \end{aligned}$$

Hence, we express that st-lim $Q_{kl}(\gamma) = 0$.

Now, we establish the connections between different convergence concepts outlined in the preceding section. Additionally, we present the first theorem, which is pivotal for our main findings, albeit without providing its proof.

Theorem 3.10. *Let $\{Q_{kl}\}$ and $\{R_{kl}\}$ be two uncertain fuzzy double sequences. It can be asserted that the subsequent statements are valid.*

(i) *If st-*lim $Q_{kl} = Q_0$, *st-*lim $R_{kl} = R_0$, *and* $\lambda, \eta \in \mathbb{C}$, *then st-*lim $(\lambda Q_{kl} + \eta R_{kl}) = \lambda Q_0 + \eta R_0$.
(ii) *If st-*lim $Q_{kl} = Q_0$ *and there exists a fuzzy double sequence* $\{R_{kl}\}$ *such that* $Q_{kl} = R_{kl}$ *for all most of* k, l, *then st-*lim $R_{kl} = R_0$.

Note that for two fuzzy sequences $\{Q_{kl}\}$ and $\{R_{kl}\}$, if $Q_{kl} = R_{kl}$ for all most of k, l, then for given $\varepsilon > 0$,

$$\delta\left((k, l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}(\gamma), R_{kl}\right) \geqslant \varepsilon\right) = \lim_{r,s \to \infty} \frac{1}{rs} \left|\{0 < k \leqslant r, 0 < l \leqslant s : d\left(Q_{kl}(\gamma), R_{kl}\right) \geqslant \varepsilon\}\right| = 0.$$

Now, we five the following definition.

An uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is considered to be strongly convergent to Q_0 provided that for each $\gamma \in \Lambda$,

$$\lim_{r,s \to \infty} \frac{1}{rs} \sum_{k,l=1,1}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) = 0.$$

Therefore, we establish the following theorem.

Theorem 3.11. *If a uncertain double sequence $\{Q_{kl}\}$ of fuzzy numbers is strongly convergent to Q_0, then it is statistically convergent to the same limit. However, the converse of the theorem holds if the uncertain sequence $\{Q_{kl}\}$ is bounded.*

Proof. Suppose $\{Q_{kl}\}$ is an uncertain double sequence of fuzzy numbers which is strongly convergent to Q_0. Then, we can deduce the following

$$\frac{1}{rs} \sum_{k,l=1,1}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) = \frac{1}{rs} \left[\sum_{k,l=1,1 \& d(Q_{kl}(\gamma), Q_0(\gamma)) < \varepsilon}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \right.$$

$$+ \sum_{k,l=1,1\&d(Q_{kl}(\gamma),Q_0(\gamma))\geqslant\varepsilon}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\Bigg]$$
$$\geqslant \frac{1}{rs}\sum_{k,l=1,1\&d(Q_{kl}(\gamma),Q_0(\gamma))\geqslant\varepsilon}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right).$$

Assuming the uncertain sequence $\{Q_{kl}\}$ is both bounded and statistically convergent to Q_0. Given an event Λ and for all $\gamma \in \Lambda$, there exists a constant T such that

$$\sup_{k,l}\left(d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\right) \leqslant T < \infty,$$

and

$$\delta\left((k,l) \in \mathbb{N}\times\mathbb{N} : d\left(Q_{kl}(\gamma), R_{kl}\right) \geqslant \varepsilon\right) = 0. \tag{3.1}$$

For each $\gamma \in \Lambda$, we obtain

$$\begin{aligned}
&\frac{1}{rs}\sum_{k,l=1,1}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\\
&\quad= \frac{1}{rs}\left[\sum_{k,l=1,1\&d(Q_{kl}(\gamma),Q_0(\gamma))<\varepsilon}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \sum_{k,l=1,1\&d(Q_{kl}(\gamma),Q_0(\gamma))\geqslant\varepsilon}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\right]\\
&\quad\leqslant \frac{1}{rs}\left[\varepsilon T \sum_{k,l=1,1\&d(Q_{kl}(\gamma),Q_0(\gamma))<\varepsilon}^{r,s} 1 \sum_{k,l=1,1\&d(Q_{kl}(\gamma),Q_0(\gamma))\geqslant\varepsilon}^{r,s} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\right].
\end{aligned}$$

By letting $r, s \to \infty$ on both sides and employing equation (3.1), we can conclude that $\{Q_{kl}\}$ exhibits strong convergence to the same limit Q_0. □

Theorem 3.12. *If the uncertain sequence $\{Q_{kl}\}$ of fuzzy numbers is statistically convergent in mean to Q_0, then it is statistically convergent in measure to Q_0.*

Proof. Suppose that the uncertain sequence $\{Q_{kl}\}$ is statistically convergent in mean to Q_0. Then, for given $\varepsilon > 0$, we have

$$\delta\left((k,l) \in \mathbb{N}\times\mathbb{N} : E\left[d\left(Q_{kl}, Q_0\right)\right] \geqslant \varepsilon\right) = 0.$$

Applying Markov's inequality, for any given $\delta^*, \varepsilon > 0$, we get

$$\begin{aligned}
&\delta\left((k,l) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\{\gamma \in \Lambda : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\}\right) \geqslant \delta^*\right)\\
&\quad= \lim_{r,s\to\infty}\frac{1}{rs}\left|\{0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M}\left(\{\gamma \in \Lambda : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\}\right) \geqslant \delta^*\}\right|\\
&\quad\leqslant \lim_{r,s\to\infty}\frac{1}{rs}\left|\left\{0 < k \leqslant r, 0 < l \leqslant s : \left(\frac{E\left[d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\right]}{\varepsilon}\right) \geqslant \delta^*\right\}\right|.
\end{aligned}$$

This result indicates that $\{Q_{kl}\}$ statistically converges in measure to the same limit. □

The opposite of the aforementioned theorem is not true. To illustrate this, we provide the following example.

Example 3.13. Consider the uncertainty space for fuzzy sequence $(\Gamma, \mathcal{L}, \mathcal{M})_Q$ to be $\mathcal{Y}$, where $\mathcal{Y} = \{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ such that $K_1(\gamma) = \sup_{\gamma_{k+l}\in\Lambda}\frac{1}{k+l+1}$ and $K_2(\gamma) = \sup_{\gamma_{k+l}\in\Lambda^c}\frac{1}{k+l+1}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} K_1(\gamma), & \text{if } K_1(\gamma) < 0.5,\\ 1 - K_2(\gamma), & \text{if } K_2(\gamma) < 0.5,\\ 0.5, & \text{if not.}\end{cases}$$

Consider the uncertainty space described above, linked to uncertain variables defined as follows

$$Q_{kl}(\gamma) = (k+l+1)\mathcal{D}(\gamma, \gamma_{k+l}) \quad (k, l = 1, 2, 3, \ldots),$$

and $Q_0 \equiv \overline{0}$. Here $\mathcal{D}(\gamma, \gamma_{k+l})$ denotes the Kronecker delta function which is given by

$$\mathcal{D}(t, x) = \begin{cases} 1, & \text{if } t = x, \\ 0, & \text{if not.} \end{cases}$$

For given $\varepsilon, \delta^* > 0$ and $k, l \geqslant 2$, we get

$$\begin{aligned} &\delta\left((k,l) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\{\gamma \in \Lambda : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\}\right) \geqslant \delta^*\right) \\ &= \lim_{r,s\to\infty} \frac{1}{rs} \left|\{0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M}\left(\{\gamma \in \Lambda : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) \geqslant \varepsilon\}\right) \geqslant \delta^*\}\right| \\ &= \lim_{r,s\to\infty} \frac{1}{rs} \left|\left\{0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M}\left(\left\{\gamma \in \Lambda : d\left(Q_{kl}(\gamma), \overline{0}\right) \geqslant \varepsilon\right\}\right) \geqslant \delta^*\right\}\right| = 0. \end{aligned}$$

This implies that the uncertain double sequence $\{Q_{kl}\}$ statistically converges in measure to 0. Additionally, the uncertainty distribution of $d(Q_{kl}, Q_0) = d\left(Q_{kl}, \overline{0}\right)$ for each $k, l \geqslant 2$ can be found out by

$$\psi_{kl}(x) = \begin{cases} 1 - 1/(k+l+1), & \text{if } 0 \leqslant x < k+l+1 \\ 1, & \text{if } x \geqslant k+l+1 \\ 0, & \text{if not.} \end{cases}$$

It is noted that for each $k, l \geqslant 2$,

$$\begin{aligned} \delta\left((k,l) \in \mathbb{N}\times\mathbb{N} : E\left[d\left(Q_{kl}, Q_0\right) - 1\right]\right) &= \lim_{r,s\to\infty} \frac{1}{rs} \left|\{0 < k \leqslant r, 0 < l \leqslant s : E\left[d\left(Q_{kl}, Q_0\right) - 1\right]\}\right| \\ &= \lim_{r,s\to\infty} \frac{1}{rs} \left[\int_0^{r+s+1} \left[1 - \psi_{kl}(x)\right] dx - 1\right] = 0. \end{aligned}$$

Hence, the uncertain sequence $\{Q_{kl}\}$ is not statistically convergent in mean to 0.

Remark 3.14. If $\{Q_{kl}\}$ statistically converges almost surely, it may or may not exhibit statistical convergence in measure.

The preceding observation becomes evident through the following example.

Example 3.15. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})_F$ to be $\mathcal{Y} = (\gamma_1, \gamma_2, \ldots)$ such that

$$L_1(\gamma) = \sup_{\gamma_{k+l} \in \Lambda} \frac{k+l}{2(k+l)+1} \quad \text{and} \quad L_2(\gamma) = \sup_{\gamma_{k+l} \in \Lambda^c} \frac{k+l}{2(k+l)+1}$$

with

$$\mathcal{M}\{\Lambda\} = \begin{cases} L_1(\gamma), & \text{if } L_1(\gamma) < 0.5, \\ 1 - L_2(\gamma), & \text{if } L_2(\gamma) < 0.5, \\ 0.5, & \text{if not.} \end{cases}$$

Define an uncertain variable as

$$Q_{kl}(\gamma) = \begin{cases} k+l, & \text{if } \gamma = \gamma_{k+l}, \\ 1, & \text{if not,} \end{cases}$$

for all $k,l \in \mathbb{N}$ and also, $Q_0 \equiv \overline{0}$. It can be readily verified that the sequence statistically converges almost surely to Q_0. However, it is observed that

$$\begin{aligned}
&\lim_{r,s\to\infty}\frac{1}{rs}\left|\left\{0<k\leqslant r, 0<l\leqslant s : \mathcal{M}\left(\{\gamma : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\geqslant \varepsilon\}\right)\geqslant \frac{1}{2}\right\}\right| \\
&\quad=\lim_{r,s\to\infty}\frac{1}{rs}\left|\left\{0<k\leqslant r, 0<l\leqslant s : \mathcal{M}\left(\left\{\gamma : d\left(Q_{kl}(\gamma), \overline{0}\right)\geqslant \varepsilon\right\}\right)\geqslant \frac{1}{2}\right\}\right| \\
&\quad=\lim_{r,s\to\infty}\frac{1}{rs}\left|\left\{0<k\leqslant r, 0<l\leqslant s : \mathcal{M}\left(\gamma_k\right)\geqslant \frac{1}{2}\right\}\right| = \frac{1}{2}.
\end{aligned}$$

Theorem 3.16. *If the sequence $\{Q_{kl}\}$ statistically converges in measure to Q_0, then it need not to be statistically convergent almost surely.*

To illustrate the above theorem, we employ the following example.

Example 3.17. With the help of Borel algebra and Lebesgue measure, consider the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})_Q$ to be $[0,1]$. Then, there exist integers w,q such that $k=2^w+s$, $l=2^q+s$, where s is an integer between 0 and $\min\{2^w,2^q\}-1$. For any $k,l\in\mathbb{N}$ we establish an uncertain variable by

$$Q_{kl}(\gamma)=\begin{cases}1, & \text{if } \frac{s}{2^{w+q}}\leqslant\gamma\leqslant\frac{s+1}{2^{w+q}},\ (k,l=1,2,3,\ldots),\\ 0, & \text{if not,}\end{cases}$$

and $Q_0\equiv\overline{0}$. For given $\varepsilon,\delta^*>0$ and $k,l\geqslant 2$, one can obtain

$$\begin{aligned}
&\lim_{r,s\to\infty}\frac{1}{rs}\left|\{0<k\leqslant r, 0<l\leqslant s : \mathcal{M}\left(\{\gamma : d\left(Q_{kl}(\gamma), Q_0(\gamma)\right)\geqslant \varepsilon\}\right)\geqslant \delta^*\}\right| \\
&\quad=\lim_{r,s\to\infty}\frac{1}{rs}\left|\left\{0<k\leqslant r, 0<l\leqslant s : \mathcal{M}\left(\left\{\gamma : d\left(Q_{kl}(\gamma), \overline{0}\right)\geqslant \varepsilon\right\}\right)\geqslant \delta^*\right\}\right| \\
&\quad=\lim_{r,s\to\infty}\frac{1}{rs}\left|\{0<k\leqslant r, 0<l\leqslant s : \mathcal{M}\left(Q_{kl}\right)\geqslant \delta^*\}\right| = \frac{1}{2^{w+q}},
\end{aligned}$$

which tends to 0, as w,q tends to ∞. As a result, the sequence $\{Q_{kl}\}$ statistically converges in measure to Q_0. Furthermore, for given $\varepsilon>0$, it can be observed that

$$\lim_{r,s\to\infty}\frac{1}{rs}\left|\{0<k\leqslant r, 0<l\leqslant s : E\left[d\left(Q_{kl},Q_0\right)\right]\geqslant\varepsilon\}\right|=0.$$

This indicates that the sequence $\{Q_{kl}\}$ statistically converges in mean to Q_0. Nevertheless, for any $\gamma\in[0,1]$ there exist an infinite number of intervals in the form $\left[\frac{s}{2^{w+q}},\frac{s+1}{2^{w+q}}\right]$ that contain γ. Consequently $\{Q_{kl}\}$ does not exhibit statistical convergence almost surely to Q_0.

Remark 3.18. If the sequence $\{Q_{kl}\}$ is statistically convergent almost surely to Q_0, then it is not statistically convergent in mean to Q_0.

We consider the following example to provide the proof.

Example 3.19. Consider the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})_Q$ to be $\mathcal{Y}$ such that with

$$\mathcal{M}\{\Lambda\}=\sum_{\gamma_{k+l}\in\lambda}\frac{1}{2^{k+l}}.$$

For $k,l=1,2,3,\ldots$, take the uncertain variable as

$$Q_{kl}(\gamma)=2^{k+l}\mathcal{D}\left(\gamma,\gamma_{k+l}\right)$$

and $Q_0 \equiv 0$. Then, the sequence $\{Q_{kl}\}$ statistically converges almost surely to Q_0. Note that the uncertainty distributions of $\|Q_{kl}\|$ are

$$Q_{kl}(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_{k+l},\ (k,l = 1,2,3,\ldots), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \psi_{kl}(x) = \begin{cases} 0, & \text{if } x \leqslant 0, \\ 1 - \frac{1}{2^{k+l}}, & \text{if } 0 \leqslant x < 2^{k+l}, \\ 1, & \text{if } x \geqslant 2^{k+l}. \end{cases}$$

Then, we get

$$\lim_{r,s\to\infty} \frac{1}{rs} \left|\{0 < k \leqslant r, 0 < l \leqslant s : E\left[d\left(Q_{kl}, Q\right)\right] \geqslant 1\}\right| = 0.$$

This gives that the sequence $\{Q_{kl}\}$ does not statistically converge in mean to Q_0.

Theorem 3.20. *The sequence $\{Q_{rs}\}$ is statistically convergent almost surely to Q_0, if for any $\varepsilon, \delta^* > 0$, we have*

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M} \left(\bigcap_{k,l=1,1}^{\infty,\infty} \bigcup_{r,s=k,l}^{\infty,\infty} d\left(Q_{kl}, Q\right) \geqslant \varepsilon \right) \geqslant \delta^* \right\} \right| = 0.$$

Proof. Assuming that the uncertain double sequence $\{Q_{rs}\}$ statistically converges almost surely to Q_0, we can deduce the following from the definition,

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : d\left(Q_{kl}, Q_0\right) \geqslant \varepsilon\right) = \lim_{r,s\to\infty} \frac{1}{rs} \left|\{0 < k \leqslant r, 0 < l \leqslant s : d\left(Q_{kl}, Q_0\right) \geqslant \varepsilon\}\right| = 0,$$

for given $\varepsilon > 0$ and an event Λ with $\mathcal{M}(\Lambda) = 1$. So, for any $\varepsilon > 0$, there are k, l such that $d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) < \varepsilon$ for each $0 < k \leqslant r$, $0 < l \leqslant s$ and $\gamma \in \Lambda$, equivalently

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M} \left(\left\{ \gamma : \bigcap_{k,l=1,1}^{\infty,\infty} \bigcup_{r,s=k,l}^{\infty,\infty} d\left(Q_{kl}(\gamma), Q_0(\gamma)\right) < \varepsilon \right\} \right) \geqslant 1 \right\} \right| = 0,$$

or, simply we write

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M} \left(\bigcap_{k,l=1,1}^{\infty,\infty} \bigcup_{r,s=k,l}^{\infty,\infty} d\left(Q_{kl}, Q_0\right) < \varepsilon \right) \geqslant 1 \right\} \right| = 0.$$

Utilizing the duality axiom of the uncertain measure, we establish that for $\delta^* > 0$,

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M} \left(\bigcap_{k,l=1,1}^{\infty,\infty} \bigcup_{r,s=k,l}^{\infty,\infty} d\left(Q_{kl}, Q_0\right) \geqslant \varepsilon \right) \geqslant \delta^* \right\} \right| = 0.$$

□

Theorem 3.21. *The sequence $\{Q_{rs}\}$ statistically converges almost surely to Q_0 if and only if for any $\varepsilon, \delta^* > 0$, we have*

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M} \left(\bigcup_{r,s=k,l}^{\infty,\infty} d\left(Q_{kl}, Q_0\right) \geqslant \varepsilon \right) \geqslant \delta^* \right\} \right| = 0.$$

Proof. Assuming that the uncertain sequence $\{Q_{rs}\}$ statistically converges almost surely to Q_0, then, for given $\delta^* > 0$, we can find a set D such that $\mathcal{M}(D) < \delta^*$ and $\{Q_{rs}\}$ statistically converges uniformly almost surely to Q_0 on $\Gamma - D$. From the definition, for given $\varepsilon > 0$, there exist $k \leqslant r$, $l \leqslant s$ such that $d(Q_{rs} - Q_0) < \varepsilon$ for $\gamma \in \Gamma - D$. This implies that

$$\bigcup_{r,s=k,l}^{\infty,\infty} \{d(Q_{rs}, Q_0) < \varepsilon\} \subset D.$$

But, using the subadditivity axiom it yields that

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M}\left(\bigcup_{r,s=k,l}^{\infty,\infty} d(Q_{kl}, Q_0) \geqslant \varepsilon \right) \right\} \right| \leqslant \delta^*(\mathcal{M}(D)) < \delta^*.$$

This concludes the proof of the first part. Secondly, we consider for any $\varepsilon > 0$ and given $\delta^* > 0$,

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r, 0 < l \leqslant s : \mathcal{M}\left(\bigcup_{r,s=k,l}^{\infty,\infty} d(Q_{kl}, Q_0) \geqslant \varepsilon \right) \geqslant \delta^* \right\} \right| = 0.$$

Now, for $w \geqslant 1$ and given $\delta^* > 0$, we can find w_k, w_l satisfying

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcup_{r,s=k,l}^{\infty,\infty} d(Q_{kl}, Q_0) \geqslant \frac{1}{w} \right) \right) < \frac{\delta^*}{2^w}.$$

Now, setting the set D as

$$D = \bigcup_{w=1}^{\infty} \bigcup_{r,s=w_k,w_l}^{\infty,\infty} \left\{ d(Q_{rs}, Q_0) \geqslant \frac{1}{w} \right\},$$

we have

$$\delta(\mathcal{M}(D)) \leqslant \sum_{w=1}^{\infty} \delta\left(\mathcal{M}\left((k,l) \in \mathbb{N} \times \mathbb{N} : \bigcup_{r,s=w_k,w_l}^{\infty,\infty} d(Q_{rs}, Q_0) \geqslant \frac{1}{r} \right) \right) < \sum_{w=1}^{\infty} \frac{\delta^*}{2^w}.$$

Thus, for any $w = 1, 2, 3, \dots$ and $r > w_k$, $s > w_l$ we have

$$\sup_{\gamma \in \Gamma - D} d(Q_{rs}, Q_0) < \frac{1}{w}.$$

This concludes that $\{Q_{rs}\}$ statistically converges almost surely to Q_0. □

Theorem 3.22. *Let the sequence $\{Q_{rs}\}$ be statistically convergent uniformly almost surely to Q_0. Then it is statistically convergent almost surely to Q_0.*

Proof. Assume that the sequence $\{Q_{rs}\}$ is statistically convergent uniformly almost surely to Q_0. Then, from Theorem 3.21, for $\varepsilon > 0$ and $\delta^* > 0$, we have

$$\lim_{r,s\to\infty} \frac{1}{rs} \left| \left\{ 0 < k \leqslant r,\ 0 < l \leqslant s : \mathcal{M}\left(\bigcup_{r,s=k,l}^{\infty,\infty} d(Q_{kl}, Q_0) \geqslant \varepsilon \right) \geqslant \delta^* \right\} \right| = 0.$$

But, it is known that

$$\delta\left(\mathcal{M}\left(\bigcap_{k,l=1,1}^{\infty,\infty} \bigcup_{r,s=k,l}^{\infty,\infty} \{d(Q_{kl}, Q_0) \geqslant \varepsilon\} \right) \right) \leqslant \delta\left(\mathcal{M}\left(\bigcup_{r,s=k,l}^{\infty,\infty} \{d(Q_{kl}, Q_0) \geqslant \varepsilon\} \right) \right).$$

Taking limits on both sides, we have

$$\delta\left(\mathcal{M}\left(\bigcap_{k,l=1,1}^{\infty,\infty} \bigcup_{r,s=k,l}^{\infty,\infty} \{d(Q_{kl}, Q_0) \geqslant \varepsilon\} \right) \right) = 0,$$

from Theorem 3.20 we follow the proof. □

4. Conclusion

In this paper, we have delved into various aspects of convergence, encompassing convergence in distribution, in the mean, and in measure. Furthermore, we have explored the realm of uniform almost sure convergence in the context of uncertain fuzzy sequences of real numbers within a statistical framework. Our investigation has revealed several compelling insights regarding these convergence concepts.

These findings underscore the significance of understanding and applying convergence principles, shedding light on their relevance in statistical and uncertain contexts. The exploration of convergence enriches our comprehension of how uncertain fuzzy sequences behave, providing valuable insights for researchers and practitioners across diverse fields.

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