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S^{JS}-metric spaces: a survey

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Abstract

The aim of this survey article, is to present in one place the recently published results on S^{JS}-metric spaces, their generalizations and applications. We start with S^{JS}-metric spaces and study their properties. Then we deal with abstract S^{JS}-topological spaces induced by S^{JS}-metric and present several classical results including Cantor's intersection theorem. Next the notion of sequentially compactness on S^{JS}-metric spaces and properties of sequentially compact S^{JS}-metric spaces are studied. Some fixed point theorems are obtained for integral type contractive mappings. Finally we prove several new results on fixed point for rational type contractive mappings, obtain Ekeland's variational principle on S^{JS}-metric spaces as an application and in the end also present results regarding best S^{JS}-proximity point with application.

Keywords: S^{JS}-metric space, S^{JS}-topological space, sequentially compact S^{JS}-metric space, S-metric space, S_b-metric space, generalized metric, Cantor's intersection property, Ekeland's variational principle, contractive type mapping, $\mathcal{Z}$-type contractive map, rational type contractive map, fixed point, best S^{JS}-proximity point.

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1. Introduction and Preliminaries

Maurice Fréchet [27] introduced metric spaces in his work "Sur quelques points du calcul fonctionnel". A metric space is a set together with a metric (a real valued distance function between points of the set) on the set and this metric also induces topological properties like open and closed sets, which lead to the study of more abstract topological spaces [41]. However soon after the introduction of the concept of metric spaces by Fréchet in his seminal paper, it was felt by researchers that these conditions of metric are too abstract and unrealistic. There are two types of extensions/generalizations of a metric; replace real number set R by some other larger set or relax one of the conditions in the definition of a metric. There are many attempts in the literature to relax/generalize them by several researchers see probabilistic metric spaces [49], 2-metric spaces [28], fuzzy metric spaces [40], modular metric space with the Fatou property [45], generalized D-metric spaces [20, 50], b-metric spaces [16], pseudometric spaces/dislocated

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metric spaces [33], cone metric spaces [34], partial metric spaces [13], generalized cone metric spaces [4], JS-metric spaces [37], and so on [21, 36]. Recently Beg et al.[6] gave a very general notion of S^{JS}-metric (see Definition 2.1) which does not satisfy the triangle inequality and symmetry, and studied its properties with several examples. In this survey article we present all these results on S^{JS}-metric spaces, their generalizations and applications published in different journals, due to Beg et al. [6–8, 52–54].

First we recall some basic notions and notations for subsequent use.

Let A be a non-empty set and $d : A \times A \to [0, \infty]$ be a mapping. For any $a \in A$, define the set

$$C(d, A, a) = \{\{a_n\} \subset A : \lim_{n\to\infty} d(a_n, a) = 0\}.$$

Jleli and Samet [37] have given the following definition regarding a generalized metric space.

Definition 1.1 ([37])**.** Let $d : A \times A \to [0, \infty]$ be a mapping which satisfies the following conditions:

(i) $d(a, b) = 0$ implies $a = b$ for all $a, b \in A$;
(ii) for every $(a, b) \in A \times A$, we have $d(a, b) = d(b, a)$;
(iii) if $(a, b) \in A \times A$ and $\{a_n\} \in C(d, A, a)$, then $d(a, b) \leqslant p \limsup_{n\to\infty} d(a_n, b)$, for some $p > 0$.

The pair (A, d) is a generalized metric space, also known as JS-metric space.

Jleli and Samet [37] observed that any metric space, b-metric space and dislocated metric spaces are JS-metric space. Our below example shows that a rectangular metric space [11] may not be a JS-metric space.

Example 1.2. Let $X = \mathbb{R}$ and $d : X^2 \to [0, \infty)$ be defined as follows. $d(x, y) = d(y, x)$ for any $x, y \in X$, $d(x, y) = 0$ if $x = y$ and for $x \neq y$.

$$d(x, y) = \begin{cases} \frac{1}{n}, & \text{if } x = 1, y = 1 + \frac{1}{n} \text{ for any } n \geqslant 2, \\ \frac{1}{n^2}, & \text{if } x = 2, y = 1 + \frac{1}{n} \text{ for any } n \geqslant 2, \\ 3, & \text{otherwise.} \end{cases}$$

Then it can be easily verified that (X, d) is a rectangular metric space but it is not a metric space, because

$$d(1, \frac{3}{2}) + d(\frac{3}{2}, 2) = \frac{3}{4} < 3 = d(1, 2).$$

Here we see that $\{1 + \frac{1}{n}\}_{n\geqslant 2} \in C(d, X, 1)$ but there exits no $C > 0$ for which

$$d(1, 2) \leqslant C \limsup_{n\to\infty} d(1 + \frac{1}{n}, 2).$$

Hence X is not a JS-metric space.

We now give the definitions of S-metric space and S_b-metric space.

Definition 1.3 ([58])**.** Let X be a non-empty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions, for each $x, y, z, w \in X$:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $S(x, y, z) \leqslant S(x, x, w) + S(y, y, w) + S(z, z, w)$.

The function S is called an S-metric and the pair (X, S) is called an S-metric space.

Definition 1.4 ([51])**.** Let X be a nonempty set and $s \geqslant 1$ be a given number. Also let a function $S_b : X^3 \to [0, \infty)$ satisfies the following conditions, for each $x, y, z, w \in X$:

(i) $S_b(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S_b(x,y,z) \leqslant s[S_b(x,x,w)+S_b(y,y,w)+S_b(z,z,w)]$.

The pair (X,S_b) is called an S_b-metric space.

A symmetric S_b-metric is a function which satisfies the conditions (i) and (ii) and also the following condition:

$$S_b(x,x,y)=S_b(y,y,x)$$

for all $x,y \in X$.

Definition 1.5 ([48])**.** Let X be a non-empty set and $S_d : X^3 \to [0,\infty)$ be a mapping which satisfies the following conditions for all $x,y,z,w \in X$:

(i) $S_d(x,y,z)=0$ implies $x=y=z$;
(ii) $S_d(x,y,z) \leqslant k[S_d(x,x,w)+S_d(y,y,w)+S_d(z,z,w)]$, where $k \geqslant 1$.

The function S_d is said to be a dislocated S_b-metric and the pair (X,S_d) is called a dislocated S_b-metric space. In the case when $k=1$, S_d is known as the dislocated S-metric.

2. S^{JS}-metric spaces

Menger [49] was the first to propose probabilistic metric spaces, a generalization of metric spaces. During the last six decades a lot of further generalizations/extension of metric spaces was introduced/proposed by the researchers; 2-metric spaces [28], pseudometric spaces/dislocated metric spaces [33], partial metric spaces [13], modular metric space with the Fatou property [45], fuzzy metric spaces [40], cone metric spaces [34], b-metric spaces [16], generalized D-metric spaces [11, 20, 50], generalized cone metric spaces [4] and so on. Sedghi et al. [58] gave the concept of S-metric spaces by modifying D-metric and G-metric spaces. Following this Souayan and Mlaiki [62] proposed the concept of S_b-metric spaces as a generalization of S-metric spaces. Afterwards Rohen et al. [51] have given the definition of S_b-metric space in a more generalized way and they renamed the usual S_b-metric space as symmetric S_b-metric space. Recently Jleli and Samet [37] introduced the idea of JS-metric spaces, which is one of the interesting generalization of usual metric spaces. They also showed that any standard metric space, b-metric space, dislocated metric space and modular metric space with the Fatou property are JS-metric space. In this section we continue to study these efforts to further weaken the hypothesis of a metric. First we present S^{JS}-metric spaces with examples and study their properties (see [6]). Let X be a nonempty set and $J : X^3 \to [0,\infty]$ be a function. Let us define the set

$$S(J,X,x)=\{\{x_n\}\subset X : \lim_{n\to\infty} J(x,x,x_n)=0\}$$

for all $x \in X$.

Definition 2.1. Let X be a nonempty set and $J : X^3 \to [0,\infty]$ satisfies the following conditions:

(J_1) $J(x,y,z)=0$ implies $x=y=z$ for any $x,y,z \in X$;
(J_2) there exists some $b>0$ such that for any $(x,y,z)\in X^3$ and $\{z_n\} \in S(J,X,z)$, we have

$$J(x,y,z) \leqslant b \limsup_{n\to\infty}(J(x,x,z_n)+J(y,y,z_n)).$$

Then the pair (X,J) is called an S^{JS}-metric space.

Additionally if J also satisfies

(J_3) $J(x,x,y)=J(y,y,x)$ for all $x,y \in X$,

then we call it a symmetric S^{JS}-metric space.

Example 2.2. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x| + |y| + |z|$ for all $x, y, z \in X$, then clearly (J_1) is satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. If $z = 0$, then for $\{z_n\} \in S(J, X, 0)$, we have

$$J(x, y, 0) \leqslant \frac{1}{2} \limsup_{n\to\infty} (J(x, x, z_n) + J(y, y, z_n))$$

for all $x, y \in X$. Then (J_2) is also satisfied. So (X, J) is an S^{JS}-metric space. Clearly it is not symmetric.

Example 2.3. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x| + |y| + 2|z|$ for all $x, y, z \in X$. Clearly the conditions (J_1) and (J_3) are satisfied. Also one can check that for any $x, y, z \in X$

$$J(x, y, z) \leqslant \limsup_{n\to\infty} (J(x, x, z_n) + J(y, y, z_n))$$

for any sequence $\{z_n\} \in S(J, X, z)$. In a similar way as in Example 2.2 we can show that condition (J_2) is also satisfied. Hence X is a symmetric S^{JS}-metric space.

Example 2.4. Let $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = 1$ if x, y, z are all finite and $J(x, y, z) = \infty$ if any one of x, y, z is ∞. Clearly the conditions (J_1) and (J_3) are satisfied. Also we see that for any $z \in X$, $S(J, X, z) = \emptyset$. Therefore the condition (J_2) is also trivially satisfied. Hence X is a symmetric S^{JS}-metric space.

Remark 2.5.

(1) Let (X, S) be an S-metric space (see Definition 1.3). Clearly S satisfies condition (J_1). Now let $(x, y, z) \in X^3$ and $\{z_n\}$ converges to z in (X, S), then $S(z, z, z_n) \to 0$ as $n \to \infty$ and from the condition (ii) we have

$$S(x, y, z) \leqslant \limsup_{n\to\infty} (S(x, x, z_n) + S(y, y, z_n)).$$

Therefore S satisfies (J_2) also. Hence X is an S^{JS}-metric space. It is also symmetric.

(2) Let (X, S_b) be an S_b-metric space with coefficient $s \geqslant 1$ (see Definition 1.4). Then clearly S_b satisfies (J_1) and it also satisfies (J_2) for $b = s$. So an S_b-metric space is an S^{JS}-metric space.

(3) If (X, S_d) is a dislocated S_b-metric space with coefficient $k \geqslant 1$ (see Definition 1.5), then clearly S_d satisfies the condition (J_1) and condition (J_2) for $b = k$. So a dislocated S_b-metric space is an S^{JS}-metric space.

Definition 2.6. Let (X, J) be an S^{JS}-metric space, then a sequence $\{x_n\} \subset X$ is said to be convergent to an element $x \in X$ if $\{x_n\} \in S(J, X, x)$.

Definition 2.7. Let (X, J) be an S^{JS}-metric space. A sequence $\{x_n\} \subset X$ is said to be Cauchy if

$$\lim_{n,m\to\infty} J(x_n, x_n, x_m) = 0.$$

Definition 2.8. An S^{JS}-metric space is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.9. Let (X, J) be an S^{JS}-metric space and $T : X \to X$ be a self mapping. Then T is called continuous at $a \in X$ if for any $\epsilon > 0$ there exists $\delta \equiv \delta(\epsilon) > 0$ such that for any $x \in X$, $J(Ta, Ta, Tx) < \epsilon$ whenever $J(a, a, x) < \delta$.

Theorem 2.10. *In an S^{JS}-metric space (X, J) if $\{x_n\}$ converges to both x and y for $x, y \in X$, then $x = y$.*

Proof. Now,

$$J(x, x, y) \leqslant b \limsup_{n\to\infty} (2J(x, x, x_n)).$$

Since $x_n \to x$, then $\lim_{n\to\infty} J(x, x, x_n) = 0$, which implies $J(x, x, y) = 0$ that is $x = y$. □

Theorem 2.11. *Let (X, J) be an S^{JS}-metric space and $\{x_n\} \subset X$ converges to some $x \in X$. Then $J(x, x, x) = 0$.*

Proof. Since $\{x_n\}$ converges to x it follows that $\{x_n\} \in S(J, X, x)$ and thus

$$J(x, x, x) \leqslant b \limsup_{n\to\infty}(2J(x, x, x_n)),$$

which implies $J(x, x, x) = 0$. □

Theorem 2.12. *In a symmetric S^{JS}-metric space (X, J) if a Cauchy sequence $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is also convergent in X.*

Proof. Let $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to $x \in X$. Now since (X, J) is symmetric, we have

$$J(x, x, x_n) = J(x_n, x_n, x) \leqslant b \limsup_{k\to\infty}(2J(x_n, x_n, x_{n_k})).$$

Taking $n, k \to \infty$ we have $\lim_{n\to\infty} J(x, x, x_n) = 0$. So $\{x_n\}$ converges to x. □

Theorem 2.13. *In an S^{JS}-metric space (X, J) if T is continuous at $a \in X$, then for any sequence $\{x_n\} \in S(J, X, a)$ implies $\{Tx_n\} \in S(J, X, Ta)$.*

Proof. Let $\epsilon > 0$ be given. Since T is continuous at a, then for $\epsilon > 0$ there exists $\delta > 0$ such that $J(a, a, x) < \delta$ implies $J(Ta, Ta, Tx) < \epsilon$.

As $\{x_n\}$ converges to $'a'$, so for $\delta > 0$ there exists $N \in \mathbb{N}$ such that $J(a, a, x_n) < \delta$ for all $n \geqslant N$. Therefore for any $n \geqslant N$, $J(Ta, Ta, Tx_n) < \epsilon$ and thus $Tx_n \to Ta$ as $n \to \infty$. □

3. S^{JS}-topological spaces

In this section we discuss S^{JS}-topological spaces induced by S^{JS}-metric and prove several classical theorems including Cantor's intersection theorem in this setting (see [6]).

Definition 3.1. Let (X, J) be an S^{JS}-metric space. The open and closed ball of center $x \in X$ and radius $r > 0$ in X are defined as

$$B_J(x, r) = \{y \in X : J(x, x, y) < r\}, \quad B_J[x, r] = \{y \in X : J(x, x, y) \leqslant r\}.$$

Remark 3.2. It may happen that in an S^{JS}-metric space X, $x \notin B_J(x, r)$ for some $r > 0$ and $x \in X$. In Example 2.2 if we take $x = 1$ and $r = 2$, then $J(1, 1, 1) = 3$ and therefore $1 \notin B_J(1, 2)$.

Theorem 3.3. *Let (X, J) be an S^{JS}-metric space. Let $\tau = \{\emptyset\} \cup \{U(\neq \emptyset) \subset X$: for any $x \in U$ there exists $r > 0$ such that $B_J(x, r) \subset U\}$. Then τ forms a topology on X, called the topology induced by J and (X, τ) is said to be a S^{JS}-topological space.*

Proof. Clearly $X \in \tau$. Now let $\{G_\alpha\}_{\alpha\in\Lambda}$, Λ being an indexing set, be a collection of members of τ and $G = \cup_{\alpha\in\Lambda} G_\alpha$. If $x \in G$, then there exists some $\beta \in \Lambda$ such that $x \in G_\beta$. So there exists $r > 0$ such that $B_J(x, r) \subset G_\beta \subset G$. Hence $G \in \tau$.

Also let $G, H \in \tau$ and $y \in G \cap H$. Then there exist $r_1, r_2 > 0$ such that $B_J(y, r_1) \subset G$ and $B_J(y, r_2) \subset H$. If we take $r = \min\{r_1, r_2\}$, then we have $B_J(y, r) \subset G \cap H$ and so $G \cap H \in \tau$. Therefore τ forms a topology on X. □

Definition 3.4. Let (X, J) be an S^{JS}-topological space and $F \subset X$. Then F is said to be closed if there exists an open set $U \subset X$ such that $F = U^c$.

Theorem 3.5. *Let (X, J) be an S^{JS}-topological space and $F \subset X$ be closed. Let $\{x_n\} \subset F$ be such that $\{x_n\} \in S(J, X, x)$, then $x \in F$.*

Proof. If possible let $x \notin F$. Then $x \in F^c = U$, where U is open. So there exists $r > 0$ such that $B_J(x,r) \subset U$. Now $\lim_{n\to\infty} J(x,x,x_n) = 0$ so for $r > 0$ there exists $N \in \mathbb{N}$ such that $J(x,x,x_n) < r$ whenever $n \geqslant N$. Thus $x_n \in B_J(x,r) \subset U$ for all $n \geqslant N$, a contradiction. Hence $x \in F$. □

Theorem 3.6. *Let (X,J) be an S^{JS}-topological space and $F \subset X$ be closed. If X is complete, then (F, J_F) is also complete.*

Proof. Let $\{x_n\} \subset F$ be Cauchy in F. Since X is complete and $\{x_n\}$ is Cauchy in X also, there exists $z \in X$ such that $\{x_n\} \in S(J,X,z)$. As F is closed, then by Theorem 3.5 we have $z \in F$. Thus $\{x_n\}$ is convergent in F. Therefore F is complete. □

Theorem 3.7. *Let (X,J) be an S^{JS}-topological space and T be continuous self mapping on X. Then for any open set U, $T^{-1}(U)$ is open.*

Proof. Let U be any open set in X, if $T^{-1}(U) = \emptyset$, then we are done. So let $T^{-1}(U) \neq \emptyset$ and $a \in T^{-1}(U)$. Then $Ta \in U$ and since U is open there exists $\epsilon > 0$ such that $B_J(Ta,\epsilon) \subset U$. T is continuous at $'a'$ so there exists $\delta > 0$ such that $J(x,x,a) < \delta$ implies $J(Tx,Tx,Ta) < \epsilon$. Therefore $T(B_J(a,\delta)) \subset B_J(Ta,\epsilon) \subset U$ implying that $B_J(a,\delta) \subset T^{-1}(U)$. Hence $T^{-1}(U)$ is open. □

Definition 3.8. Let (X,J) be an S^{JS}-metric space and $A \subset X$. Then $\operatorname{diam}(A) = \sup\{J(a,a,b) : a,b \in X\}$.

Definition 3.9. In an S^{JS}-topological space (X,J), a sequence $\{F_n\}$ of subsets of X is said to be decreasing if $F_1 \supset F_2 \supset F_3 \supset \ldots$.

Following theorem gives conditions under which the intersection of such a sequence is non empty.

Theorem 3.10 (Cantor's intersection property)**.** *Let (X,J) be a complete S^{JS}-metric space and $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X such that $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$. Then the intersection $\cap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. Let $x_n \in F_n$ be arbitrary for all $n \in \mathbb{N}$. Since $\{F_n\}$ is decreasing, we have $\{x_n, x_{n+1}, \ldots\} \subset F_n$ for all $n \in \mathbb{N}$.

Now for any $n,m \in \mathbb{N}$ with $n,m \geqslant k$ we have $J(x_n,x_n,x_m) \leqslant \operatorname{diam}(F_k)$, $k \geqslant 1$. Let $\epsilon > 0$ be given. Then there exists some $p \in \mathbb{N}$ such that $\operatorname{diam}(F_p) < \epsilon$ since $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$. From this it follows that $J(x_n,x_n,x_m) < \epsilon$ whenever $n,m \geqslant p$. So $\{x_n\}$ is Cauchy in X. By the completeness of X there exists $z \in X$ such that $\{x_n\} \in S(J,X,z)$. Since $\{x_n, x_{n+1}, \ldots\} \subset F_n$ and F_n is closed for each $n \in \mathbb{N}$, using Theorem 3.5 we have $z \in \cap_{n=1}^{\infty} F_n$.

Next we prove the uniqueness of z. Let $y \in \cap_{n=1}^{\infty} F_n$ be another point, then $J(z,z,y) > 0$. As $\operatorname{diam}(F_n) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\operatorname{diam}(F_n) < J(z,z,y) \leqslant \operatorname{diam}(F_n)$$

for all $n \geqslant N_0$, a contradiction. Hence $\cap_{n=1}^{\infty} F_n = \{z\}$ and this completes the proof of our theorem. □

Definition 3.11. Let (X,J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. Then a closed set F (if exists) is said to be the closure of A if it is largest which satisfies $A \subset F \subset A \cup \{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X,J,x)\}$. We denote F as $\overline{A}$.

Remark 3.12. If (X,J) is an S^{JS}-metric space and $A(\neq \emptyset) \subset X$ is closed, then by Theorem 3.5 we have $\overline{A} = A$.

Theorem 3.13. *Let (X,J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. Then $\{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X,J,x)\} = \{x \in X :$ for all $r > 0, B_J(x,r) \cap A \neq \emptyset\}$.*

Proof. Let $y \in \{x \in X : \text{for all } r > 0, B_J(x, r) \cap A \neq \emptyset\}$. Then $B_J(y, \frac{1}{n}) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. So there exists $y_n \in B_J(y, \frac{1}{n}) \cap A$ for all $n \in \mathbb{N}$ and we have, $\{y_n\} \in S(X, J, y)$. Thus $y \in \{x \in X : \text{there exists } \{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$.

Conversely let, $z \in \{x \in X : \text{there exists } \{x_n\} \subset A \text{ such that } \{x_n\} \in S(X, J, x)\}$. Then there exists $\{z_n\} \subset A$ such that $J(z, z, z_n) \to 0$ as $n \to \infty$. Let us choose a $r > 0$. Then there exists $m \in \mathbb{N}$ such that $z_n \in B_J(z, r)$ for all $n \geqslant m$. So $B_J(z, r) \cap A \neq \emptyset$. Hence $z \in \{x \in X : \text{for all } r > 0, B_J(x, r) \cap A \neq \emptyset\}$. □

Remark 3.14. Clearly from Theorem 3.3 we have $A \subset \overline{A} \subset A \cup \{x \in X : \text{for all } r > 0, B_J(x, r) \cap A \neq \emptyset\}$.

Theorem 3.15. *Let* (X, J) *be an* S^{JS}*-metric space and* A, B *be two nonempty subsets of* X *with* $A \subset B$. *Then* $\overline{A} \subset \overline{B}$.

Proof. Clearly $\overline{A}$ and $\overline{B}$ are largest closed sets, respectively, satisfying the following

$$A \subset \overline{A} \subset A \cup \{x \in X : \text{ there exists } \{x_n\} \subset A \text{ such that } \{x_n\} \in S(X, J, x)\},$$
$$B \subset \overline{B} \subset B \cup \{x \in X : \text{ there exists } \{x_n\} \subset B \text{ such that } \{x_n\} \in S(X, J, x)\}.$$

Now, $A \cup B \subset \overline{A} \cup \overline{B} \subset (A \cup B) \cup (\{x \in X : \text{there exists } \{x_n\} \subset A \text{ such that } \{x_n\} \in S(X, J, x)\} \cup \{x \in X : \text{there exists } \{x_n\} \subset B \text{ such that } \{x_n\} \in S(X, J, x)\})$ implies that $B \subset \overline{A} \cup \overline{B} \subset B \cup \{x \in X : \text{there exists } \{x_n\} \subset B$ such that $\{x_n\} \in S(X, J, x)\}$. Since $\overline{A} \cup \overline{B}$ is closed, it follows that $\overline{A} \cup \overline{B} \subset \overline{B}$. Therefore we have $\overline{A} \cup \overline{B} = \overline{B}$ and thus $\overline{A} \subset \overline{B}$. □

Theorem 3.16. *Let* (X, J) *be a symmetric* S^{JS}*-metric space and* $A(\neq \emptyset) \subset X$ *for which* $\overline{A}$ *exists. Then* $\operatorname{diam}(\overline{A}) \leqslant L \operatorname{diam}(A)$, *where* $L = \max\{1, 2b, 4b^2\}$.

Proof. Let $x, y \in \overline{A}$. Then we have to consider three cases.

Case 1. If $x, y \in A$, then

$$J(x, x, y) \leqslant \operatorname{diam}(A). \tag{3.1}$$

Case 2. If $x \in A$ and $y \in \{x \in X : \text{there exists } \{x_n\} \subset A \text{ such that } \{x_n\} \in S(X, J, x)\}$, then there exists a sequence $\{y_n\} \subset A$ such that $\{y_n\} \in S(X, J, y)$ and we have

$$J(x, x, y) \leqslant 2b \limsup_{n\to\infty} J(x, x, y_n) \leqslant 2b \operatorname{diam}(A). \tag{3.2}$$

Case 3. If $x, y \in \{p \in X : \text{there exists } \{p_n\} \subset A \text{ such that } \{p_n\} \in S(X, J, p)\}$, then there exists sequences $\{x_n\}, \{y_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)$, $\{y_n\} \in S(X, J, y)$ and we have

$$\begin{aligned} J(x, x, y) &\leqslant 2b \limsup_{n\to\infty} J(x, x, y_n) \\ &= 2b \limsup_{n\to\infty} J(y_n, y_n, x) \leqslant 2b \limsup_{n\to\infty}(2b \limsup_{m\to\infty} J(y_n, y_n, x_m)) \leqslant 4b^2 \operatorname{diam}(A). \end{aligned} \tag{3.3}$$

Therefore from (3.1), (3.2), and (3.3) we get $\operatorname{diam}(\overline{A}) \leqslant L \operatorname{diam}(A)$, $L = \max\{1, 2b, 4b^2\}$. □

Theorem 3.17 (Converse of Theorem 3.10)**.** *Let* (X, J) *be a symmetric* S^{JS}*-metric space in which every nonempty subset has a closure and* $\{F_n\}$ *be a decreasing sequence of nonempty closed subsets of* X *with* $\operatorname{diam}(F_n) \to 0$ *as* $n \to \infty$. *If* $\cap_{n=1}^{\infty} F_n$ *contains exactly one point, then* X *is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Let us choose $G_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is a Cauchy sequence therefore $\operatorname{diam}(G_n) \to 0$ as $n \to \infty$. Also $\{\overline{G_n}\}$ is a decreasing sequence of nonempty closed subsets of X (using Theorem 3.15 such that $\operatorname{diam}(\overline{G_n}) \to 0$ as $n \to \infty$ (from Theorem 3.16). Hence from the given condition we see that $\cap_{n=1}^{\infty} \overline{G_n} = \{z\}$, $z \in X$.

Now $J(z, z, x_n) \leqslant \operatorname{diam}(\overline{G_n}) \to 0$ as $n \to \infty$. So $\{x_n\}$ is convergent and X is complete. □

Example 3.18. Let us consider the symmetric S^{JS}-metric space given in Example 2.3. Then we have for any $x \in X$ and for any $r > 0$,

$$B_J(x,r) = \begin{cases} \emptyset, & \text{if } |x| \geqslant \frac{r}{2} \\ (-(\frac{r}{2} - |x|), (\frac{r}{2} - |, x|)), & \text{if } |x| < \frac{r}{2}, \end{cases}$$

and

$$B_J[x,r] = \begin{cases} \emptyset, & \text{if } |x| > \frac{r}{2} \\ [-(\frac{r}{2} - |x|), (\frac{r}{2} - |, x|)], & \text{if } |x| \leqslant \frac{r}{2}. \end{cases}$$

Here we see that the topology τ is given by$\tau = \{\emptyset\} \cup \{B(\neq \emptyset) : B \subset X\backslash\{0\}\} \cup \{B(\neq \emptyset) : 0 \in B$ and there exists $r > 0$ such that $\left(-\frac{r}{2}, \frac{r}{2}\right) \subset B\}$.

Clearly any nonempty subset of X containing 0 is closed.

If $A(\neq \emptyset) \subset X$, $0 \notin A$ and there does not exist a sequence $\{x_n\} \subset A$ converging to 0 in X, then there must exists some $r > 0$ such that $0 \in \left(-\frac{r}{2}, \frac{r}{2}\right) \subset X\backslash A$ and therefore we have A is closed. If $A(\neq \emptyset) \subset X$ is not closed, $0 \notin A$ and there exists a sequence $\{x_n\} \subset A$ converging to 0 in X, then $\overline{A} = A \cup \{0\}$. So in (X, J) any nonempty subset of X has closure.

Example 3.19. (Supporting example for Theorem 3.17) Let us consider the symmetric S^{JS}-metric space given in Example 2.3. Also let $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X such that $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$. If $0 \notin F_m$ for some $m \in \mathbb{N}$. Then $0 \notin F_k$ for all $k \geqslant m$. Now let $x_k \in F_k$ for all $k \geqslant m$. Then $\{x_m, x_{m+1}, \dots\} \subset F_m$ and also $J(x_k, x_k, x_k) \leqslant \operatorname{diam}(F_k) \to 0$ as $m \leqslant k \to \infty$. Thus $|x_k| \to 0$ as $k \to \infty$ and we get $J(0, 0, x_k) = 2|x_k| \to 0$ as $m \leqslant k \to \infty$. Since F_m is closed so by Theorem 3.5 we get $0 \in F_m$, a contradiction.

Therefore $0 \in F_n$ for all $n \in \mathbb{N}$. Now if $t(\neq 0) \in \cap_{n=1}^{\infty} F_n$, then $J(t,t,t) \leqslant \operatorname{diam}(F_n) \to 0$ as $n \to \infty$ implying that $t = 0$, a contradiction. Therefore $\cap_{n=1}^{\infty} F_n = \{0\}$. Here we see that (X, J) is complete.

The condition, S^{JS}-metric space X is symmetric is a sufficient condition in Theorem 3.17. Which can be shown from our next example.

Example 3.20. If we consider the S^{JS}-metric space given in Example 2.2, then it is not symmetric and the topology τ is given by $\tau = \{\emptyset\} \cup \{B(\neq \emptyset) : B \subset X\backslash\{0\}\} \cup \{B(\neq \emptyset) : 0 \in B$ and there exists $r > 0$ such that $0 \in (-r, r) \subset B\}$. Clearly any nonempty subset of X containing 0 is closed. If $A(\neq \emptyset) \subset X$, $0 \notin A$ and there does not exist a sequence $\{x_n\} \subset A$ converging to 0 in X, then there must exists some $r > 0$ such that $0 \in (-r, r) \subset X\backslash A$ and therefore we have A is closed. If $A(\neq \emptyset) \subset X$ is not closed, $0 \notin A$ and there exists a sequence $\{x_n\} \subset A$ converging to 0 in X, then $\overline{A} = A \cup \{0\}$. So in (X, J) any nonempty subset of X has closure and we can prove that for any decreasing sequence $\{F_n\}$ of nonempty closed subsets of X such that $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$, $\cap_{n=1}^{\infty} F_n = \{0\}$, in a similar way as in Example 3.18.

Definition 3.21. Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. Then $\operatorname{int}(A)$ is the largest open set contained in A.

Definition 3.22. Let (X, J) be an S^{JS}-metric space. A subset A of X is said to be nowhere dense in X if $\overline{A}$ exists and $\operatorname{int}(\overline{A}) = \emptyset$.

Theorem 3.23. *Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$. If $\overline{A}$ exists, then* $\operatorname{int}(X\backslash A) = X\backslash\overline{A}$.

Proof. Since $\overline{A}$ exists, then $A \subset \overline{A} \subset A \cup \{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$. Let us denote the set $\{x \in X :$ there exists $\{x_n\} \subset A$ such that $\{x_n\} \in S(X, J, x)\}$ by A'. Then $(X\backslash A) \cap (X\backslash A') \subset X\backslash\overline{A} \subset X\backslash A$. Now $X\backslash\overline{A}$ is open so $X\backslash\overline{A} \subset \operatorname{int}(X\backslash A)$. If $\operatorname{int}(X\backslash A) = \emptyset$, then we are done. So let $\operatorname{int}(X\backslash A) \neq \emptyset$ and $x \in \operatorname{int}(X\backslash A)$. Then there exists some $r > 0$ such that $B_J(x, r) \subset \operatorname{int}(X\backslash A) \subset X\backslash A$. So $B_J(x, r) \cap A = \emptyset$ and we have $x \in X\backslash A'$ (using Theorem 3.13). It implies that $x \in (X\backslash A) \cap (X\backslash A') \subset X\backslash\overline{A}$. Therefore $\operatorname{int}(X\backslash A) \subset X\backslash\overline{A}$, which shows that $\operatorname{int}(X\backslash A) = X\backslash\overline{A}$. □

Theorem 3.24. *Let (X, J) be an S^{JS}-metric space and $A(\neq \emptyset) \subset X$ be a nowhere dense set in X. Then for any open set $U \neq \emptyset$ there exists an open set $V(\neq \emptyset) \subset U$ such that $V \cap A = \emptyset$.*

Proof. Since $\text{int}(\overline{A}) = \emptyset$, then $\overline{A} \neq X$. So $\text{int}(X\backslash A) = X\backslash\overline{A} \neq \emptyset$. Let U be a nonempty open set in X. Then $U \cap \text{int}(X\backslash A) \neq \emptyset$ because if $U \cap \text{int}(X\backslash A) = \emptyset$, then $U \cap (X\backslash\overline{A}) = \emptyset$ implying that $U \subset \overline{A}$, a contradiction. Let $V = U \cap \text{int}(X\backslash A)$. Then V is open and $V \subset \text{int}(X\backslash A) \subset X\backslash A$. Therefore $V \cap A = \emptyset$. □

Definition 3.25. An S^{JS}-metric space (X, J) is said to have property (c) if every nonempty subset of X has a closure.

Conjecture 3.26. A complete S^{JS}-metric space (X, J) with property (c) is not expressable as a countable union of nowhere dense sets.

4. Fixed point of integral type contractive mappings

Nowadays fixed point theory is one of the most important and recent trends of research area in mathematics for its numerous applications. Fixed point theory has various applications in different branches of mathematics viz. boundary value problems, nonlinear differential and integral equations, nonlinear matrix equations, homotopy theory etc. The main purpose of fixed point theory is to deal with several mappings either of contractive type or nonexpansive type in nature over various generalized spaces and to investigate the existence of their fixed points therein. Branciari [12] introduced integral type contractive mappings and proved some fixed point theorems. Following this, researchers have considered various types of contractive mappings of integral type in several topological spaces and proved fixed point theorems therein [56]. In addition to fixed point, researchers are also interested in investigating the existence of common fixed points of two or more mappings, coincidence points of mappings and coupled fixed points of mappings etc. See [19, 26, 30, 31, 38, 63] to make further enrichment of the area of fixed point theory.

Sedghi et al. [58] introduced the concept of S-metric space by modifying D-metric and G-metric spaces. Following this article Souayan and Mlaiki [62] introduced the concept of S_b-metric space as a generalization of S-metric space and established some fixed point theorems on it. Rohen et al. [51] have given the definition of S_b-metric space in a more generalized way and they renamed the usual S_b-metric space as symmetric S_b-metric space. Jleli and Samet [37] introduced a generalized metric space commonly known as JS-metric space, which is one of the interesting generalizations of usual metric spaces. They showed that any standard metric space, b-metric space [16], dislocated metric space [33] and Modular metric space with the Fatou property [45] are also JS-metric space. Moreover they have considered some generalized contractive type mappings in this newly introduced space and proved some fixed point theorems on it. Following this literature Senapati and Dey [60] proved some coupled fixed point theorems in the setting of partially ordered JS-metric spaces. Recently Beg et al. [6] introduced the notion of S^{JS}-metric space and proved several interesting classical results in these spaces. They also gave examples to show that s-metric spaces and S_b-metric spaces are S^{JS}-metric spaces. The aim of this section is to give some fixed point theorems together with common fixed point and coupled fixed point theorems for a class of integral type contractive mappings in the setting of S^{JS}-metric space (see [54]).

Let us consider the following set. $\Phi = \{\varphi : [0, \infty) \to [0, \infty) : \varphi$ is bounded, Lebesgue-integrable, summable and for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0\}$.

In an S^{JS}-metric space (X, J), $d_J : X^2 \to [0, \infty]$ stands for the function defined as $d_J(x, y) = J(x, x, y)$ for any $x, y \in X$.

Lemma 4.1. *Let $\varphi \in \Phi$ and $\{a_\lambda : \lambda \in \Lambda\} \subset \mathbb{R}^+$ be a nonempty set. If for some $M > 0$, $\int_0^{a_\lambda} \varphi(t)dt \leqslant M$ for all $\lambda \in \Lambda$, then $\int_0^{a} \varphi(t)dt \leqslant M$, where $a = \sup\{a_\lambda : \lambda \in \Lambda\} < \infty$.*

Proof. For any $n \in \mathbb{N}$ there exists $\alpha \in \Lambda$ such that $a_\alpha + \frac{1}{n} > a$. Therefore

$$\int_0^{a} \varphi(t)dt \leqslant \int_0^{a_\alpha+\frac{1}{n}} \varphi(t)dt = \int_0^{a_\alpha} \varphi(t)dt + \int_{a_\alpha}^{a_\alpha+\frac{1}{n}} \varphi(t)dt \leqslant M + \int_{a_\alpha}^{a_\alpha+\frac{1}{n}} \varphi(t)dt. \tag{4.1}$$

Since φ is bounded, there exists $L > 0$ such that $\varphi(t) \leqslant L$ for all $t \in [0, \infty)$. Therefore from (4.1) for any $n \in \mathbb{N}$ we have $\int_0^a \varphi(t)dt \leqslant M + L\frac{1}{n}$. Hence $\int_0^a \varphi(t)dt \leqslant M$. □

Theorem 4.2. *Let* (X, J) *be a complete* S^{JS}*-metric space and* $T : X \to X$ *be a self mapping. Also let* T *satisfies*

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leqslant k \int_0^{d_J(x,y)} \varphi(t)dt \tag{4.2}$$

for some $\varphi \in \Phi$, $k \in [0, 1)$ *and for all* $x, y \in X$. *If there exists* $x_0 \in X$ *such that* $\delta(J, T, x_0) = \sup\{d_J(T^i x_0, T^j x_0) : i, j \geqslant 1\} < \infty$, *then* T *has at least one fixed point in* X.

Proof. Now since T satisfies (4.1), for any $n \in \mathbb{N}$ we have

$$\int_0^{d_J(T^{n+i}x_0, T^{n+j}x_0)} \varphi(t)dt \leqslant k \int_0^{d_J(T^{n-1+i}x_0, T^{n-1+j}x_0)} \varphi(t)dt$$

for all $i, j \geqslant 1$. Let us take $\delta(J, T^{p+1}, x_0) = \sup\{d_J(T^{p+i}x_0, T^{p+j}x_0) : i, j \in \mathbb{N}\}$ for any non-negative integer p and for any $x_0 \in X$. Then for all $i, j \geqslant 1$,

$$\int_0^{d_J(T^{n+i}x_0, T^{n+j}x_0)} \varphi(t)dt \leqslant k \int_0^{\delta(J,T^n,x_0)} \varphi(t)dt.$$

Since $\delta(J, T^{p+1}, x_0) \leqslant \delta(J, T, x_0) < \infty$ for any $p \geqslant 1$, from Lemma 4.1 it implies that

$$\int_0^{\delta(J,T^{n+1},x_0)} \varphi(t)dt \leqslant k \int_0^{\delta(J,T^n,x_0)} \varphi(t)dt$$

for any $n \in \mathbb{N}$. It further implies

$$\int_0^{\delta(J,T^{n+1},x_0)} \varphi(t)dt \leqslant k^n \int_0^{\delta(J,T,x_0)} \varphi(t)dt$$

for all $n \geqslant 1$. Taking $n \to \infty$ we have $\int_0^{\delta(J,T^{n+1},x_0)} \varphi(t)dt \to 0$. Since $\varphi \in \Phi$, we get $\lim_{n\to\infty} \delta(J, T^{n+1}, x_0) = 0$. Now for any $1 \leqslant n < m$ it follows that $d_J(T^n x_0, T^m x_0) \leqslant \delta(J, T^n, x_0)$ which tends to 0 as n tends to ∞. Thus $\{T^n x_0\}$ is a Cauchy sequence in X. By the completeness of X there exists some $z \in X$ such that $\{T^n x_0\} \in S(J, X, z)$. Now for any $n \in \mathbb{N}$ we have

$$\int_0^{d_J(Tz,T^{n+1}x_0)} \varphi(t)dt \leqslant k \int_0^{d_J(z,T^n x_0)} \varphi(t)dt.$$

Taking $n \to \infty$ we get $\int_0^{d_J(Tz,T^{n+1}x_0)} \varphi(t)dt \to 0$ and therefore $\lim_{n\to\infty} d_J(Tz, T^{n+1}x_0) = 0$. From Theorem 2.10 it follows that $Tz = z$. Hence T has a fixed point in X. □

Theorem 4.3. *If* z *and* z' *are two fixed points of* T *in Theorem* 4.2 *such that* $d_J(z, z') < \infty$, *then* $z = z'$.

Proof. Since z and z' are fixed points of T satisfying condition (4.2), then we obtain

$$\int_0^{d_J(z,z')} \varphi(t)dt = \int_0^{d_J(Tz,Tz')} \varphi(t)dt \leqslant k \int_0^{d_J(z,z')} \varphi(t)dt.$$

It implies that $\int_0^{d_J(z,z')} \varphi(t)dt = 0$ as $0 \leqslant k < 1$. Since $\varphi \in \Phi$ we get $d_J(z, z') = 0$. Hence $z = z'$. □

Corollary 4.4. *Let* (X, S) *be a complete* S*-metric space and* $T : X \to X$ *satisfies*

$$S(Tx, Tx, Ty) \leqslant LS(x, x, y)$$

for all $x, y \in X$*, where* $0 \leqslant L < 1$*. Then* T *has a unique fixed point in* X*.*

Proof. If we set $d_S(x, y) = S(Tx, Tx, Ty)$ for all $x, y \in X$, then for any $x_0 \in X$ and for $1 \leqslant i < j$ we have

$$\begin{aligned}
d_S(T^i x_0, T^j x_0) &= S(T^i x_0, T^i x_0, T^j x_0) \\
&\leqslant 2d_S(T^i x_0, T^{i+1} x_0) + S(T^{i+1} x_0, T^{i+1} x_0, T^j x_0) \\
&\leqslant 2\left[d_S(T^i x_0, T^{i+1} x_0) + d_S(T^{i+1} x_0, T^{i+2} x_0)\right] + S(T^{i+2} x_0, T^{i+2} x_0, T^j x_0) \\
&\ \ \vdots \\
&\leqslant 2[d_S(T^i x_0, T^{i+1} x_0) + d_S(T^{i+1} x_0, T^{i+2} x_0) + \cdots + d_S(T^{j-2} x_0, T^{j-1} x_0)] \\
&\quad + S(T^{j-1} x_0, T^{j-1} x_0, T^j x_0) \\
&\leqslant 2[d_S(T^i x_0, T^{i+1} x_0) + d_S(T^{i+1} x_0, T^{i+2} x_0) + \cdots + d_S(T^{j-1} x_0, T^j x_0)] \\
&\leqslant 2[L^i + L^{i+1} + \cdots + L^{j-1}] d_S(x_0.Tx_0) \\
&\leqslant 2L^i \frac{1 - L^{j-i}}{1 - L} d_S(x_0.Tx_0) \leqslant 2\frac{L^i}{1 - L} d_S(x_0.Tx_0).
\end{aligned}$$

For any $i, j \geqslant 1$ we can arrange them as $1 \leqslant i < j$ and so

$$\sup\{d_S(T^i x_0, T^j x_0) : i, j \geqslant 1\} \leqslant \sup\{2\frac{L^i}{1 - L} d_S(x_0.Tx_0) : i \geqslant 1\} = \frac{2Ld_S(x_0, Tx_0)}{1 - L} < \infty.$$

Now if we take $\phi(t) = 1$ for all $t \geqslant 0$, then we get

$$\int_0^{d_S(Tx,Ty)} \varphi(t)dt \leqslant L \int_0^{d_S(x,y)} \varphi(t)dt$$

for all $x, y \in X$. Also from Remark 2.5 it follows that any S-metric space is a symmetric S^{JS}-metric space. So all conditions of Theorems 4.2 and 4.3 are satisfied and hence T has a unique fixed point in X. □

Remark 4.5. Above Corollary 4.4 is a theorem proved in Sedghi et al. [58].

Proposition 4.6. *In an* S^{JS}*-metric space* (X, J) *if two mappings* $T, S : X \to X$ *satisfy* $T(X) \subset S(X)$*, then for any* $x_0 \in X$ *there exists a sequence* $\{y_n\}$*, where* $y_n = T(x_n) = S(x_{n+1})$ *for all non-negative integers* n*.*

Proof. Let $x_0 \in X$. Then $Tx_0 \in T(X) \subset S(X)$ and therefore there exists $x_1 \in X$ such that $Tx_0 = Sx_1$. Next $Tx_1 \in T(X) \subset S(X)$, so there exists some $x_2 \in X$ such that $Tx_1 = Sx_2$. Proceeding similarly we can construct a sequence $\{y_n\}$ in such a way that $y_n = Tx_n = Sx_{n+1}$ for any $n \geqslant 0$. □

Theorem 4.7. *Let* (X, J) *be a complete* S^{JS}*-metric space and* $T, S : X \to X$ *be two commutative mappings such that* S *is continuous and* $T(X) \subset S(X)$*. Also let* T *and* S *satisfy the following condition:*

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leqslant k \int_0^{d_J(Sx,Sy)} \varphi(t)dt \tag{4.3}$$

for all $x, y \in X$*, where* $0 \leqslant k < 1$ *and* $\varphi \in \Phi$*. If there exists* $x_0 \in X$ *such that* $\delta(J, T, S, x_0) = \sup\{d_J(y_i, y_j) : i, j \in \{0\} \cup \mathbb{N}\} < \infty$*, then* T *and* S *have atleast one common fixed point in* X*.*

Proof. Let us denote $\delta(J,T,S,p+1,x_0)=\sup\{d_J(y_{p+i},y_{p+j}):i,j\in\{0\}\cup\mathbb{N}\}$ for any $p\geqslant 0$. Since T satisfies condition (4.3) we obtain for any $n\in\mathbb{N}$,

$$\int_0^{d_J(y_{n+i},y_{n+j})}\varphi(t)dt=\int_0^{d_J(Tx_{n+i},Tx_{n+j})}\varphi(t)dt\leqslant k\int_0^{d_J(Sx_{n+i},Sx_{n+j})}\varphi(t)dt=k\int_0^{d_J(y_{n-1+i},y_{n-1+j})}\varphi(t)dt$$

for all $i,j\geqslant 0$, which in turn implies that $\int_0^{d_J(y_{n+i},y_{n+j})}\varphi(t)dt\leqslant k\int_0^{\delta(J,T,S,n,x_0)}\varphi(t)dt$ for all $i,j\geqslant 0$. Since $\delta(J,T,S,n+1,x_0)\leqslant\delta(J,T,S,x_0)<\infty$, then by Lemma 4.1 it follows that $\int_0^{\delta(J,T,S,n+1,x_0)}\varphi(t)dt\leqslant k\int_0^{\delta(J,T,S,n,x_0)}\varphi(t)dt$ for any $n\geqslant 1$. Therefore for any $n\in\mathbb{N}$ we obtain

$$\int_0^{\delta(J,T,S,n+1,x_0)}\varphi(t)dt\leqslant k^n\int_0^{\delta(J,T,S,x_0)}\varphi(t)dt,$$

which implies that $\lim_{n\to\infty}\int_0^{\delta(J,T,S,n+1,x_0)}\varphi(t)dt=0$. Since $\varphi\in\Phi$ it follows that $\lim_{n\to\infty}\delta(J,T,S,n+1,x_0)=0$. Now for any $1\leqslant n<m$ we have $d_J(y_n,y_m)\leqslant\delta(J,T,S,n,x_0)$ which tends to 0 as $n\to\infty$. So $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists $z\in X$ such that $\{y_n\}\in S(J,X,z)$. So $Tx_n=Sx_{n+1}\to z$ as $n\to\infty$. Since S is continuous, $\{Sy_n\}$ converges to Sz. Now for any $n\in\mathbb{N}$ $Sy_n=S(Tx_n)=T(Sx_n)=Ty_{n-1}$ and from (4.3) it follows that

$$\int_0^{d_J(Tz,Ty_n)}\varphi(t)dt\leqslant k\int_0^{d_J(Sz,Sy_n)}\varphi(t)dt.$$

Therefore $\lim_{n\to\infty}d_J(Tz,Ty_n)=0$ since $\varphi\in\Phi$ and we get $Tz=Sz$. Also using (4.3) we get

$$\int_0^{d_J(Tz,T^2z)}\varphi(t)dt\leqslant k\int_0^{d_J(Sz,S(Tz))}\varphi(t)dt=k\int_0^{d_J(Sz,T(Sz))}\varphi(t)dt=k\int_0^{d_J(Tz,T^2z)}\varphi(t)dt.$$

Since $k\in[0,1)$ we obtain $\int_0^{d_J(Tz,T^2z)}\varphi(t)dt=0$, which implies $T^2z=Tz$. Now, $S(Tz)=T(Sz)=T(Tz)=Tz$. Hence Tz is a common fixed point of T and S. □

Theorem 4.8. *If* u *and* u' *are two common fixed points of* T *and* S *in Theorem 4.7 such that* $d_J(u,u')<\infty$*, then* $u=u'$.

Proof. Given that u and u' are two common fixed points of T and S, so from (4.3) we obtain

$$\int_0^{d_J(u,u')}\varphi(t)dt=\int_0^{d_J(Tu,Tu')}\varphi(t)dt\leqslant k\int_0^{d_J(Su,Su')}\varphi(t)dt=k\int_0^{d_J(u,u')}\varphi(t)dt.$$

Since $0\leqslant k<1$ we have $\int_0^{d_J(u,u')}\varphi(t)dt=0$ implying that $u=u'$. □

Corollary 4.9. *Let* T *and* S *be two commuting self mappings of a complete* S^{JS}*-metric space* (X,J)*. Suppose that* S *is continuous,* $T(X)\subset S(X)$ *and* $\sup\{d_J(x,Tx):x\in X\}<\infty$*. Also let* T *and* S *satisfy the following condition for some positive integer* p:

$$\int_0^{d_J(T^px,T^py)}\varphi(t)dt\leqslant k\int_0^{d_J(Sx,Sy)}\varphi(t)dt$$

for all $x,y\in X$*, for some* $k\in[0,1)$ *and* $\varphi\in\Phi$*. If there exists* $x_0\in X$ *such that* $\delta(J,T^p,S,x_0)<\infty$*, then* T *and* S *have atleast one common fixed point in* X.

Proof. Clearly T^p and S commute with each other and also $T^p(X)\subset T(X)\subset S(X)$. So all conditions of Theorem 4.7 are satisfied and thus T^p and S have a common fixed point in X, say z. Then $T^pz=Sz=z$. Since $T^p(Tz)=Tz=T(Sz)=S(Tz)$, it follows that Tz is also a common fixed point of T and S in X. By the given condition $d_J(z,Tz)<\infty$ and thus from Theorem 4.8 it follows that $Tz=z$. Hence z is a common fixed point of T and S. □

Example 4.10. Let us take the complete S^{JS}-metric space (X, J) given in Example 2.2. Let $T : X \to X$ be defined by $Tx = \frac{x}{2}$ for all $x \in X$ and $\varphi(t) = \frac{t}{t+1}$ for all $t \in [0, \infty)$. Now for any $a \geqslant 0$ we get

$$\int_0^a \varphi(t)dt = \int_0^a \frac{t}{t+1}dt = a - \log(a+1).$$

Also $\frac{1}{2}\log(a+1) \leqslant \log(\frac{1}{2}a+1)$ for any $a \geqslant 0$, $d_J(x,y) = 2|x| + |y|$ and $d_J(Tx,Ty) = |x| + \frac{1}{2}|y|$ for all $x, y \in X$. Then clearly

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leqslant \frac{1}{2}\int_0^{d_J(x,y)} \varphi(t)dt$$

for all $x, y \in X$. Also for any $x_0 \in X \backslash \{-\infty, \infty\}$, $x_n = T^n x_0 = \frac{x_0}{2^n}$ for all $n \geqslant 1$ and therefore $\delta(J, T, x_0) \leqslant \frac{3}{2}|x_0|$ and we see that 0 is a fixed point of T in X. Other fixed points of T are $-\infty$ and ∞.

Example 4.11. Let us consider $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x-1| + |y-1| + |z-1|$ for all $x, y, z \in X$. Then it is clearly an S^{JS}-metric space which is not symmetric. Also let $T : X \to X$ be defined by $Tx = \frac{x+1}{2}$ for all $x \in X$. If we take $\varphi(t) = t^2$ for all $t \geqslant 0$, then we have

$$\int_0^a \varphi(t)dt = \int_0^a t^2dt = \frac{a^3}{3} \text{ for all } a \geqslant 0.$$

Moreover $d_J(Tx, Ty) = |x-1| + \frac{1}{2}|y-1|$ and $d_J(x, y) = 2|x-1| + |y-1|$ for all $x, y \in X$. Therefore

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leqslant \frac{1}{4}\int_0^{d_J(x,y)} \varphi(t)dt$$

for any $x, y \in X$. For any $x_0 \in X \backslash \{-\infty, \infty\}$, $x_n = T^n x_0 = \frac{x_0}{2^n} + 1 - \frac{1}{2^n}$ for all $n \geqslant 1$ and thus $\delta(J, T, x_0) \leqslant \frac{3}{2}|x_0 - 1|$ and we see that 1 is the only finite fixed point of T in X.

Example 4.12. Let us consider $X = \mathbb{R} \cup \{-\infty, \infty\}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |y + z - 2x| + |x - z| + |y - z|$ for all $x, y, z \in X$. Then it is clearly a symmetric S^{JS}-metric space. Also let $T : X \to X$ be defined by $Tx = \frac{x}{4} + \frac{1}{4}$ for all $x \in X$. If we take $\varphi(t) = 1$ for all $t \geqslant 0$, then we can verify in a similar way as in Example 4.10 that

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leqslant \frac{1}{4}\int_0^{d_J(x,y)} \varphi(t)dt$$

for any $x, y \in X$. Also for any $x_0 \in X \backslash \{-\infty, \infty\}$, $\delta(J, T, x_0) \leqslant \frac{|3x_0 - 1|}{4}$ and we have $\frac{1}{3}$ is the only finite fixed point of T in X.

Example 4.13. Let us consider the complete S^{JS}-metric space (X, J) given in Example 2.2. Let us define $T, S : X \to X$ by $Tx = \frac{x}{6}$ and $Sx = \frac{x}{2}$ for all $x \in X$. Also let us take φ defined in Example 4.10. Then for any $x, y \in X$, T and S satisfy

$$\int_0^{d_J(Tx,Ty)} \varphi(t)dt \leqslant \frac{1}{3}\int_0^{d_J(Sx,Sy)} \varphi(t)dt.$$

Also for any $x_0 \in X \backslash \{-\infty, \infty\}$ the sequence $\{y_n\}_{n \geqslant 0}$ is given by $y_n = \frac{x_0}{2.3^{n+1}}$ for all $n \in \{0\} \cup \mathbb{N}$ and we have $\delta(J, T, S, x_0) \leqslant \frac{1}{2}|x_0|$. Here 0 is the unique finite common fixed point of T and S in X. Also $-\infty$ and ∞ are common fixed points of T and S in X.

The notion of coupled fixed point was introduced in 1987 by Guo and Lakshmikantham (see [31]). Next we prove a coupled fixed point theorem in the setting of S^{JS}-metric space. First we define coupled fixed point of a mapping.

Definition 4.14. Let X be a non-empty set and $f : X^2 \to X$ be a mapping. A point $(a, b) \in X^2$ is said to be a coupled fixed point of f if $f(a, b) = a$ and $f(b, a) = b$.

For any $(a, b) \in X^2$ we can construct two iterative sequences using f in the following way

$$f^2(a, b) = f(f(a, b), f(b, a)), \qquad f^2(b, a) = f(f(b, a), f(a, b)),$$
$$f^3(a, b) = f(f^2(a, b), f^2(b, a)), \qquad f^3(b, a) = f(f^2(b, a), f^2(a, b)).$$

Proceeding in the similar manner we can get

$$f^{n+1}(a, b) = f(f^n(a, b), f^n(b, a)), \qquad f^{n+1}(b, a) = f(f^n(b, a), f^n(a, b)),$$

for any $n \geqslant 1$. So we get two iterative sequences $\{f^n(a, b)\}$ and $\{f^n(b, a)\}$.

Theorem 4.15. *Let (X, J) be a complete S^{JS}-metric space and $f : X^2 \to X$. Also let f satisfies*

$$\int_0^{d_J(f(x,y),f(u,v))} \varphi(t)dt \leqslant k \int_0^{\frac{1}{2}[d_J(x,u)+d_J(y,v)]} \varphi(t)dt$$

for all $x, y, u, v \in X$, for some $k \in [0, 1)$ and $\varphi \in \Phi$. If there exists $(x_0, y_0) \in X^2$ such that $\delta(J, f,(x_0, y_0)) = \{d_J(f^i(x_0, y_0), f^j(x_0, y_0)) : i, j \geqslant 1\} < \infty$ and $\delta(J, f,(y_0, x_0)) = \{d_J(f^i(y_0, x_0), f^j(y_0, x_0)) : i, j \geqslant 1\} < \infty$, then f has at least one coupled fixed point in X.

Proof. For any $n \in \mathbb{N}$ from the given contractive condition we have

$$\begin{aligned}\int_0^{d_J(f^{n+i}(x_0,y_0),f^{n+j}(x_0,y_0))} \varphi(t)dt &= \int_0^{d_J\left(f(f^{n-1+i}(x_0,y_0),f^{n-1+i}(y_0,x_0)),f(f^{n-1+j}(x_0,y_0),f^{n-1+j}(y_0,x_0))\right)} \varphi(t)dt \\ &\leqslant k \int_0^{\frac{1}{2}[d_J(f^{n-1+i}(x_0,y_0),f^{n-1+j}(x_0,y_0))+d_J(f^{n-1+i}(y_0,x_0),f^{n-1+j}(y_0,x_0))]} \varphi(t)dt\end{aligned}$$

for any $i, j \geqslant 1$. Let us take $\delta(J, f^{q+1},(x_0, y_0)) = \sup\{d_J(f^{q+i}(x_0, y_0), f^{q+j}(x_0, y_0) : i, j \in \mathbb{N}\}$ and $\{d_J(f^{q+i}(y_0, x_0), f^{q+j}(y_0, x_0) : i, j \in \mathbb{N}\}$ for any $q \geqslant 0$. Then for all $i, j \in \mathbb{N}$,

$$\int_0^{d_J(f^{n+i}(x_0,y_0),f^{n+j}(x_0,y_0))} \varphi(t)dt \leqslant k \int_0^{\frac{1}{2}[\delta(J,f^n,(x_0,y_0))+\delta(J,f^n,(y_0,x_0))]} \varphi(t)dt$$

for any $n \geqslant 1$. Since $\delta(J, f^{q+1},(x_0, y_0)) \leqslant \delta(J, f,(x_0, y_0)) < \infty$ for any $q \geqslant 1$, then using Lemma 4.1 we get

$$\int_0^{\delta(J,f^{n+1},(x_0,y_0))} \varphi(t)dt \leqslant k \int_0^{\frac{1}{2}[\delta(J,f^n,(x_0,y_0))+\delta(J,f^n,(y_0,x_0))]} \varphi(t)dt \tag{4.4}$$

for any $n \in \mathbb{N}$. Similarly we can obtain for any $n \geqslant 1$,

$$\int_0^{\delta(J,f^{n+1},(y_0,x_0))} \varphi(t)dt \leqslant k \int_0^{\frac{1}{2}[\delta(J,f^n,(x_0,y_0))+\delta(J,f^n,(y_0,x_0))]} \varphi(t)dt. \tag{4.5}$$

Let $M_n = \max\{\delta(J, f^n,(x_0, y_0)), \delta(J, f^n,(y_0, x_0))\}$ for all $n \in \mathbb{N}$. Then from (4.4) and (4.5) we have

$$\int_0^{M_{n+1}} \varphi(t)dt \leqslant k \int_0^{M_n} \varphi(t)dt$$

for all $n \geqslant 1$. Then we obtain $\int_0^{M_{n+1}} \varphi(t)dt \leqslant k^n \int_0^{M_1} \varphi(t)dt$ for all $n \in \mathbb{N}$. Since $M < \infty$ it follows that

$$\lim_{n\to\infty} \int_0^{M_{n+1}} \varphi(t)dt = 0.$$

This shows that $\lim_{n\to\infty} M_n = 0$ as $\varphi \in \Phi$. Therefore we get $\lim_{n\to\infty} \delta(J, f^n, (x_0, y_0)) = 0 = \lim_{n\to\infty} \delta(J, f^n, (y_0, x_0))$. Now for any $1 \leqslant n < m$ we get $d_J(f^n(x_0, y_0), f^m(x_0, y_0)) \leqslant \delta(J, f^n, (x_0, y_0)) \to 0$ as $n \to \infty$. So $\{f^n(x_0, y_0)\}$ is Cauchy in X and since X is complete so there exists $z_1 \in X$ such that $f^n(x_0, y_0) \to z_1$ as n tending to ∞. In a similar way we can find $z_2 \in X$ such that $f^n(y_0, x_0) \to z_2$ as n tending to ∞. Now

$$\begin{aligned}\int_0^{d_J(f(z_1,z_2),f^n(x_0,y_0))} \varphi(t)dt &= \int_0^{d_J\left(f(z_1,z_2),f(f^{n-1}(x_0,y_0),f^{n-1}(y_0,x_0))\right)} \varphi(t)dt \\ &\leqslant k \int_0^{\frac{1}{2}[d_J(z_1,f^{n-1}(x_0,y_0))+d_J(z_2,f^{n-1}(y_0,x_0))]} \varphi(t)dt.\end{aligned}$$

Since $f^n(x_0, y_0) \to z_1$ and $f^n(y_0, x_0) \to z_2$ as $n \to \infty$ so we obtain

$$\lim_{n\to\infty} \int_0^{d_J(f(z_1,z_2),f^n(x_0,y_0))} \varphi(t)dt = 0.$$

Therefore it follows that $\lim_{n\to\infty} f^n(x_0, y_0) = f(z_1, z_2)$. Then by Theorem 2.10 we get $f(z_1, z_2) = z_1$. In a similar manner we have $f(z_2, z_1) = z_2$. Hence (z_1, z_2) is a coupled fixed point of f in X. □

Theorem 4.16. *If* (z_1, z_2) *and* (z_1', z_2') *are two coupled fixed points of* f *in Theorem 4.2 such that* $d_J(z_1, z_1') < \infty$ *and* $d_J(z_2, z_2') < \infty$*, then* $(z_1, z_2) = (z_1', z_2')$.

Proof. Now

$$\int_0^{d_J(z_1,z_1')} \varphi(t)dt = \int_0^{d_J\left(f(z_1,z_2),f(z_1',z_2')\right)} \varphi(t)dt \leqslant k \int_0^{\frac{1}{2}[d_J(z_1,z_1')+d_J(z_2,z_2')]} \varphi(t)dt \tag{4.6}$$

and also

$$\int_0^{d_J(z_2,z_2')} \varphi(t)dt = \int_0^{d_J\left(f(z_2,z_1),f(z_2',z_1')\right)} \varphi(t)dt \leqslant k \int_0^{\frac{1}{2}[d_J(z_1,z_1')+d_J(z_2,z_2')]} \varphi(t)dt. \tag{4.7}$$

If we set $L = \max\{d_J(z_1, z_1'), d_J(z_2, z_2')\}$, then from (4.6) and (4.7) we can get

$$\int_0^L \varphi(t)dt \leqslant k \int_0^L \varphi(t)dt.$$

Since $k \in [0, 1)$, then we obtain $\int_0^L \varphi(t)dt = 0$ and therefore $L = 0$. Then $d_J(z_1, z_1') = 0 = d_J(z_2, z_2')$ that is $z_1 = z_1'$ and $z_2 = z_2'$. Hence $(z_1, z_2) = (z_1', z_2')$. □

Example 4.17. Let us consider the complete S^{JS}-metric space (X, J) defined in Example 2.2 and φ as defined in Example 4.10. Let $f : X^2 \to X$ be defined by $f(x, y) = \frac{x+y}{3}$ for all $x, y \in X$. Then for any $x, y, u, v \in X$ we have

$$\int_0^{d_J(f(x,y),f(u,v))} \varphi(t)dt \leqslant \frac{2}{3} \int_0^{\frac{1}{2}[d_J(x,u)+d_J(y,v)]} \varphi(t)dt.$$

For any $a, b \in X\backslash\{-\infty, \infty\}$ we get $\delta(J, f, (a, b)) \leqslant \frac{4}{3}|a + b|$ and also $\delta(J, f, (b, a)) \leqslant \frac{4}{3}|a + b|$. Here we see that $(0, 0)$, $(-\infty, -\infty)$ and (∞, ∞) are all coupled fixed points of f.

5. Fixed point on S^{JS}-metric spaces with two metrics

Maia [46] generalized the classical Banach contraction principle in the setting of a metric space with two metrics and proved that if T is a contraction mapping with respect to some non complete metric on a nonempty set X, while X is complete with respect to some metric, then T has a fixed point under certain conditions. In the past few years Maia's theorem and its applications in study of differential equations has been generalized in many directions by several researchers, see Agarwal and O'Regan [1], Smet [57], Khan et al. [42], Rus et al. [55], Soni [61] and references therein. In this section, we consider a nonempty set X together with two S^{JS}-metrics and prove several fixed point results for $\mathcal{Z}$-contractive map, Geraghty type contractive map and interpolative Hardy-Rogers type contractive mapping (see[53]). Examples are presented to high light the significance of newly obtained fixed point theorems.

Definition 5.1 ([15])**.** A function $\zeta : [0,\infty)^2 \to \mathbb{R}$ is called a simulation function, if it satisfies the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $s,t > 0$;
(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$,

then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

In an S^{JS}-metric space (X, F), $D_F : X^2 \to [0,\infty]$ stands for the function defined as $D_F(x,y) = F(x,x,y)$ for any $x,y \in X$. Set of all simulation functions is denoted by $\mathcal{Z}$. Also we denote by (X,F,J) a nonempty set X together with two S^{JS}- metrics $F, J : X^3 \to [0,\infty]$. An example of (X,F,J) is as follows.

Example 5.2. Let $X = [-\infty,\infty]$ and $F, J : X^3 \to \infty$ be defined by $F(x,y,z) = |x-\sqrt{2}| + |y-\sqrt{2}| + |z-\sqrt{2}|$ and $J(x,y,z) = |x| + |y| + |z|$ for all $x,y,z \in X$. Then (X,F,J) is an S^{JS}-metric space with two S^{JS}-metric, where both F and J are purely S^{JS}-metrics, neither S-metrics nor S_b-metrics.

Before proving our main fixed point results we need to extend the notion of Z-contractive map [43], Geraghty contractive map [29] and Interpolative Hardy-Roger type contractive map [39] to the case of an S^{JS}-metric space.

Definition 5.3. Let $T : X \to X$ be a map defined on an S^{JS}-metric space (X, F), such that for any $x,y \in X$, $D_F(Tx,Ty) = \infty \Rightarrow D_F(x,y) = \infty$. Then T is said to be an $S^{JS} - \mathcal{Z}$-contractive if there exists $\zeta \in \mathcal{Z}$ such that for $x,y \in X$, $D_F(x,y) < \infty$ implies $\zeta(D_F(Tx,Ty), D_F(x,y)) \geqslant 0$.

Definition 5.4. Let (X, F) be an S^{JS}-metric space. A map $T : X \to X$ is said to be an S^{JS}-Geraghty type contractive map if the map T satisfies the following contractive condition:

$$D_F(Tx,Ty) \leqslant \beta(D_F(x,y))D_F(x,y) \text{ for all } x,y \in X \text{ with } D_F(x,y) > 0,$$

where $\beta : (0,\infty] \to [0,1)$ is a function, satisfying (a) $\beta(\infty) = 0$ and (b) for any sequence $\{t_n\} \subset (0,\infty]$, $\beta(t_n) \to 1$ implies $t_n \to 0$.

Definition 5.5. Let (X, F) be an S^{JS}-metric space and $T : X \to X$. The map T is said to be an S^{JS}-Interpolative Hardy-Rogers type contractive map if there exists $\mu \in [0,1)$, $\xi,\eta,\zeta \in (0,1)$ with $\xi+\eta+\zeta < 1$ such that

$$D_F(Tx,Ty) \leqslant \mu D_F(x,y)^{\xi} D_F(x,Tx)^{\eta} D_F(y,Ty)^{\zeta} \left[\frac{1}{2}(D_F(x,Ty) + D_F(y,Tx))\right]^{1-\xi-\eta-\zeta} \tag{5.1}$$

for all $x,y \in X\backslash \mathrm{Fix}^*(T)$, where $\mathrm{Fix}^*(T) = \{x \in X : D_F(x,Tx) = 0\} \subset \mathrm{Fix}(T)$, Fix(T) is the set of all fixed points of T.

Definition 5.6. Let (X, F) be an S^{JS}-metric space and T be a self mapping on X. Then X is called T-orbitally complete if for any $x_0 \in X$ whenever $\{x_m\} \subset \mathcal{O}(T, x_0) = \{x_0, Tx_0, T^2x_0, \ldots\}$, is a Cauchy sequence, then there exists an element $x \in X$ such that $\{x_m\} \in S(F, X, x)$.

Definition 5.7. A self mapping T on an S^{JS}-metric space (X, F) is said to be orbitally continuous if for any $x_0 \in X$, $\{T^{n_i}x_0\}_{i\geqslant 1} \in S(F, X, u)$, $u \in X$, implies $\{TT^{n_i}x_0\}_{i\geqslant 1} \in S(F, X, Tu)$.

Now we state and prove our main results.

Theorem 5.8. *Let (X, F, J) be a S^{JS}-metric space with two metrics F and J. Assume that for $T : X \to X$ the following conditions are satisfied:*

(1) $F(x, y, z) = \infty$ *if and only if* $J(x, y, z) = \infty$, *otherwise* $F(x, y, z) \leqslant J(x, y, z) < \infty$ *for all* $x, y, z \in X$;
(2) (X, F) *is* T*-orbitally complete;*
(3) T *is orbitally continuous with respect to* F;
(4) T *is* $S^{JS} - \mathcal{Z}$*-contractive with respect to* J;
(5) *there exists* $x_0 \in X$ *such that* $\delta(J, T, x_0) = \sup\{d_J(T^ix_0, T^jx_0) : i, j \geqslant 1\} < \infty$ *and* $D_J(T^px_0, T^qx_0) > 0$ *for all* $p, q \geqslant 1$ $(p \neq q)$.

Then T has a fixed point in X. Moreover if $z, z' \in X$ are two fixed points of T such that $D_J(z, z') < \infty$, then $z = z'$.

Proof. Let us consider the Picard iterating sequence $\{x_n\}$ by $x_n = T^nx_0$ for all $n \in \mathbb{N}$. Then for any $(i, j) \in \mathbb{N}^2$ with $i \neq j$ we have,

$$0 \leqslant \zeta(D_J(x_{n+i}, x_{n+j}), D_J(x_{n-1+i}, x_{n-1+j})) < D_J(x_{n-1+i}, x_{n-1+j}) - D_J(x_{n+i}, x_{n+j})$$

implies $D_J(x_{n+i}, x_{n+j}) < D_J(x_{n-1+i}, x_{n-1+j})$ for all $n \in \mathbb{N}$. So $\{D_J(x_{n+i}, x_{n+j})\}_{n\in\mathbb{N}}$ is a decreasing bounded sequence for any $i, j (i \neq j) \geqslant 1$. Thus there exists $\lambda \geqslant 0$ such that $\lim_{n\to\infty} d_J(x_{n+i}, x_{n+j}) = \lambda$ for all $i, j (i \neq j) \geqslant 1$. If $\lambda > 0$, then for the sequences $t_n = D_J(x_{n+3}, x_{n+2})$ and $s_n = D_J(x_{n+2}, x_{n+1})$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \lambda$ and thus

$$0 \leqslant \limsup_{n\to\infty} \zeta(D_J(x_{n+3}, x_{n+2}), D_J(x_{n+2}, x_{n+1})) = \limsup_{n\to\infty} \zeta(t_n, s_n) < 0,$$

a contradiction. Therefore $\limsup_{n\to\infty} D_J(x_{n+i}, x_{n+j}) = 0$ for all $i, j \geqslant 1$ with $i \neq j$. Hence $\{x_n\}$ is Cauchy in (X, J). Now due to condition (1) it can be easily seen that $\{x_n\}$ is Cauchy in (X, F). Since (X, F) is T-orbitally complete, $\{x_n\}$ converges to some $\{x_n\} \in S(F, X, z)$. From condition (3) we get $\{T^{n+1}x_0\}$ converges to Tz. Hence we have $Tz = z$.

If possible, let, z, z' be two fixed points of T such that $D_J(z, z') < \infty$. If $D_J(z, z') > 0$, then we get

$$0 \leqslant \zeta(D_J(Tz, Tz'), D_J(z, z')) < D_J(z, z') - D_J(Tz, Tz') = D_J(z, z') - D_J(z, z') = 0, \text{ a contradiction.}$$

So we get $D_J(z, z') = 0$ implies $z = z'$. □

Theorem 5.9. *Let (X, F, J) be a S^{JS}-metric space with two metrics F and J. Assume that for $T : X \to X$ the following conditions are satisfied:*

(1) $F(x, y, z) = \infty$ *if and only if* $J(x, y, z) = \infty$, *otherwise* $F(x, y, z) \leqslant J(x, y, z) < \infty$ *for all* $x, y, z \in X$;
(2) (X, F) *is* T*-orbitally complete;*
(3) T *is orbitally continuous with respect to* F;
(4) T *is* S^{JS}*-Geraghty type contractive mapping with respect to* J;
(5) *there exists* $x_0 \in X$ *such that* $\delta(J, T, x_0) = \sup\{d_J(T^ix_0, T^jx_0) : i, j \geqslant 1\} < \infty$ *and* $D_J(T^px_0, T^qx_0) > 0$ *for all* $p, q \geqslant 1$ $(p \neq q)$.

Then T has a fixed point z in X. Moreover if $z' \in X$ is another fixed point of T such that $D_J(z, z') < \infty$, then $z = z'$.

Proof. Let us construct the Picard iterating sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Then for any particular $(i,j) \in \mathbb{N}^2$, we have

$$\begin{aligned} D_J(x_{n+i}, x_{n+j}) = D_J(Tx_{n-1+i}, Tx_{n-1+j}) &\leqslant \beta(D_J(Tx_{n-1+i}, Tx_{n-1+j}))D_J(Tx_{n-1+i}, Tx_{n-1+j}) \\ &< D_J(Tx_{n-1+i}, Tx_{n-1+j}) \leqslant \delta(D_J, T, x_0) < \infty \text{ for all } n \in \mathbb{N}. \end{aligned}$$

So $\{D_J(x_{n+i}, x_{n+j})\}_{n\geqslant 0}$ is a decreasing sequence of reals, which is bounded below. So $\{D_J(x_{n+i}, x_{n+j})\}_{n\geqslant 0}$ converges in $[0,\infty)$. We show that $D_J(x_{n+i}, x_{n+j}) \to 0$ as $n \to \infty$. If possible let $D_J(x_{n+i}, x_{n+j}) \to q$ as $n \to \infty$ for some $q > 0$. Then we have

$$1 > \beta(D_J(x_{n-1+i}, x_{n-1+j})) \geqslant \frac{D_J(x_{n+i}, x_{n+j})}{D_J(x_{n-1+i}, x_{n-1+j})} \to 1 \text{ as } n \to \infty.$$

Thus $\beta(D_J(x_{n-1+i}, x_{n-1+j})) \to 1$ as $n \to \infty$, a contradiction. Therefore $D_J(x_{n+i}, x_{n+j}) \to 0$ as $n \to \infty$. Since $(i,j) \in \mathbb{N}^2$ is arbitrary we get $\{x_n\}$ is Cauchy in (X, J). Now from condition (1) it can be easily seen that $\{x_n\}$ is Cauchy in (X, F). Since (X, F) is T-orbitally complete, $\{x_n\}$ converges to some $z \in X$ in (X, F). From condition (3) we get $\{T^{n+1}x_0\} \in S(F, X, Tz)$. Hence we have $Tz = z$.

If possible, let, there exists $z' \in X$ such that $Tz' = z'$ and $D_J(z, z') < \infty$. If $D_J(z, z') > 0$, then we get

$$D_J(z, z') = D_J(Tz, Tz') \leqslant \beta(D_J(z, z'))D_J(z, z') < D_J(z, z'), \text{ a contradiction.}$$

So we get $D_J(z, z') = 0$ implies $z = z'$. □

Theorem 5.10. *Let (X, F, J) be a S^{JS}-metric space with two metrics F and J. Assume that for $T : X \to X$ the following conditions are satisfied:*

(1) $F(x, y, z) = \infty$ *if and only if* $J(x, y, z) = \infty$, *otherwise* $F(x, y, z) \leqslant J(x, y, z) < \infty$ *for all* $x, y, z \in X$;
(2) (X, F) *is* T-*orbitally complete;*
(3) T *is orbitally continuous with respect to* F;
(4) T *is* S^{JS}-*Interpolative Hardy-Rogers type contractive mapping with respect to* J;
(5) *there exists* $x_0 \in X$ *such that* $\delta(J, T, x_0) = \sup\{d_J(T^i x_0, T^j x_0) : i, j \geqslant 1\} < \infty$.

Then T has a fixed point w in X.

Proof. Let us construct the Picard iterating sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. If for some $m \in \mathbb{N}$, $D_F(x_m, x_{m+1}) = 0$, then we have $Tx_m = x_m$ and T has a fixed point trivially. So without loss of generality we assume that $D_F(x_l, x_{l+1}) > 0$ for all $l \geqslant 1$. Let us take $\delta(J, T^{p+1}, x_0) = \sup\{d_J(T^{p+i}x_0, T^{p+j}x_0) : i, j \in \mathbb{N}\}$ for any non-negative integer p. Clearly $\delta(J, T^{p+1}, x_0) \leqslant \delta(J, T, x_0) < \infty$ for any $p \geqslant 1$. Then for all $i, j \geqslant 1$,

$$\begin{aligned} D_J(T^{n+i}x_0, T^{n+j}x_0) &\leqslant \mu D_J(T^{n-1+i}x_0, T^{n-1+j}x_0)^{\xi} D_J(T^{n-1+i}x_0, T^{n+i}x_0)^{\eta} D_J(T^{n-1+j}x_0, T^{n+j}x_0)^{\zeta} \\ &\quad \times \left[\frac{1}{2}(D_J(T^{n-1+i}x_0, T^{n+j}x_0) + D_J(T^{n-1+j}x_0, T^{n+i}x_0))\right]^{1-\xi-\eta-\zeta} \\ &\leqslant \mu\delta(J, T^n, x_0)^{\xi}\delta(J, T^n, x_0)^{\eta}\delta(J, T^n, x_0)^{\zeta}\left[\frac{1}{2}(\delta(J, T^n, x_0) + \delta(J, T^n, x_0))\right]^{1-\xi-\eta-\zeta} \\ &= \mu\delta(J, T^n, x_0) \text{ for all } n \geqslant 1. \end{aligned}$$

Therefore $\delta(J, T^{n+1}, x_0) \leqslant \mu\delta(J, T^n, x_0)$ for all $n \geqslant 1$. Thus $\delta(J, T^{n+1}, x_0) \leqslant \mu^n\delta(J, T, x_0)$ for all $n \in \mathbb{N}$. From which it follows that $\delta(J, T^{n+1}, x_0) \to 0$ as $n \to \infty$. So $\lim_{n\to\infty} D_J(T^n x_0, T^{n+k}x_0) \leqslant \lim_{n\to\infty} \delta(J, T^n, x_0) = 0$ that is $\lim_{n\to\infty} D_J(T^n x_0, T^{n+k}x_0) = 0$, $k \geqslant 1$. Hence by applying condition (1) we have $\lim_{n\to\infty} D_F(T^n x_0, T^{n+k}x_0) = 0$, $k \in \mathbb{N}$. So $\{x_n\}$ is Cauchy in (X, F). Since (X, F) is T-orbitally complete it follows that $\{x_n\}$ is convergent in (X, F) that is there exists some $w \in X$ such that $\{T^n x_0\} \in S(F, X, w)$. From condition (3) we get $\{T^{n+1}x_0\} \in S(F, X, Tw)$. Thus $Tw = w$. □

Example 5.11. Let $X = [0,1]$, $F(x,y,z) = |x-z| + |y-z|$ and $J(x,y,z) = |x| + |y| + 2|z|$ for all $x,y,z \in X$. Then both F and J are S^{JS}-metrics on X. Let $T : X \to X$ be defined as $Tx = \frac{x^2}{3(1+x)}$ for all $x \in X$ and $\zeta : [0,\infty)^2 \to \mathbb{R}$ be defined by

$$\zeta(t,s) = \begin{cases} \frac{s}{2} - t, & \text{if } 0 \leqslant s < 1, \\ s - \frac{1}{3} - t, & \text{if } s \geqslant 1. \end{cases}$$

Then one can verify that T satisfies all conditions of Theorem 5.8. Here 0 is the unique fixed point of T in X.

Example 5.12. Let $X = [0,1]$ and F, J be the S^{JS}-metrics defined on X as in above example. Also let $T : X \to X$ be defined by $Tx = \frac{x}{e}$ for all $x \in X$ and $\beta : (0,\infty] \to [0,1)$ be defined by $\beta(t) = e^{-t}$ for all $t \in (0,\infty]$. Then one can easily verify that all conditions of Theorem 5.9 are satisfied and T has a unique fixed point in X.

Example 5.13. Let $X = \{-1,0,1\}$, $F : X^3 \to [0,\infty]$ be defined by $F(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in X$ and $J : X^3 \to [0,\infty]$ be defined by $J(x,y,z) = |x| + |y| + 2|z|$ for all $x,y,z \in X$. Then both F and J are S^{JS}-metrics on X. Let $T : X \to X$ be defined by $T(-1) = 1$ and $T(0) = 0 = T(1)$. Then $\mathrm{Fix}^*(T) = \{0\}$. Therefore $X \backslash \mathrm{Fix}^*(T) = \{-1,1\}$. Now if we choose $\xi = \frac{1}{4} = \eta$, $\zeta = \frac{1}{3}$ and $\mu \in [\frac{1}{\sqrt{2}}, 1)$ be any fixed element, then we see that T satisfies contractive condition (5.1) with respect to J. Also one can verify that T satisfies all the additional conditions of Theorem 5.10. Therefore T has a fixed point in X.

In this paper we considered two S^{JS}-metrics on a nonempty set X. So we can take several combinations of metric type structures on a nonempty set X.

Remark 5.14. We can consider the following combinations:

(1) F and J both are S-metrics;
(2) both F and J are S_b-metrics;
(3) one is S-metric another is S_b-metric;
(4) one is either S-metric or S_b-metric and another is purely S^{JS}-metric neither S-metric nor S_b-metric;
(5) both F and J are purely S^{JS}-metrics neither S-metric nor S_b-metric.

6. Sequentially compact S^{JS}-metric space

Compactness [41] plays an extremely important role in mathematical analysis. A generalization of compactness is sequentially compact, if every infinite sequence of points sampled from the space has an infinite subsequence that converges to some point of the space. Various equivalent notions of compactness, including sequential compactness and limit point compactness, can be developed in general metric spaces. In general topological spaces, however, different notions of compactness are not necessarily equivalent. The most useful notion, which is the standard definition of the unqualified term compactness, is phrased in terms of the existence of finite families of open sets that "cover" the space in the sense that each point of the space lies in some set contained in the family. This more subtle notion, exhibits compact spaces as generalizations of finite sets. In spaces that are compact in this sense, it is often possible to patch together information that holds locally into corresponding statements that hold throughout the space. Thus compactness is a sort of generalization of the notion of finiteness. The power of compactness is that it provide a finite structure for infinite sets in situation where finiteness makes life easier (such as in optimization problems). In this section we continue this line of research and define the concept of sequential compactness on S^{JS}-metric spaces, study their properties and show its applications in fixed point theory (see [52]).

Now we define sequentially compact S^{JS}-metric space and study their topological properties.

Definition 6.1. Let (X, J) and (Y, J') be two S^{JS}-metric spaces and $T : X \to Y$ be a mapping. Then T is called continuous at $a \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in X$, $J'(Ta, Ta, Tx) < \epsilon$ whenever $J(a, a, x) < \delta$.

Definition 6.2. Let (X, J) be an S^{JS}-metric space. A family $\{A_\alpha\}_{\alpha\in\Lambda}$, Λ be an indexing set, of nonempty subsets of X is said to have the finite intersection property if for any finite sub-collection of subsets from $\{A_\alpha\}_{\alpha\in\Lambda}$ has nonempty intersection.

Definition 6.3. An S^{JS}-metric space (X, J) is said to be sequentially compact if every sequence $\{x_n\}$ in X has a convergent subsequence.

Theorem 6.4. *If (X, J) is a sequentially compact S^{JS}-metric space, then every countable family of closed sets with finite intersection property has non-empty intersection.*

Proof. Let $\{F_i\}_{i=1}^{\infty}$ be a countable family of closed sets with finite intersection property. Also let $x_i \in F_1 \cap F_2 \cap \cdots \cap F_i$ for all $i \in \mathbb{N}$. Then $\{x_n\}$ is a sequence in X. Now since X is sequentially compact so $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to some $x \in X$.

Now if $x \notin \cap_{i=1}^{\infty} F_i$, then $x \notin F_j$ for some $j \in \mathbb{N}$. Therefore $x \in X\backslash F_j$ and so there exists some $r > 0$ such that $B_J(x, r) \subset X\backslash F_j$. Since $J(x, x, x_{n_k}) \to 0$ as $k \to \infty$ so there exists $N \in \mathbb{N}$ such that $J(x, x, x_{n_k}) < r$ for all $k \geqslant N$ and thus $x_{n_k} \in B_J(x, r) \subset X\backslash F_j$ for all $k \geqslant N$ but $x_m \in F_j$ for any $m \geqslant j$. So if we choose $k_0 = \max\{N, j\}$, then we have $x_{n_{k_0}} \in F_j \cap (X\backslash F_j)$, a contradiction.

Hence $x \in \cap_{i=1}^{\infty} F_i$ and thus $\cap_{i=1}^{\infty} F_i \neq \emptyset$. □

Proposition 6.5. *Let (X, J) be a sequentially compact S^{JS}-metric space and $F \subset X$ be closed. Then F is also sequentially compact.*

Proof. Let $\{x_n\} \subset F$ be an arbitrary sequence. Since X is sequentially compact, then $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \in S(J, X, x)$ for some $x \in X$. Now as F is closed therefore by Theorem 3.5 we have $x \in F$. Hence F is also sequentially compact. □

Proposition 6.6. *Let (X, J) be an S^{JS}-metric space and $A \subset X$ be sequentially compact. Then A is closed.*

Proof. To prove A is closed we have to show $X\backslash A$ is open. Without loss of generality let us take $X\backslash A \neq \emptyset$. Let $x \in X\backslash A$ and let us assume that $B_J(x, \frac{1}{n}) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. Also let $x_n \in B_J(x, \frac{1}{n}) \cap A$ for all $n \geqslant 1$. Since $J(x, x, x_n) < \frac{1}{n} \to 0$ as $n \to \infty$, so $\{x_n\}$ converges to x. Now A is sequentially compact so $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \in S(J, X, y)$ for some $y \in A$. Therefore by Theorem 2.10 we have $x = y$, a contradiction. So there exists some $N \in \mathbb{N}$ such that $B_J(x, \frac{1}{N}) \subset X\backslash A$. Hence A is closed. □

Proposition 6.7.

(1) *The union of a finite collection of sequentially compact subsets of an S^{JS}-metric space is sequentially compact.*

(2) *The intersection of an arbitrary family of sequentially compact subsets of an S^{JS}-metric space is sequentially compact.*

Proof.

(a) Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of sequentially compact subsets of an S^{JS}-metric space (X, J). Let $A = \cup_{i=1}^{n} A_i$ and $\{x_n\} \subset A$. Therefore there exists at least one A_i, $1 \leqslant i \leqslant n$ such that A_i contains infinitely many terms of $\{x_n\}$. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}|_{A_i}$ which converges to some $x \in A_i$. Then x also belongs to A and hence A is also sequentially compact.

(b) Let $\{B_\gamma\}_{\gamma\in\Lambda}$ be an arbitrary collection of sequentially compact subsets of (X, J). Let $\{x_n\} \subset B = \cap_{\gamma\in\Lambda} B_\gamma$. Then $\{x_n\} \subset B_\gamma$ for all $\gamma \in \Lambda$. In particular, since B_γ is sequentially compact so $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to some $x \in B_\gamma$. Now since each B_γ is closed by Proposition 6.6, we get B is closed in X. So $x \in B$ and this proves that B is sequentially compact. □

Proposition 6.8. *Let (X, J) and (Y, J') be two S^{JS}-metric spaces and $T : X \to Y$ be a continuous mapping. If X is sequentially compact, then $T(X)$ is also sequentially compact.*

Proof. Let $\{y_n\} \subset T(X)$ be an arbitrary sequence. Then there exists a sequence $\{x_n\}$ in X such that $y_n = Tx_n$ for all $n \in \mathbb{N}$. Since X is sequentially compact so $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in X which converges to some $x \in X$. Since T is continuous therefore it can be easily seen that $\{Tx_{n_k}\} \in S(J', Y, Tx)$. Hence $T(X)$ is also sequentially compact. □

Following the literatures [22, 44, 59], we now prove some contractive fixed point theorems on a sequentially compact S^{JS}- metric space.

Theorem 6.9. *Let (X, J) be a sequentially compact S^{JS}-metric space such that for any continuous map $G : X \to X$, $J(x, x, Gx)$ is a continuous function on X. Suppose that T is a continuous map on X such that for all $x, y \in X$ with $J(x, x, y) > 0$ we have*

$$J(Tx, Tx, Ty) < \max\{J(x, x, Tx), J(y, y, Ty), J(x, x, y)\}.$$

If for some $z \in X$, $J(z, z, Tz) < \infty$, then T has a fixed point u in X with $d_J(u, u) = 0$. Moreover if u' is another fixed point of T such that $d_J(u', u') < \infty$ and $d_J(u, u') < \infty$, then $u = u'$.

Proof. Let us denote $d_J(x, y) = J(x, x, y)$ for all $x, y \in X$ and $z_n = T^n z$ for all $n \geqslant 0$. If $d_J(z_n, z_{n+1}) = 0$ for some $n \in \mathbb{N} \cup \{0\}$, then $z_n = z_{n+1}$ and we have $z_n = z_{n+k}$ for all $k \geqslant 1$. Therefore $d_J(z_n, z_{n+k}) = 0$ for all $k \in \mathbb{N}$. So clearly $\{z_m\}$ converges to $u = z_n$ and since T is continuous it follows that $Tu = u$. So we assume that $d_j(z_n, z_{n+1}) > 0$ for all $n \geqslant 0$. Also let $V(y) = d_J(y, Ty)$, then by our assumption V is continuous. Now,

$$V(z_{n+1}) = d_j(z_{n+1}, z_{n+2}) < \max\{d_j(z_n, z_{n+1}), d_j(z_{n+1}, z_{n+2}), d_j(z_n, z_{n+1})\}$$

implies $V(z_{n+1}) < V(z_n)$ for all $n \in \mathbb{N} \cup \{0\}$ that is for any $n \geqslant 0$ we have

$$V(z_{n+1}) < V(z_n) < V(z_0) = V(z) < \infty.$$

Thus there exists some $r \geqslant 0$ such that $\lim_{n\to\infty} V(z_n) = r$. Since the sequence $\{z_n\}$ has a convergent subsequence which converges to some $u \in X$ and V is continuous it follows that $V(u) = r$. Also since T is continuous we get $V(Tu) = r$. Now if $r > 0$, then we see that

$$r = V(Tu) = d_J(Tu, T^2u) < \max\{d_J(u, Tu), d_J(Tu, T^2u), d_J(u, Tu)\} = r,$$

a contradiction. Therefore $r = 0$ and we have $d_J(u, Tu) = d_J(Tu, T^2u) = 0$. So u is a fixed point of T. Now we prove that the sequence $\{z_n\}$ converges to u. For any given $\epsilon > 0$, there exists a positive integer M such that $\{V(z_M), d_J(z_M, u)\} < \epsilon$. Thus we get,

$$\begin{aligned} d_J(z_n, u) = d_J(Tz_{n-1}, Tu) &< \max\{d_J(z_{n-1}, u), d_J(z_{n-1}, z_n), d_J(u, Tu)\} \\ &= \max\{V(z_{n-1}), d_J(z_{n-1}, u))\} \\ &< \max\{V(z_{n-1}), V(z_{n-2}), d_J(z_{n-2}, u)\} \\ &= \max\{V(z_{n-2}), d_J(z_{n-2}, u)\} \\ &\vdots \\ &< \max\{V(z_M), d_J(z_M, u)\} < \epsilon \end{aligned}$$

for all $n > M$. Hence $\lim_{n\to\infty} z_n = u$ and this completes our proof. Clearly $d_J(u, u) = 0$.

Uniqueness. Let u' be another fixed point of T such that $d_J(u, u') < \infty$. Since $d_J(u', u') < \infty$ it follows that $d_J(u', u') = 0$. If possible, let, $d_J(u, u') > 0$, then

$$d_J(u, u') = d_J(Tu, Tu') < \max\{d_J(u, u), d_J(u', u'), d_J(u, u')\} = d_J(u, u'),$$

a contradiction. Hence $d_J(u, u') = 0$ and we get $u = u'$. □

Corollary 6.10. *Let (X, J) be a sequentially compact S^{JS}-metric space such that for any continuous map $G : X \to X$, $J(x, x, Gx)$ is a continuous function on X. Also let T be a mapping on X such that for all $x, y \in X$ with $J(x, x, y) > 0$ we have*

$$J(Tx, Tx, Ty) < J(x, x, y).$$

Then for any z in X with $J(z, z, Tz) < \infty$, $\{T^n z\}$ converges to u and u is a fixed point of T.

Proof. Clearly T satisfies all conditions of Theorem 6.9. So the result of this Corollary follows immediately. □

Corollary 6.11. *Let (X, J) be a sequentially compact S^{JS}-metric space such that for any continuous map $G : X \to X$, $J(x, x, Gx)$ is a continuous function on X. Also let T be a continuous mapping on X such that for all $x, y \in X$ with $J(x, x, y) > 0$ we have*

$$J(Tx, Tx, Ty) < \frac{1}{2}[J(x, x, Tx) + J(y, y, Ty)].$$

Then for any z in X with $J(z, z, Tz) < \infty$, $\{T^n z\}$ converges to u and u is a fixed point of T.

Proof. Clearly T satisfies all conditions of Theorem 6.9. So the result of this Corollary follows immediately. □

Corollary 6.12. *Let (X, J) be a sequentially compact S^{JS}-metric space such that for any continuous map $G : X \to X$, $J(x, x, Gx)$ is a continuous function on X and T be a continuous mapping on X such that for all $x, y \in X$ with $J(x, x, y) > 0$ we have*

$$J(Tx, Tx, Ty) < \max\{J(x, x, Tx), J(y, y, Ty)\}.$$

Then for any z in X with $J(z, z, Tz) < \infty$, $\{T^n z\}$ converges to v and v is a fixed point of T.

Proof. Clearly T satisfies all conditions of Theorem 6.9. So the result of this Corollary follows immediately. □

Definition 6.13 ([58])**.** Let X be a non-empty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions: for each $x, y, z, w \in X$,

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $S(x, y, z) \leqslant S(x, x, w) + S(y, y, w) + S(z, z, w)$.

The function S is called an S-metric and the pair (X, S) is called an S-metric space.

Obviously every S-metric space is a S^{JS}-metric space. Converse is not always true.

Corollary 6.14. *Let (X, S) be a sequentially compact S-metric space and T be a continuous mapping on X such that for all $x, y \in X$ with $x \neq y$ we have*

$$S(Tx, Tx, Ty) < \max\{S(x, x, Tx), S(y, y, Ty), S(x, x, y)\}.$$

Then for any y in X, $\{T^n y\}$ converges to v and v is a fixed point of T.

Proof. If G is a continuous mapping, then for any $x \in X$ and any sequence $\{x_n\}$ in X converging to x we get, $|S(x_n, x_n, Gx_n) - S(x, x, Gx)| \leqslant 2[S(x_n, x_n, x) + S(Gx_n, Gx_n, Gx)] \to 0$ as $n \to \infty$. Therefore $S(x, x, Gx)$ is a continuous function on X whenever G is continuous. Therefore all conditions of Theorem 6.9 are satisfied and it follows that v is a fixed point of T. □

Example 6.15. Let $X = [0, \infty)$ and $J : X^3 \to X$ be defined by $J(x, y, z) = |x - y| + |x - z|$ for all $x, y, z \in X$ and $T : X \to X$ be defined by $Tx = x - \tan^{-1} x$ for all $x \in X$. Then T satisfies $J(Tx, Tx, Ty) < J(x, x, y)$ for all $x, y \in X$ and also for $x_0 = 0$ the iterative sequence $\{T^n x_0\}$ converges to 0 and we see that 0 is the unique fixed point of T.

Example 6.16. Let $X = [0,5]$ and $J : X^3 \to X$ be defined by $J(x,y,z) = |x-y| + |x-z|$ for all $x,y,z \in X$ and $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{3x}{4}, & \text{if } x \in [0,4], \\ 10 - \frac{7x}{4}, & \text{if } x \in [4,5]. \end{cases}$$

Then T is continuous and satisfies the contractive condition of Corollary 6.14 but it does not satisfy $J(Tx,Tx,Ty) < J(x,x,y)$ for all $x,y \in X$ since $J(T4,T4,T5) = \frac{7}{4} \not< J(4,4,5)$. Thus all conditions of Corollary 6.14 are satisfied. Also for $z_0 = 0$ the iterative sequence $\{T^n z_0\}$ converges to 0 and we see that 0 is the unique fixed point of T.

7. Ekeland's variational principle

In his classic paper Ekeland [23] proved a theorem (Ekeland's variational principle) that asserts that there exists nearly optimal solutions to some optimization problems. Ekeland's variational principle can be applied when the lower level set of a minimization problems is not compact, so that the Bolzano–Weierstrass theorem cannot be used. Ekeland's principle relies on Cantor intersection theorem and axiom of choice. Ekeland's principle also leads to an elegant proof of the famous Caristi fixed point theorem [14]. For further generalizations and applications of Ekeland's variational principle we refere to [9, 25, 32, 47] and their references. The aim of this section is first to give a variant of Ekeland's variational principle in S^{JS}-metric spaces and, then derive Caristi fixed point theorem as an application (see [7]). These results generalize/extend several results from the existing literature.

Definition 7.1. In an S^{JS}-metric space (X,J), a mapping $\psi : X \to \overline{\mathbb{R}}$ is said to be lower semi-continuous at $t_0 \in X$ if for any $\epsilon > 0$ there exits some $\delta_\epsilon > 0$ such that $\psi(t_0) < \psi(t) + \epsilon$ for all $t \in B_J(t_0, \delta_\epsilon)$.

Definition 7.2. Let (X,J) be an S^{JS}-metric space and $\{A_n\}$ be a decreasing sequence of nonempty subsets of X. Then $\{A_n\}$ is said to have vanishing diameter property (νd-property) if for each $i \in \mathbb{N}$ there exists some fixed $a_i \in A_i$ such that $J(x,x,a_i) \leqslant J(a_i,a_i,a_i) + r_i$ for all $x \in A_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$.

Definition 7.3. An S^{JS}-metric space (X,J) is said to have vanishing diameter property if for any decreasing sequence of nonempty subsets $\{A_n\}$ of X with νd-property we have $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$.

We now establish Ekeland's variational principle in an S^{JS}-metric space. Let us denote $d_J(x,y) = J(x,x,y)$ for all $x,y \in X$.

Theorem 7.4. *Let (X,J) be a complete S^{JS}-metric space with coefficient $s > 1$, such that d_J is continuous in both variables, $\sup\{J(x,x,x) : x \in X\} < \infty$ and X has vanishing diameter property. Now let, $f : X \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper and lower bounded mapping. Then for every $x_0 \in X$ and $\epsilon > 0$ with*

$$f(x_0) \leqslant \inf_{x \in X} f(x) + \epsilon,$$

there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that:

(i) $x_n \to x_\epsilon$ *as* $n \to \infty$;
(ii) *for all* $n \geqslant 1$, $J(x_\epsilon,x_\epsilon,x_n) - J(x_n,x_n,x_n) \leqslant \frac{\epsilon}{2^n}$;
(iii) *for all* $x \neq x_\epsilon$, $f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x,x,x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon,x_\epsilon,x_n)$;
(iv) $f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon,x_\epsilon,x_n) \leqslant f(x_0) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n,x_n,x_n) \leqslant \inf_{x \in X} f(x) + \epsilon + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n,x_n,x_n)$.

Proof. Consider the set

$$S_f(x_0) = \{x \in X : f(x) + d_J(x,x_0) \leqslant f(x_0) + d_J(x_0,x_0)\}.$$

Since $x_0 \in S_f(x_0)$, then $S_f(x_0)$ is nonempty. Let $\{z_n\} \subset S_f(x_0)$ such that $\{z_n\}$ converges to some $z \in X$. Then $f(z_n) + d_J(z_n, x_0) \leqslant f(x_0) + d_J(x_0, x_0)$ for all $n \in \mathbb{N}$. Now f is lower semi-continuous at $z \in X$, so for any $\epsilon_1 > 0$, $f(z) < f(t) + \frac{\epsilon_1}{2}$ for all $t \in B_J(z, \delta_{\epsilon_1})$ for $\delta_{\epsilon_1} > 0$. Also $\{z_n\}$ converges to some z, so there exists $N_1 \geqslant 1$ such that $z_n \in B_J(z, \delta_{\epsilon_1})$ for all $n \geqslant N_1$. Therefore $f(z) < f(z_n) + \frac{\epsilon_1}{2}$ for all $n \geqslant N_1$. Now continuity of d_J implies that $d_J(z_n, x_0) \to d_J(z, x_0)$ as $n \to \infty$. Thus for all $n \geqslant N_2$,

$$d_J(z, x_0) - \frac{\epsilon_1}{2} < d_J(z_n, x_0) < d_J(z, x_0) + \frac{\epsilon_1}{2}.$$

Therefore for all $n \geqslant N = \max\{N_1, N_2\}$ we get,

$$f(z) + d_J(z, x_0) < f(z_n) + d_J(z_n, x_0) + \epsilon_1 \text{ for all } n \geqslant N \leqslant f(x_0) + d_J(x_0, x_0) + \epsilon_1.$$

Since $\epsilon_1 > 0$ is arbitrary, thus $f(z) + d_J(z, x_0) \leqslant f(x_0) + d_J(x_0, x_0)$. Therefore $z \in S_f(x_0)$. Hence $S_f(x_0)$ is closed. Also for any $y \in S_f(x_0)$ we get

$$d_J(y, x_0) - d_J(x_0, x_0) \leqslant f(x_0) - f(y) \leqslant f(x_0) - \inf_{x \in X} f(x) \leqslant \epsilon.$$

We choose $x_1 \in S_f(x_0)$ such that $f(x_1) + d_J(x_1, x_0) \leqslant \inf_{x \in S_f(x_0)}\{f(x) + d_J(x, x_0)\} + \frac{\epsilon}{2s}$ and let

$$S_f(x_1) = \{x \in X : f(x) + d_J(x, x_0) + \frac{1}{s} d_J(x, x_1) \leqslant f(x_1) + d_J(x_1, x_0) + \frac{1}{s} d_J(x_1, x_1)\}.$$

Thus $x_1 \in S_f(x_1)$ and in a similar way as above we can prove that $S_f(x_1)$ is also closed. Inductively, we can suppose that $x_{n-1} \in S_f(x_{n-2})$ (for $n > 2$) was already chosen and we consider

$$S_f(x_{n-1}) = \{x \in S_f(x_{n-2}) : f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i) \leqslant f(x_{n-1}) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x_{n-1}, x_i)\}.$$

Let us choose $x_n \in S_f(x_{n-1})$ such that

$$f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x_n, x_i) \leqslant \inf_{x \in S_f(x_{n-1})} \{f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i)\} + \frac{\epsilon}{2^n s^n}$$

and we define the set

$$S_f(x_n) = \{x \in S_f(x_{n-1}) : f(x) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x, x_i) \leqslant f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\}.$$

Clearly $x_n \in S_f(x_n)$ and $S_f(x_n)$ is also closed. Now for each $y \in S_f(x_n)$ we get

$$\begin{aligned} \frac{1}{s^n} d_J(y, x_n) &\leqslant \{f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\} - \{f(y) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(y, x_i)\} \\ &\leqslant \{f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\} - \inf_{x \in S_f(x_{n-1})} \{f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i)\} \\ &\leqslant \frac{1}{s^n} d_J(x_n, x_n) + \frac{\epsilon}{2^n s^n}. \end{aligned} \tag{7.1}$$

Therefore for any $y \in S_f(x_n)$ we have

$$d_J(y, x_n) - d_J(x_n, x_n) \leqslant \frac{\epsilon}{2^n} \text{ for all } n \in \mathbb{N}.$$

Thus the decreasing sequence of nonempty closed subsets $\{S_f(x_n)\}_{n\geqslant 0}$ has νd-property. Since X has νd-property therefore $\operatorname{diam}(S_f(x_n)) \to 0$ as $n \to \infty$. Thus by Cantor's intersection theorem (see Theorem 3.10) we have $\cap_{n=0}^{\infty} S_f(x_n) = \{x_\epsilon\}$. Now $d_J(x_\epsilon, x_n) \leqslant \operatorname{diam}(S_f(x_n)) \to 0$ as $n \to \infty$ and we have $x_n \to x_\epsilon$ as $n \to \infty$. From (7.1) we see that

$$J(x_\epsilon, x_\epsilon, x_n) - J(x_n, x_n, x_n) \leqslant \frac{\epsilon}{2^n} \text{ for all } n \in \mathbb{N}.$$

Now

$$\begin{aligned} f(x_1) + d_J(x_1, x_0) &\leqslant f(x_0) + d_J(x_0, x_0), \\ f(x_2) + d_J(x_2, x_0) + \frac{1}{s} d_J(x_2, x_1) &\leqslant f(x_1) + d_J(x_1, x_0) + \frac{1}{s} d_J(x_1, x_1), \\ &\leqslant f(x_0) + d_J(x_0, x_0) + \frac{1}{s} d_J(x_1, x_1), \\ &\vdots \\ f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} d_J(x_m, x_i) &\leqslant f(x_0) + \sum_{i=0}^{m-1} \frac{1}{s^i} d_J(x_i, x_i) \text{ for all } m > 1. \end{aligned}$$

Also $x_\epsilon \in S_f(x_q)$ for all $q \in \mathbb{N}$, therefore

$$f(x_\epsilon) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_\epsilon, x_i) \leqslant f(x_q) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_q, x_i) \leqslant f(x_0) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_i, x_i) \text{ for all } q \geqslant 1,$$

which in turn implies that

$$f(x_\epsilon) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_\epsilon, x_i) \leqslant f(x_0) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_i, x_i) \leqslant \inf_{x\in X} f(x) + \epsilon + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_i, x_i).$$

Moreover for all $x \neq x_\epsilon$, we have $x \notin \cap_{n=0}^{\infty} S_f(x_n)$ and thus there exists $m \in \mathbb{N}$ such that $x \notin S_f(x_m)$. So $x \notin S_f(x_q)$ for all $q \geqslant m$. Therefore

$$f(x) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x, x_i) > f(x_q) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_q, x_i) \geqslant f(x_\epsilon) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_\epsilon, x_i) \text{ for all } q \geqslant m.$$

Hence we see that

$$f(x) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x, x_i) > f(x_\epsilon) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_\epsilon, x_i).$$

□

Next we have the following consequence of Ekeland's variational principle in S^{JS}-metric spaces.

Corollary 7.5. *Let (X, J) be a complete S^{JS}-metric space with coefficient $s > 1$, such that d_J is continuous in both variables, $\sup\{J(x, x, x) : x \in X\} < \infty$ and X has vanishing diameter property. Now let, $f : X \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper and lower bounded mapping. Then for every $\epsilon > 0$ there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that*

(i) $x_n \to x_\epsilon$ *as* $n \to \infty$;
(ii) $f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) \geqslant f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n)$ *for every* $x \in X$;
(iii) $f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) \leqslant \inf_{x\in X} f(x) + \epsilon + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n)$.

As an application of Theorem 7.4 we now prove Caristi's fixed point theorem in the context of S^{JS}-metric spaces.

Theorem 7.6. *Let (X, J) be a complete S^{JS}-metric space with coefficient $s > 1$, such that d_J is continuous in both variables, $\sup\{J(x, x, x) : x \in X\} < \infty$ and X has vanishing diameter property. Let $T : X \to X$ be an operator for which there exists a lower semi-continuous mapping, proper and lower bounded mapping $f : X \to \overline{\mathbb{R}}$ such that*

$$J(u, u, v) + sJ(u, u, Tu) \geqslant J(Tu, Tu, v) \tag{7.2}$$

and

$$\frac{s^2}{s-1} J(u, u, Tu) \leqslant f(u) - f(Tu) \text{ for all } u, v \in X. \tag{7.3}$$

Then T has at least one fixed point in X.

Proof. Let us assume that for all $x \in X$, $Tx \neq x$. Using Corollary 7.5 for f, we obtain that for each $\epsilon > 0$ there exists a sequence $\{x_n\} \subset X$ such that $x_n \to x_\epsilon$ as $n \to \infty$ and

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) \text{ for every } x \neq x_\epsilon.$$

If in the above inequality, we put $x = T(x_\epsilon)$, then, since $T(x_\epsilon) \neq x_\epsilon$, we get that

$$\begin{aligned} f(x_\epsilon) - f(Tx_\epsilon) &< \sum_{n=0}^{\infty} \frac{1}{s^n} [d_J(Tx_\epsilon, x_n) - d_J(x_\epsilon, x_n)] \\ &< \sum_{n=0}^{\infty} \frac{1}{s^n} s d_J(x_\epsilon, Tx_\epsilon) \text{ (Using (7.2))} = s \sum_{n=0}^{\infty} \frac{1}{s^n} d_J(x_\epsilon, Tx_\epsilon) = \frac{s^2}{s-1} d_J(x_\epsilon, Tx_\epsilon). \end{aligned}$$

Also from (7.3) we get $\frac{s^2}{s-1} d_J(x_\epsilon, Tx_\epsilon) \leqslant f(x_\epsilon) - f(Tx_\epsilon)$, a contradiction. Therefore there exists at least one $x^* \in X$ such that $Tx^* = x^*$. □

Definition 7.7 ([62])**.** Let X be a nonempty set and $s \geqslant 1$ be a given number. Also let a function $S_b : X^3 \to [0, \infty)$ satisfies the following conditions, for each $x, y, z, w \in X$:

(i) $S_b(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $S_b(x, y, z) \leqslant s[S_b(x, x, w) + S_b(y, y, w) + S_b(z, z, w)]$.

The pair (X, S_b) is called an S_b-metric space.

[62, Theorem 2.4] by Souayah and Mlaiki follows from our Theorem 7.4 as immediate Corollary 7.8.

Corollary 7.8. *Let (X, S_b) be a complete S_b-metric space with coefficient $s > 1$, such that the S_b-metric is continuous and $f : X \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper and lower bounded mapping. Then for every $x_0 \in X$ and $\epsilon > 0$ with*

$$f(x_0) \leqslant \inf_{x \in X} f(x) + \epsilon,$$

there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that

(i) $x_n \to x_\epsilon$ *as* $n \to \infty$;
(ii) $S_b(x_\epsilon, x_\epsilon, x_n) \leqslant \frac{\epsilon}{2^n}$ *for all* $n \geqslant 1$;
(iii) $f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x, x, x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n)$ *for every* $x \neq x_\epsilon$;
(iv) $f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \leqslant f(x_0) \leqslant \inf_{x \in X} f(x) + \epsilon$.

Proof. Let $\{A_n\}$ be a decreasing sequence of nonempty subsets of X such that it has νd-property. Then for each $i \in \mathbb{N}$ there exists some fixed $a_i \in A_i$ such that $S_b(x, x, a_i) \leqslant S_b(a_i, a_i, a_i) + r_i = r_i$ for all $x \in A_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$. Let $x^{(i)}, y^{(i)}, z^{(i)} \in A_i$ be arbitrary. Then

$$S_b(x^{(i)}, y^{(i)}, z^{(i)}) \leqslant s[S_b(x^{(i)}, x^{(i)}, a_i) + S_b(y^{(i)}, y^{(i)}, a_i) + S_b(z^{(i)}, z^{(i)}, a_i)] \leqslant 3sr_i.$$

It implies $\text{diam}(A_i) \leqslant 3sr_i$. Since this is true for all $i \in \mathbb{N}$ we get $\text{diam}(A_i) \to 0$ as $r_i \to \infty$. Thus (X, S_b) has vanishing diameter property. Therefore all the conditions of Theorem 7.4 are satisfied and the result follows immediately. □

Corollary 7.9. *Let (X, S_b) be a complete S_b-metric space with coefficient $s > 1$, such that the S_b-metric is continuous and let $T : X \to X$ be an operator for which there exists a lower semi-continuous, proper and lower bounded mapping $f : X \to \overline{\mathbb{R}}$, such that:*

$$S_b(u, u, v) + sS_b(u, u, Tu) \geqslant S_b(Tu, Tu, v) \quad \textit{and} \quad \frac{s^2}{s-1} S_b(u, u, Tu) \leqslant f(u) - f(Tu) \textit{ for all } u, v \in X.$$

Then T has at least one fixed point in X.

Proof. Using Theorem 7.6 and Corollary 7.8 we get the required proof. □

Remark 7.10. [10, Theorem 2.2] is a particular case of our Theorem 7.4.

8. Fixed point of rational type contractive mappings

In this section we prove some fixed point results for rational type contractive mappings in the setting of S^{JS}-metric space.

Theorem 8.1. *Let (X, J) be a complete S^{JS}-metric space and $T : X \to X$ be a mapping which satisfies the following conditions:*

(i) *there exists $\nu_1, \nu_2 \in [0, 1)$ such that $\nu_1 + \nu_2 < 1$ and for all $x, y \in X$,*

$$d_J(Tx, Ty) \leqslant \nu_1 d_J(x, y) + \nu_2 \frac{d_J(y, Ty)[1 + d_J(x, Tx)]}{1 + d_J(x, y)}; \tag{8.1}$$

(ii) *for a mapping $\varphi : [0, \infty)^2 \to [0, \infty)$ with $\varphi(0, 0) = 0$ and which is continuous at $(0, 0)$, we have*

$$d_J(Tx, Ty) \leqslant \varphi(d_J(x, Tx), d_J(y, Ty)) \textit{ for all } x, y \in X.$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be taken as arbitrary and let us construct the Picard iterating sequence $\{x_n\}_{n \geqslant 1}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} d_J(x_n, x_{n+1}) = d_J(Tx_{n-1}, Tx_n) &\leqslant \nu_1 d_J(x_{n-1}, x_n) + \nu_2 \frac{d_J(x_n, x_{n+1})[1 + d_J(x_{n-1}, x_n)]}{1 + d_J(x_{n-1}, x_n)} \\ &= \nu_1 d_J(x_{n-1}, x_n) + \nu_2 d_J(x_n, x_{n+1}) \text{ for all } n \geqslant 1. \end{aligned}$$

Therefore we have $d_J(x_n, x_{n+1}) \leqslant \mu d_J(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, where $\mu = \frac{\nu_1}{1-\nu_2}$. So $\lim_{n\to\infty} d_J(x_{n-1}, x_n) = 0$. Now using condition (ii) we get

$$d_J(x_n, x_m) = d_J(Tx_{n-1}, Tx_{m-1}) \leqslant \varphi(d_J(x_{n-1}, x_n), d_J(x_{m-1}, x_m)) \to 0 \text{ as } n, m \to \infty.$$

Thus $\{x_n\}_{n\geqslant 1}$ is Cauchy in X and by the completeness of X there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now,

$$d_J(Tu, x_{n+1}) = d_J(Tu, Tx_n) \leqslant \nu_1 d_J(u, x_n) + \nu_2 \frac{d_J(x_n, x_{n+1})[1 + d_J(u, Tu)]}{[1 + d_J(u, x_n)]} \to 0 \text{ as } n \to \infty.$$

Therefore by Theorem 2.10 we see that $Tu = u$. Let ν be a fixed point of T in X. Then

$$d_J(\nu, \nu) = d_J(T\nu, T\nu) \leqslant \nu_1 d_J(\nu, \nu) + \nu_2 \frac{d_J(\nu, T\nu)[1 + d_J(\nu, T\nu)]}{1 + d_J(\nu, \nu)} = (\nu_1 + \nu_2) d_J(\nu, \nu).$$

This implies $d_J(\nu, \nu) = 0$. Hence from condition (ii) we have $d_J(u, \nu) = d_J(Tu, T\nu) \leqslant \varphi(d_J(u, Tu), d_J(\nu, T\nu)) = \varphi(0, 0) = 0$. From this it follows that $u = \nu$ and T has a unique fixed point in X. □

Example 8.2. Let $X = [0, 2]$ and $J(x, y, z) = |y + z - 2x| + |x - z| + |y - z|$ for all $x, y, z \in X$. Then (X, J) is a symmetric S^{JS}-metric space. Let $F : X \to X$ be defined by

$$F(x) = \begin{cases} \frac{x+1}{2}, & \text{if } 0 \leqslant x \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Case-I: If $x, y \in [0, 1]$, then

$$d_J(Tx, Ty) = 3|Tx - Ty| = 3\left|\frac{x+1}{2} - \frac{y+1}{2}\right| = \frac{3}{2}|x - y| = \frac{1}{2} d_J(x, y).$$

Case-II: If $x, y \in (1, 2]$, then

$$d_J(Tx, Ty) = 0.$$

Case-III: If $x \in [0, 1]$ and $y \in (1, 2]$, then

$$\begin{aligned} d_J(Tx, Ty) &= 3|Tx - Ty| = 3\left|\frac{x+1}{2} - 0\right| = \frac{3}{2}(x + 1), \\ d_J(x, y) &= 3(y - x), \\ \frac{d_J(y, Ty)[1 + d_J(x, Tx)]}{1 + d_J(x, y)} &= \frac{3y[1 + \frac{3}{2}(1 - x)]}{1 + 3(y - x)}. \end{aligned}$$

Then by a routine calculation we see that

$$d_J(Tx, Ty) \leqslant \frac{1}{2} d_J(x, y) + \frac{1}{4} \frac{d_J(y, Ty)[1 + d_J(x, Tx)]}{1 + d_J(x, y)}.$$

Therefore from all the cases we get

$$d_J(Tx, Ty) \leqslant \frac{1}{2} d_J(x, y) + \frac{1}{4} \frac{d_J(y, Ty)[1 + d_J(x, Tx)]}{1 + d_J(x, y)} \text{ for all } x, y \in X.$$

Also one can check that $d_J(Tx, Ty) \leqslant d_J(x, Tx) + d_J(y, Ty)$ for all $x, y \in X$. Therefore all conditions of Theorem 8.1 are satisfied and $x = 1$ is the unique fixed point of T.

Theorem 8.3. *Let (X, J) be a complete S^{JS}-metric space and $T : X \to X$ be a mapping which satisfies the following conditions:*

(i) *there exists $\nu \in [0, 1)$ such that for all $x, y \in X$,*

$$d_J(Tx, Ty) \leqslant \nu \max\left\{d_J(x, y), \frac{d_J(y, Ty)[1 + d_J(x, Tx)]}{1 + d_J(x, y)}, \frac{d_J(x, Tx)[1 + d_J(y, Ty)]}{1 + d_J(Tx, Ty)}\right\};$$

(ii) *for a mapping* $\varphi : [0,\infty)^2 \to [0,\infty)$ *with* $\varphi(0,0) = 0$ *and which is continuous at* $(0,0)$*, we have*

$$d_J(Tx, Ty) \leqslant \varphi(d_J(x, Tx), d_J(y, Ty)) \text{ for all } x, y \in X;$$

(iii) *the mapping* T *is continuous in* X.

Then T *has a fixed point in* X *which is unique.*

Proof. Let $x_0 \in X$ be arbitrarily chosen and we construct the Picard iterating sequence $\{x_n\}_{n\geqslant 1}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then we see that

$$\begin{aligned} d_J(x_n, x_{n+1}) &= d_J(Tx_{n-1}, Tx_n) \\ &\leqslant \nu \max\{d_J(x_{n-1}, x_n), \frac{d_J(x_n, x_{n+1})[1 + d_J(x_{n-1}, x_n)]}{1 + d_J(x_{n-1}, x_n)}, \frac{d_J(x_{n-1}, x_n)[1 + d_J(x_n, Tx_n)]}{1 + d_J(Tx_{n-1}, Tx_n)}\} \\ &= \nu \max\{d_J(x_{n-1}, x_n), \frac{d_J(x_n, x_{n+1})[1 + d_J(x_{n-1}, x_n)]}{1 + d_J(x_{n-1}, x_n)}, \frac{d_J(x_{n-1}, x_n)[1 + d_J(x_n, x_{n+1})]}{1 + d_J(x_n, x_{n+1})}\} \\ &= \nu \max\{d_J(x_{n-1}, x_n), d_J(x_n, x_{n+1})\} \text{ for all } n \geqslant 1. \end{aligned}$$

Therefore we have $d_J(x_n, x_{n+1}) \leqslant \nu d_J(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} d_J(x_{n-1}, x_n) = 0$. Now using condition (ii) we get

$$d_J(x_n, x_m) = d_J(Tx_{n-1}, Tx_{m-1}) \leqslant \varphi(d_J(x_{n-1}, x_n), d_J(x_{m-1}, x_m)) \to 0 \text{ as } n, m \to \infty.$$

Thus $\{x_n\}_{n\geqslant 1}$ is Cauchy in X and therefore by the completeness of X there exists some $\nu \in X$ such that $x_n \to \nu$ as $n \to \infty$. Now since T is continuous, it follows that $x_{n+1} = Tx_n \to T\nu$ as $n \to \infty$. Hence $T\nu = \nu$ and ν is a fixed point of T. The uniqueness of fixed point can be achieved in a similar way as Theorem 8.1. □

9. Best S^{JS}-proximity point

In fixed point theory the existence of solution of nonlinear equations of the form $Tx = x$, in which T is a self mapping defined on a subset of a topological space is studied. But if we consider T as a non-self mapping then it may so happen that the equation $Tx = x$ has no solution, in this case one can find a point x which is very much nearer to the concept of fixed point of T. In this case actually we seek for an approximate solution $x \in A$ for the equation $d(x, Tx) = d(A, B)$, where A, B are nonempty subsets of a set X with a distance function d and $T : A \to B$ is a given mapping. Such approximate solutions are called proximal points for T. For a self mapping over a metric space a proximal point or a best proximity point is a fixed point. After Fan [24], seminal paper on classical best approximation theorem several researchers had improved, generlaized and extended the Fan's result in many directions. A best proximity theorem for contractive mappings has been given by Basha [2]. Basha [3], also studied necessary and sufficient condition for deriving the existence of a best proximity point for a non-self proximal contraction maps of the first and second kind. Subsequently several researchers have proved various proximity point theorems in different topological spaces (see [5, 17, 18]). Here we study the notion of proximal S^{JS}-quasi-contraction mappings of first and second kind and also proximal $S^{JS} - \mathcal{Z}$-contraction mappings of first and second kind on a partially ordered S^{JS}-metric space to prove some proximity point theorems on it (see [8]). An application to variational inequality problem is also shown generalizing several results from the existing literature.

Let (X, J) be an S^{JS}-metric space such that $(X, \sqsubseteq)$ be a partially ordered set and A, B be two nonempty subsets of X. Denote $d_J(x, y) = J(x, x, y)$ for all $x, y \in X$. Define

$$\begin{aligned} d_J(A, B) &= \inf\{d_J(x, y) : x \in A \text{ and } y \in B\}, \\ A_0 &= \{x \in A : d_J(x, y) = d_J(A, B) \text{ for some } y \in B\}, \\ B_0 &= \{y \in B : d_J(x, y) = d_J(A, B) \text{ for some } x \in A\}. \end{aligned}$$

Definition 9.1. A point $u \in A$ is called a best S^{JS}-proximity point of the mapping $T : A \to B$ if $d_J(u, Tu) = d_J(A, B)$.

Definition 9.2. A mapping $T : A \to B$ is called proximal S^{JS}-quasi-contraction of the first kind if there exists $k \in [0, 1)$ such that for all $x, y, u, v \in A$, $x \sqsubseteq y$ or $\sqsubseteq x$, $d_J(u, Tx) = d_J(A, B)$ and $d_J(v, Ty) = d_J(A, B)$, imply $d_J(u, v) \leqslant k \max\{d_J(x, y), d_J(x, u), d_J(y, v), d_J(x, v), d_J(y, u)\}$.

Definition 9.3. A mapping $T : A \to B$ is called proximal S^{JS}-quasi-contraction of the second kind if there exists $k \in [0, 1)$ such that for all $x, y, u, v \in A$, $x \sqsubseteq y$ or $y \sqsubseteq x$, $d_J(u, Tx) = d_J(A, B)$ and $d_J(v, Ty) = d_J(A, B)$, implies $d_J(Tu, Tv) \leqslant k \max\{d_J(Tx, Ty), d_J(Tx, Tu), d_J(Ty, Tv), d_J(Tx, Tv), d_J(Ty, Tu)\}$.

Definition 9.4. A mapping $T : A \to B$ is called proximal $S^{JS} - \mathcal{Z}$-contraction of the first kind if for all $x, y, u, v \in A$,

$$d_J(u, Tx) = d_J(A, B), \quad d_J(v, Ty) = d_J(A, B), \quad \text{and} \quad d_J(u, v) = \infty \quad \Rightarrow \quad d_J(x, y) = \infty,$$

and there exists $\zeta \in \mathcal{Z}$ such that $x \sqsubseteq y$ or $y \sqsubseteq x$, $d_J(x, y) < \infty$, $d_J(u, Tx) = d_J(A, B)$, and $d_J(v, Ty) = d_J(A, B) \Rightarrow \zeta(d_J(u, v), d_J(x, y)) \geqslant 0$.

Definition 9.5. A mapping $T : A \to B$ is called proximal $S^{JS} - \mathcal{Z}$-contraction of the second kind if for all $x, y, u, v \in A$,

$$d_J(u, Tx) = d_J(A, B), \quad d_J(v, Ty) = d_J(A, B), \quad \text{and} \quad d_J(Tu, Tv) = \infty \quad \Rightarrow \quad d_J(Tx, Ty) = \infty,$$

and there exists $\zeta \in \mathcal{Z}$ such that $x \sqsubseteq y$ or $y \sqsubseteq x$, $d_J(Tx, Ty) < \infty$, $d_J(u, Tx) = d_J(A, B)$, and $d_J(v, Ty) = d_J(A, B) \Rightarrow \zeta(d_J(Tu, Tv), d_J(Tx, Ty)) \geqslant 0$.

Definition 9.6. A mapping $T : A \to B$ is called proximal S^{JS}-order-preserving if, for all $x, y, u, v \in A$, it satisfies the following condition:

$$x \sqsubseteq y, \quad d_J(u, Tx) = d_J(A, B), \quad \text{and} \quad d_J(v, Ty) = d_J(A, B) \quad \Rightarrow \quad u \sqsubseteq v.$$

Theorem 9.7. *Let $(X, \sqsubseteq)$ be a nonempty ordered set such that (X, J) be an S^{JS}-metric space. Let A, B be two nonempty closed subsets of X such that A_0 is nonempty. Also let $T : A \to B$ satisfies:*

(i) T *is proximal S^{JS}-order-preserving such that $T(A_0) \subset B_0$;*
(ii) *there exists elements $x_0 \in A_0$ and $x_1 \in A$ such that either $x_0 \sqsubseteq x_1$ or $x_1 \sqsubseteq x_0$ and $d_J(x_1, Tx_0) = d_J(A, B)$.*

Then there exists a sequence $\{x_n\}$ in A_0 such that either $x_n \sqsubseteq x_{n+1}$ and $d_J(x_{n+1}, Tx_n) = d_J(A, B)$ for all non-negative integers n or $x_{n+1} \sqsubseteq x_n$ and $d_J(x_{n+1}, Tx_n) = d_J(A, B)$ for all non-negative integers n.

Proof. By the given condition (ii) there exists two elements $x_0 \in A_0$ and $x_1 \in A$ such that $d_J(x_1, Tx_0) = d_J(A, B)$. Let us take $x_0 \sqsubseteq x_1$. Since $Tx_0 \in T(A_0) \subset B_0 \subset B$, then $x_1 \in A_0$. Now $Tx_1 \in T(A_0) \subset B_0$ so there exists $x_2 \in A$ such that $d_J(x_2, Tx_1) = d_J(A, B)$. Then clearly $x_2 \in A_0$. Since T is proximal S^{JS}-order-preserving it follows that $x_1 \sqsubseteq x_2$. Proceeding in a similar way we can construct a sequence $\{x_n\}$ in A_0 such that $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ with $d_J(x_{n+1}, Tx_n) = d_J(A, B)$ for all $n \geqslant 0$.

Proof is similar for the case $x_1 \sqsubseteq x_0$. □

We call the above sequence $\{x_n\}$ as proximally ordered sequence corresponding to the mapping T and $(x_0, x_1) \in A_0^2$.

Theorem 9.8. *Let $(X, \sqsubseteq)$ a non-empty partially ordered set and let (X, J) be a complete symmetric S^{JS}-metric space. Let A and B be nonempty closed subsets of X such that A_0 be nonempty. Also let $T : A \to B$ satisfies:*

(a) T *is a proximal S^{JS}-quasi-contraction of the first kind;*
(b) T *is proximal S^{JS}-order-preserving such that $T(A_0) \subset B_0$;*

(c) *for any proximally ordered convergent sequence* $\{x_n\}$ *which converges to* z*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $Tx_{n_k} \to Tz$ *as* $k \to \infty$*;*

(d) *there exists elements* $x_0 \in A_0$ *and* $x_1 \in A$ *such that either* $x_0 \sqsubseteq x_1$ *or* $x_1 \sqsubseteq x_0$ *and* $d_J(x_1, Tx_0) = d_J(A, B)$*;*

(e) $d_J(x, y)$ *is continuous in both the variables;*

(f) *for the proximally ordered sequence corresponding to* T *and* $(x_0, x_1) \in A_0 \times A$ *we have* $\delta(J, T, x_0, x_1) = \sup\{d_J(x_i, x_j) : i, j \geqslant 0\} < \infty$.

Then there exists $z \in A_0$ *such that* $d_J(z, Tz) = d_J(A, B)$.

Proof. Let us denote $\delta(J, T^n, x_0, x_1) = \sup\{d_J(x_{n-1+i}, x_{n-1+j}) : i, j \geqslant 0\}$ for all $n \geqslant 1$. Then clearly $\delta(J, T^n, x_0, x_1) \leqslant \delta(J, T, x_0, x_1) < \infty$ for all $n \in \mathbb{N}$. Now as $\{x_m\}$ is a proximally ordered sequence and $d_J(x_{n+i}, Tx_{n-1+i}) = d_J(A, B)$, $d_J(x_{n+j}, Tx_{n-1+j}) = d_J(A, B)$ for all $n \geqslant 1$ and for all $i, j \geqslant 0$, thus

$$\begin{aligned} &d_J(x_{n+i}, x_{n+j}) \\ &\quad \leqslant k \max\{d_J(x_{n-1+i}, x_{n-1+j}), d_J(x_{n-1+i}, x_{n+i}), d_J(x_{n-1+j}, x_{n+j}), d_J(x_{n-1+i}, x_{n+j}), d_J(x_{n-1+j}, x_{n+i})\} \\ &\quad \leqslant k\delta(J, T^n, x_0, x_1) \end{aligned}$$

for any $n \geqslant 1$ and for any $i, j \geqslant 0$ implies that

$$\delta(J, T^{n+1}, x_0, x_1) \leqslant k\delta(J, T^n, x_0, x_1) \leqslant \cdots \leqslant k^n \delta(J, T, x_0, x_1)$$

for all $n \geqslant 1$. Therefore $d_J(x_n, x_{n+p}) \leqslant \delta(J, T^{n+1}, x_0, x_1) \to 0$ as $n \to \infty$ and for any $p \geqslant 1$. Thus $\{x_n\}$ is Cauchy in X, since X is complete it follows that there exists $z \in X$ such that $\{x_n\} \in S(J, X, z)$. By the given condition (c) we get a subsequence $\{x_{n_k}\}$ such that $Tx_{n_k} \to Tz$ as $k \to \infty$. Since A is closed, then $z \in A$ and $Tz \in B$. From the continuity of d_J we have $d_J(z, Tz) = \lim_{k\to\infty} d_J(x_{n_k+1}, Tx_{n_k}) = d_J(A, B)$. Clearly we have $z \in A_0$. □

Theorem 9.9. *Let* $(X, \sqsubseteq)$ *a non-empty partially ordered set and let* (X, J) *be a complete symmetric* S^{JS}*-metric space. Let* A *and* B *be nonempty closed subsets of* X *such that* A_0 *be nonempty. Also let* $T : A \to B$ *satisfies:*

(a) T *is a proximal* S^{JS}*-quasi-contraction of the second kind;*

(b) T *is proximal* S^{JS}*-order-preserving such that* $T(A_0) \subset B_0$*;*

(c) *for any proximally ordered sequence* $\{x_n\}$ *if* $\{Tx_n\}$ *converges to some* $y \in B$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $x_{n_k} \to z$ *as* $k \to \infty$ *with* $y = Tz$*;*

(d) *there exists elements* $x_0 \in A_0$ *and* $x_1 \in A$ *such that either* $x_0 \sqsubseteq x_1$ *or* $x_1 \sqsubseteq x_0$ *and* $d_J(x_1, Tx_0) = d_J(A, B)$*;*

(e) $d_J(x, y)$ *is continuous in both the variables;*

(f) *for the proximally ordered sequence corresponding to* T *and* $(x_0, x_1) \in A_0 \times A$ *we have* $\delta_T(J, T, x_0, x_1) = \sup\{d_J(Tx_i, Tx_j) : i, j \geqslant 0\} < \infty$.

Then there exists a best S^{JS}*-proximity point of* T.

Proof. Let us consider $\delta_T(J, T^n, x_0, x_1) = \sup\{d_J(Tx_{n-1+i}, Tx_{n-1+j}) : i, j \geqslant 0\}$ for all $n \geqslant 1$. Obviously $\delta_T(J, T^n, x_0, x_1) \leqslant \delta_T(J, T, x_0, x_1) < \infty$ for all $n \in \mathbb{N}$. Now as $\{x_m\}$ is a proximally ordered sequence and $d_J(x_{n+i}, Tx_{n-1+i}) = d_J(A, B)$, $d_J(x_{n+j}, Tx_{n-1+j}) = d_J(A, B)$ for all $n \geqslant 1$ and for all $i, j \geqslant 0$, thus by condition (a) we have

$$\begin{aligned} d_J(Tx_{n+i}, Tx_{n+j}) \leqslant k \max\{&d_J(Tx_{n-1+i}, Tx_{n-1+j}), d_J(Tx_{n-1+i}, Tx_{n+i}), d_J(Tx_{n-1+j}, Tx_{n+j}), \\ &d_J(Tx_{n-1+i}, Tx_{n+j}), d_J(Tx_{n-1+j}, Tx_{n+i})\} \leqslant k\delta_T(J, T^n, x_0, x_1) \end{aligned}$$

for any $n \geqslant 1$ and for any $i, j \geqslant 0$ implies that

$$\delta_T(J, T^{n+1}, x_0, x_1) \leqslant k\delta_T(J, T^n, x_0, x_1) \leqslant \cdots \leqslant k^n \delta_T(J, T, x_0, x_1)$$

for all $n \geqslant 1$. Therefore $d_J(Tx_n, Tx_{n+p}) \leqslant \delta_T(J, T^{n+1}, x_0, x_1) \to 0$ as $n \to \infty$ and for any $p \geqslant 1$. Thus $\{Tx_n\}$ is Cauchy in X, since X is complete it follows that there exists some $y \in X$ such that $\{Tx_n\}$ converges to y. By the given condition (c) we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z$ as $k \to \infty$ for some $z \in X$ with $y = Tz$. Since A is closed, then $z \in A$ and $Tz \in B$. From the continuity of d_J we have $d_J(z, Tz) = \lim_{k\to\infty} d_J(x_{n_k}, Tx_{n_k-1}) = d_J(A, B)$. Thus z is a best S^{JS}-proximity point of T. Clearly we have $z \in A_0$. □

Theorem 9.10. *Let $(X, \sqsubseteq)$ a non-empty partially ordered set and let (X, J) be a complete symmetric S^{JS}-metric space. Let A and B be nonempty closed subsets of X such that A_0 be nonempty. Also let $T : A \to B$ satisfies:*

(a) *T is a proximal $S^{JS} - \mathcal{Z}$-contraction of the first kind;*
(b) *T is proximal S^{JS}-order-preserving such that $T(A_0) \subset B_0$;*
(c) *for any proximally ordered convergent sequence $\{x_n\}$ which converges to z, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to Tz$ as $k \to \infty$;*
(d) *there exists elements $x_0 \in A_0$ and $x_1 \in A$ such that either $x_0 \sqsubseteq x_1$ or $x_1 \sqsubseteq x_0$ and $d_J(x_1, Tx_0) = d_J(A, B)$;*
(e) *$d_J(x, y)$ is continuous in both the variables;*
(f) *for the proximally ordered sequence corresponding to T and $(x_0, x_1) \in A_0 \times A$ we have $\delta(J, T, x_0, x_1) = \sup\{d_J(x_i, x_j) : i, j \geqslant 0\} < \infty$ and $x_p \neq x_q$ for all $p, q \geqslant 0$ $(p \neq q)$.*

Then there exists $z \in A_0$ such that $d_J(z, Tz) = d_J(A, B)$.

Proof. Here we see that for any $n \geqslant 0$ and for all $i, j(i \neq j) \in \mathbb{N} \cup \{0\}$, $0 < d_J(x_{n+i}, x_{n+j}) \leqslant \delta(J, T, x_0, x_1) < \infty$. Because if for some $n \geqslant 0$ and $i, j(i \neq j) \in \mathbb{N} \cup \{0\}$, $d_J(x_{n+i}, x_{n+j}) = 0$, then we get $x_{n+i} = x_{n+j}$, a contradiction. Since $\{x_n\}$ is a proximally ordered sequence and $d_J(x_{n+i}, Tx_{n-1+i}) = d_J(A, B)$, $d_J(x_{n+j}, Tx_{n-1+j}) = d_J(A, B)$ for all $n \geqslant 1$ and for all $i, j \geqslant 0$ with $i \neq j$, thus by condition (a) we have

$$0 \leqslant \zeta(d_J(x_{n+i}, x_{n+j}), d_J(x_{n-1+i}, x_{n-1+j})) < d_J(x_{n-1+i}, x_{n-1+j}) - d_J(x_{n+i}, x_{n+j})$$

implies $d_J(x_{n+i}, x_{n+j}) < d_J(x_{n-1+i}, x_{n-1+j})$ for all $n \in \mathbb{N}$ and for all $i, j(i \neq j) \geqslant 0$. So $\{d_J(x_{n+i}, x_{n+j})\}_{n\in\mathbb{N}}$ is a decreasing bounded sequence for any $i, j(i \neq j) \geqslant 0$. Thus there exists $\lambda \geqslant 0$ such that $\lim_{n\to\infty} d_J(x_{n+i}, x_{n+j}) = \lambda$ for all $i, j(i \neq j) \geqslant 0$. If $\lambda > 0$, then for the sequences $t_n = d_J(x_{n+3}, x_{n+2})$ and $s_n = d_J(x_{n+2}, x_{n+1})$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \lambda$ and thus

$$0 \leqslant \limsup_{n\to\infty} \zeta(d_J(x_{n+3}, x_{n+2}), d_J(x_{n+2}, x_{n+1})) = \limsup_{n\to\infty} \zeta(t_n, s_n) < 0,$$

a contradiction. Therefore $\limsup_{n\to\infty} d_J(x_{n+i}, x_{n+j}) = 0$ for all $i, j \geqslant 0$ with $i \neq j$. Hence $\{x_n\}$ is Cauchy in X. Since X is complete it follows that there exists some $z \in X$ such that $\{x_n\} \in S(J, X, z)$. By the given condition (c) we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to Tz$ as $k \to \infty$. Since A is closed, then $z \in A$ and $Tz \in B$. From the continuity of d_J we have $d_J(z, Tz) = \lim_{k\to\infty} d_J(x_{n_k+1}, Tx_{n_k}) = d_J(A, B)$. Clearly we have $z \in A_0$. □

Theorem 9.11. *Let $(X, \sqsubseteq)$ a nonempty partially ordered set and let (X, J) be a complete symmetric S^{JS}-metric space. Let A and B be nonempty closed subsets of X such that A_0 be nonempty. Also let $T : A \to B$ satisfies:*

(a) *T is a proximal $S^{JS} - \mathcal{Z}$-contraction of the second kind;*
(b) *T is proximal S^{JS}-order-preserving such that $T(A_0) \subset B_0$;*
(c) *for any proximally ordered sequence $\{x_n\}$ if $\{Tx_n\}$ converges to some $y \in B$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z$ as $k \to \infty$ with $y = Tz$;*
(d) *there exists elements $x_0 \in A_0$ and $x_1 \in A$ such that either $x_0 \sqsubseteq x_1$ or $x_1 \sqsubseteq x_0$ and $d_J(x_1, Tx_0) = d_J(A, B)$;*
(e) *$d_J(x, y)$ is continuous in both the variables;*
(f) *for the proximally ordered sequence corresponding to T and $(x_0, x_1) \in A_0 \times A$ we have $\delta_T(J, T, x_0, x_1) = \sup\{d_J(Tx_i, Tx_j) : i, j \geqslant 0\} < \infty$ and $Tx_p \neq Tx_q$ for all $p, q \geqslant 0$ $(p \neq q)$.*

Then there exists a best S^{JS}*-proximity point of* T.

Proof. For any $n \geqslant 0$ and for all $i, j (i \neq j) \in \mathbb{N} \cup \{0\}$ we have $0 < d_J(Tx_{n+i}, Tx_{n+j}) \leqslant \delta_T(J, T, x_0, x_1) < \infty$, otherwise for some $n \geqslant 0$ and $i, j (i \neq j) \in \mathbb{N} \cup \{0\}$, $d_J(Tx_{n+i}, Tx_{n+j}) = 0$ implies that $Tx_{n+i} = Tx_{n+j}$, a contradiction. Since $\{x_n\}$ is a proximally ordered sequence and $d_J(x_{n+i}, Tx_{n-1+i}) = d_J(A, B)$, $d_J(x_{n+j}, Tx_{n-1+j}) = d_J(A, B)$ for all $n \geqslant 1$ and for all $i, j (i \neq j) \geqslant 0$, thus by condition (a) we have

$$0 \leqslant \zeta(d_J(Tx_{n+i}, Tx_{n+j}), d_J(Tx_{n-1+i}, Tx_{n-1+j})) < d_J(Tx_{n-1+i}, Tx_{n-1+j}) - d_J(Tx_{n+i}, Tx_{n+j})$$

implies $d_J(Tx_{n+i}, Tx_{n+j}) < d_J(Tx_{n-1+i}, Tx_{n-1+j})$ for all $n \in \mathbb{N}$ and for all $i, j (i \neq j) \geqslant 0$. So $\{d_J(Tx_{n+i}, Tx_{n+j})\}_{n \in \mathbb{N}}$ is a decreasing bounded sequence for any $i, j \geqslant 0$ with $i \neq j$. Thus there exists $\eta \geqslant 0$ such that $\lim_{n\to\infty} d_J(Tx_{n+i}, Tx_{n+j}) = \eta$ for all $i, j (i \neq j) \geqslant 0$. If $\eta > 0$, then for the sequences $t_n = d_J(Tx_{n+3}, Tx_{n+2})$ and $s_n = d_J(Tx_{n+2}, Tx_{n+1})$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \eta$ and thus

$$0 \leqslant \limsup_{n\to\infty} \zeta(d_J(Tx_{n+3}, Tx_{n+2}), d_J(Tx_{n+2}, Tx_{n+1})) = \limsup_{n\to\infty} \zeta(t_n, s_n) < 0,$$

a contradiction. Therefore $\limsup_{n\to\infty} d_J(Tx_{n+i}, Tx_{n+j}) = 0$ for all $i, j (i \neq j) \geqslant 0$. Thus $\{Tx_n\}$ is Cauchy in X. As X is complete it follows that there exists some $y \in X$ such that $\{Tx_n\}$ converges to y. By the given condition (c) we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z$ as $k \to \infty$ for some $z \in X$ with $y = Tz$. Since A is closed, then $z \in A$ and $Tz \in B$. From the continuity of d_J we have $d_J(z, Tz) = \lim_{k\to\infty} d_J(x_{n_k}, Tx_{n_k-1}) = d_J(A, B)$. Thus z is a best S^{JS}-proximity point of T. Clearly we have $z \in A_0$. □

Now we state some corollaries relating to our main theorems.

Corollary 9.12 (proximal S^{JS}-Banach-contraction type of the first kind). *Let* $(X, \sqsubseteq)$ *a nonempty partially ordered set and let* (X, J) *be a complete symmetric* S^{JS}*-metric space. Let* A *and* B *be nonempty closed subsets of* X *such that* A_0 *be nonempty. Also let* $T : A \to B$ *satisfies:*

(a) $x \sqsubseteq y$, $d_J(u, Tx) = d_J(A, B)$, *and* $d_J(v, Ty) = d_J(A, B) \Rightarrow d_J(u, v) \leqslant k d_J(x, y)$ *for all* $x, y, u, v \in A$ *and for some* $k \in [0, 1)$;
(b) T *is proximal* S^{JS}*-order-preserving such that* $T(A_0) \subset B_0$;
(c) *for any proximally ordered convergent sequence* $\{x_n\}$ *which converges to* z*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $Tx_{n_k} \to Tz$ *as* $k \to \infty$;
(d) *there exists elements* $x_0 \in A_0$ *and* $x_1 \in A$ *such that either* $x_0 \sqsubseteq x_1$ *or* $x_1 \sqsubseteq x_0$ *and* $d_J(x_1, Tx_0) = d_J(A, B)$;
(e) $d_J(x, y)$ *is continuous in both the variables;*
(f) *for the proximally ordered sequence corresponding to* T *and* $(x_0, x_1) \in A_0 \times A$ *we have* $\delta(J, T, x_0, x_1) = \sup\{d_J(x_i, x_j) : i, j \geqslant 0\} < \infty$.

Then there exists $z \in A_0$ *such that* $d_J(z, Tz) = d_J(A, B)$.

Proof. Applying Theorem 9.8 we get our required result. □

Corollary 9.13 (proximal S^{JS}-Kannan-contraction type of the first kind). *Let* $(X, \sqsubseteq)$ *a nonempty partially ordered set and let* (X, J) *be a complete symmetric* S^{JS}*-metric space. Let* A *and* B *be nonempty closed subsets of* X *such that* A_0 *be nonempty. Also let* $T : A \to B$ *satisfies:*

(a) $x \sqsubseteq y$, $d_J(u, Tx) = d_J(A, B)$, *and* $d_J(v, Ty) = d_J(A, B) \Rightarrow d_J(u, v) \leqslant k[d_J(x, u) + d_J(y, v)]$ *for all* $x, y, u, v \in A$ *and for some* $k \in [0, \frac{1}{2})$;
(b) T *is proximal* S^{JS}*-order-preserving such that* $T(A_0) \subset B_0$;
(c) *for any proximally ordered convergent sequence* $\{x_n\}$ *which converges to* z*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $Tx_{n_k} \to Tz$ *as* $k \to \infty$;
(d) *there exists elements* $x_0 \in A_0$ *and* $x_1 \in A$ *such that either* $x_0 \sqsubseteq x_1$ *or* $x_1 \sqsubseteq x_0$ *and* $d_J(x_1, Tx_0) = d_J(A, B)$;
(e) $d_J(x, y)$ *is continuous in both the variables;*

(f) *for the proximally ordered sequence corresponding to* T *and* $(x_0, x_1) \in A_0 \times A$ *we have* $\delta(J, T, x_0, x_1) = \sup\{d_J(x_i, x_j) : i, j \geqslant 0\} < \infty$.

Then there exists $z \in A_0$ *such that* $d_J(z, Tz) = d_J(A, B)$.

Proof. We can obtain this Corollary by applying Theorem 9.8. □

Corollary 9.14 (proximal S^{JS}-Chatterjee-contraction type of the first kind)**.** *Let* $(X, \sqsubseteq)$ *a nonempty partially ordered set and let* (X, J) *be a complete symmetric* S^{JS}*-metric space. Let* A *and* B *be nonempty closed subsets of* X *such that* A_0 *be nonempty. Also let* $T : A \to B$ *satisfies:*

(a) $x \sqsubseteq y$, $d_J(u, Tx) = d_J(A, B)$, *and* $d_J(v, Ty) = d_J(A, B) \Rightarrow d_J(u, v) \leqslant k[d_J(x, v) + d_J(y, u)]$ *for all* $x, y, u, v \in A$ *and for some* $k \in [0, \frac{1}{2})$;
(b) T *is proximal* S^{JS}*-order-preserving such that* $T(A_0) \subset B_0$;
(c) *for any proximally ordered convergent sequence* $\{x_n\}$ *which converges to* z*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $Tx_{n_k} \to Tz$ *as* $k \to \infty$;
(d) *there exists elements* $x_0 \in A_0$ *and* $x_1 \in A$ *such that either* $x_0 \sqsubseteq x_1$ *or* $x_1 \sqsubseteq x_0$ *and* $d_J(x_1, Tx_0) = d_J(A, B)$;
(e) $d_J(x, y)$ *is continuous in both the variables;*
(f) *for the proximally ordered sequence corresponding to* T *and* $(x_0, x_1) \in A_0 \times A$ *we have* $\delta(J, T, x_0, x_1) = \sup\{d_J(x_i, x_j) : i, j \geqslant 0\} < \infty$.

Then there exists $z \in A_0$ *such that* $d_J(z, Tz) = d_J(A, B)$.

Proof. Using Theorem 9.8 we obtain this Corollary. □

Corollary 9.15. *Let* $(X, \sqsubseteq)$ *a nonempty partially ordered set and let* (X, J) *be a complete symmetric* S^{JS}*-metric space. Let* A *and* B *be nonempty closed subsets of* X *such that* A_0 *be nonempty. Also let* $T : A \to B$ *satisfies:*

(a) $x \sqsubseteq y$, $d_J(u, Tx) = d_J(A, B)$, *and* $d_J(v, Ty) = d_J(A, B) \Rightarrow d_J(u, v) \leqslant d_J(x, y) - \varphi(d_J(x, y))$ *for all* $x, y, u, v \in A$ *and for some lower semi-continuous mapping* $\varphi : [0, \infty) \to [0, \infty)$ *such that* $\varphi^{-1}(0) = 0$;
(b) T *is proximal* S^{JS}*-order-preserving such that* $T(A_0) \subset B_0$;
(c) *for any proximally ordered convergent sequence* $\{x_n\}$ *which converges to* z*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $Tx_{n_k} \to Tz$ *as* $k \to \infty$;
(d) *there exists elements* $x_0 \in A_0$ *and* $x_1 \in A$ *such that either* $x_0 \sqsubseteq x_1$ *or* $x_1 \sqsubseteq x_0$ *and* $d_J(x_1, Tx_0) = d_J(A, B)$;
(e) $d_J(x, y)$ *is continuous in both the variables;*
(f) *for the proximally ordered sequence corresponding to* T *and* $(x_0, x_1) \in A_0 \times A$ *we have* $\delta(J, T, x_0, x_1) = \sup\{d_J(x_i, x_j) : i, j \geqslant 0\} < \infty$ *and* $x_p \neq x_q$ *for all* $p, q (p \neq q) \geqslant 0$.

Then there exists $z \in A_0$ *such that* $d_J(z, Tz) = d_J(A, B)$.

Proof. We can get our required result by considering $\zeta(t, s) = s - \varphi(s) - t$ for all $s, t \geqslant 0$ in Theorem 9.10. □

Corollary 9.16. *Let* $(X, \sqsubseteq)$ *a nonempty partially ordered set and let* (X, J) *be a complete symmetric* S^{JS}*-metric space. Let* A *and* B *be nonempty closed subsets of* X *such that* A_0 *be nonempty. Also let* $T : A \to B$ *satisfies:*

(a) $x \sqsubseteq y$, $d_J(u, Tx) = d_J(A, B)$, *and* $d_J(v, Ty) = d_J(A, B) \Rightarrow d_J(u, v) \leqslant \varphi(d_J(x, y)) d_J(x, y)$ *for all* $x, y, u, v \in A$ *and for some mapping* $\varphi : [0, \infty) \to [0, 1)$ *such that* $\limsup_{t \to r^+} \varphi(t) < 1$ *for all* $r > 0$;
(b) T *is proximal* S^{JS}*-order-preserving such that* $T(A_0) \subset B_0$;
(c) *for any proximally ordered convergent sequence* $\{x_n\}$ *which converges to* z*, there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $Tx_{n_k} \to Tz$ *as* $k \to \infty$;
(d) *there exists elements* $x_0 \in A_0$ *and* $x_1 \in A$ *such that either* $x_0 \sqsubseteq x_1$ *or* $x_1 \sqsubseteq x_0$ *and* $d_J(x_1, Tx_0) = d_J(A, B)$;
(e) $d_J(x, y)$ *is continuous in both the variables;*
(f) *for the proximally ordered sequence corresponding to* T *and* $(x_0, x_1) \in A_0 \times A$ *we have* $\delta(J, T, x_0, x_1) = \sup\{d_J(x_i, x_j) : i, j \geqslant 0\} < \infty$ *and* $x_p \neq x_q$ *for all* $p, q \geqslant 0$ *and* $p \neq q$.

Then there exists $z \in A_0$ *such that* $d_J(z, Tz) = d_J(A, B)$.

Proof. If we set $\zeta(t, s) = s\varphi(s) - t$ for all $s, t \geqslant 0$ in Theorem 9.10 we can obtain this Corollary. □

Next we give some examples which support our Theorems.

Example 9.17. Let $X = \mathbb{R}^2$ and $J : X^3 \to [0, \infty)$ be defined by $J(x, y, z) = |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$ for all $x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2)$ in X. Then clearly (X, J) is an S^{JS}-metric space. Also let $\sqsubseteq$ be defined on X by $(a, b) \sqsubseteq (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Now let us consider the subsets $A = \{(0, x) : x \in [0, 1]\}$ and $B = \{(1, y) : y \in [0, 1]\}$ of X and $T : A \to B$ be defined by

$$T(0, a) = \begin{cases} (1, \frac{a}{3}), & \text{if } 0 \leqslant a < \frac{1}{2}, \\ (1, \frac{a}{5}), & \text{if } \frac{1}{2} < a \leqslant 1. \end{cases}$$

If $x = (0, \alpha), y = (0, \beta), u = (0, \gamma)$, and $v = (0, \delta)$ are taken from A, then always either $x \sqsubseteq y$ or $y \sqsubseteq x$ clearly.

Case 1: If $0 \leqslant \alpha, \beta < \frac{1}{2}$, then we get,

$$d_J(u, Tx) = d_J(A, B) \quad \Rightarrow \quad d_J((0, \gamma), (1, \frac{\alpha}{3})) = 1 \quad \Rightarrow \quad 1 + |\gamma - \frac{\alpha}{3}| = 1 \quad \Rightarrow \quad \gamma = \frac{\alpha}{3}.$$

Similarly we can get $\delta = \frac{\beta}{3}$ from $d_J(v, Ty) = d_J(A, B)$. Therefore we have $d_J(u, v) = d_J((0, \gamma), (0, \delta)) = |\gamma - \delta| = \frac{1}{3}|\alpha - \beta| = \frac{1}{3} d_J(x, y)$.

Case 2: If $0 \leqslant \alpha < \frac{1}{2}$ and $\frac{1}{2} \leqslant \beta \leqslant 1$, then we get $\gamma = \frac{\alpha}{3}$ from the relation $d_J(u, Tx) = d_J(A, B)$ and

$$d_J(v, Ty) = d_J(A, B) \quad \Rightarrow \quad d_J((0, \delta), (1, \frac{\beta}{5})) = 1 \quad \Rightarrow \quad 1 + |\delta - \frac{\beta}{5}| = 1 \quad \Rightarrow \quad \delta = \frac{\beta}{5}.$$

Thus we get

$$d_J(u, v) = d_J((0, \gamma), (0, \delta)) = |\gamma - \delta| = |\frac{\alpha}{3} - \frac{\beta}{5}| \leqslant \frac{3}{4} \max\{d_J(x, y), d_J(u, x), d_J(v, y), d_J(u, y), d_J(v, x)\}.$$

Case 3: If $\frac{1}{2} \leqslant \alpha, \beta \leqslant 1$, then we get,

$$d_J(u, Tx) = d_J(A, B) \quad \Rightarrow \quad d_J((0, \gamma), (1, \frac{\alpha}{5})) = 1 \quad \Rightarrow \quad 1 + |\gamma - \frac{\alpha}{5}| = 1 \quad \Rightarrow \quad \gamma = \frac{\alpha}{5}.$$

Similarly we can get $\delta = \frac{\beta}{5}$ from $d_J(v, Ty) = d_J(A, B)$. Therefore we have $d_J(u, v) = d_J((0, \gamma), (0, \delta)) = |\gamma - \delta| = \frac{1}{5}|\alpha - \beta| = \frac{1}{5} d_J(x, y)$. Therefore $x \sqsubseteq y$, $d_J(u, Tx) = d_J(A, B)$, and $d_J(v, Ty) = d_J(A, B) \Rightarrow d_J(u, v) \leqslant \frac{3}{4} \max\{d_J(x, y), d_J(x, u), d_J(y, v), d_J(x, v), d_J(y, u)\}$ for all $x, y, u, v \in A$ and hence T is a proximal S^{JS}-quasi-contraction of the first kind.

Any proximal S^{JS}-quasi-contraction of the second kind may not be proximal S^{JS}-quasi-contraction of the first kind. For this we cite an example which is given below.

Example 9.18. Let us consider the S^{JS}-metric space (X, J), where $X = \mathbb{R}^2$ and $J : X^3 \to [0, \infty)$ be defined by $J(x, y, z) = |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$ for all $x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2)$ in X. Also let $\sqsubseteq$ be defined on X by $(a, b) \sqsubseteq (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Now let us consider the subsets $A = \{(-1, x) : x \in [0, 1]\}$ and $B = \{(1, y) : y \in [0, 1]\}$ of X and $T : A \to B$ be defined by space. Also let $\sqsubseteq$ be defined on X by $(a, b) \sqsubseteq (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Now let us consider the subsets $A = \{(0, x) : x \in [0, 1]\}$ and $B = \{(1, y) : y \in [0, 1]\}$ of X and $T : A \to B$ be defined by

$$T(-1, x) = \begin{cases} (1, 0), & \text{if } x \in Q, \\ (1, 1), & \text{if } x \notin Q. \end{cases}$$

Let us take $u = (-1, \alpha), v = (-1, \beta), x = (-1, \gamma)$, and $y = (-1, \delta)$, then always either $x \sqsubseteq y$ or $y \sqsubseteq x$ clearly.

Case 1: If $\gamma, \delta \in \mathbb{Q}$, then

$$d_J(u, Tx) = d_J(A, B) \;\Rightarrow\; d_J((-1, \alpha), (1, 0)) = 2 \;\Rightarrow\; 2 + |\alpha| = 2 \;\Rightarrow\; \alpha = 0.$$

Similarly we can show that $d_J(v, Ty) = d_J(A, B)$ implies $\beta = 0$. Thus $d_J(Tu, Tv) = d_J((1, 0), (1, 0)) = 0$.

Case 2: Let $\gamma, \delta \in \mathbb{R}\backslash\mathbb{Q}$, then

$$d_J(u, Tx) = d_J(A, B) \;\Rightarrow\; d_J((-1, \alpha), (1, 1)) = 2 \;\Rightarrow\; 2 + |\alpha - 1| = 2 \;\Rightarrow\; \alpha = 1.$$

Similarly we can show that $d_J(v, Ty) = d_J(A, B)$ implies $\beta = 1$. Thus $d_J(Tu, Tv) = d_J((1, 0), (1, 0)) = 0$.

Case 3: If $\gamma \in \mathbb{Q}$ and $\delta \in \mathbb{R}\backslash\mathbb{Q}$, then by a routine verification we get $\alpha = 0$ and $\beta = 1$. Therefore $d_J(Tu, Tv) = d_J((1, 0), (1, 0)) = 0$. Hence T is a proximal S^{JS}-quasi-contraction of the second kind. But in Case 3 if we take $x = (-1, 1)$ and $y_\epsilon = (-1, 1 + \epsilon)$, where $\epsilon \in (\mathbb{R}\backslash\mathbb{Q})^+$ with $\epsilon \to 0$, then $d_J(u, v_\epsilon) = d_J((-1, 0), (-1, 1)) = 1$, $d_J(x, y_\epsilon) = \epsilon$, $d_J(x, u) = 1$, $d_J(y_\epsilon, v_\epsilon) = \epsilon$, $d_J(x, v_\epsilon) = 0$, and $d_J(y_\epsilon, u) = 1 + \epsilon$ and therefore $\max\{d_J(x, y_\epsilon), d_J(u, x), d_J(v_\epsilon, y_\epsilon), d_J(u, y_\epsilon), d_J(v_\epsilon, x)\} = 1 + \epsilon$ for all $\to 0$. So we cannot find any $k \in [0, 1)$ such that $d_J(u, v_\epsilon) \leqslant k \max\{d_J(x, y_\epsilon), d_J(u, x), d_J(v_\epsilon, y_\epsilon), d_J(u, y_\epsilon), d_J(v_\epsilon, x)\}$ for all $\epsilon \in (\mathbb{R}\backslash\mathbb{Q})^+$ with $\epsilon \to 0$. Therefore T is not a proximal S^{JS}-quasi-contraction of the first kind.

Example 9.19. Consider the S^{JS}-metric space (X, J) similar to Example 9.18. Take $A = \{(0, x) : x \in [0, 1]\}$ and $B = \{(1, y) : y \in [0, 1]\}$ and define $T : A \to B$ by

$$T(0, a) = \begin{cases} (1, \frac{a}{5}), & \text{if} \quad 0 \leqslant a < \frac{1}{2}, \\ (1, \frac{a}{4}), & \text{if} \;\; \frac{1}{2} \leqslant a < 1. \end{cases}$$

Define the order relation $\sqsubseteq$ on X by $(a, b) \sqsubseteq (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Then $x \sqsubseteq y$, $d_J(u, Tx) = d_J(A, B)$, and $d_J(v, Ty) = d_J(A, B) \;\Rightarrow\; d_J(u, v) \leqslant \frac{7}{12} \max\{d_J(x, y), d_J(x, u), d_J(y, v), d_J(x, v), d_J(y, u)\}$ for all $x, y, u, v \in A$ and we see that T is a proximal S^{JS}-quasi-contraction of the first kind. Also if we take $x_0 = (0, \frac{1}{2})$ and $x_1 = (0, \frac{1}{8})$, then $x_1 \sqsubseteq x_0$ and $d_J(x_1, Tx_0) = 1 = d_J(A, B)$. Here $\{x_m\}$ is defined by $x_{i+1} = (0, \frac{1}{8.5^i})$ for all $i \geqslant 0$ and thus $\delta(J, T, x_0, x_1) \leqslant \frac{1}{2} < \infty$. Moreover, another conditions of Theorem 9.8 are also satisfied by T. Hence applying Theorem 9.8 we see that $(0, 0)$ is the unique best proximity point of T.

Example 9.20. Let us consider the S^{JS}-metric space (X, J) defined as above (see Example 9.18). Also let $A = \{(x, 0) : x \in [0, 1]\}$, $B = \{(y, 1) : y \in [0, 1]\}$, and $T : A \to B$ be defined by

$$T(a, 0) = (\frac{a^2}{3(1 + a)}, 1)$$

for all $a \in [0, 1]$. Let the order relation $\sqsubseteq$ be defined on X by $(a, b) \sqsubseteq (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. In addition let us take the simulation function given in Example 5.11. Let us take $u = (\alpha, 0), v = (\beta, 0), x = (\gamma, 0)$, and $y = (\delta, 0)$, then always either $x \sqsubseteq y$ or $y \sqsubseteq x$ clearly. Now let us consider the following cases.

Case 1: If $0 < \gamma, \delta < 1$, then

$$d_J(u, Tx) = d_J(A, B) \;\Rightarrow\; d_J((\alpha, 0), (\frac{\gamma^2}{3(1 + \gamma)}, 1)) = 1 \;\Rightarrow\; 1 + |\alpha - \frac{\gamma^2}{3(1 + \gamma)}| = 1 \;\Rightarrow\; \alpha = \frac{\gamma^2}{3(1 + \gamma)}.$$

Similarly we get $\beta = \frac{\delta^2}{3(1+\delta)}$ using the relation $d_J(v, Ty) = d_J(A, B)$. Then

$$d_J(u, v) = d_J((\alpha, 0), (\beta, 0)) = |\alpha - \beta| = |\frac{\gamma^2}{3(1 + \gamma)} - \frac{\delta^2}{3(1 + \delta)}| = \frac{|\gamma - \delta|}{3} \frac{\gamma\delta + \gamma + \delta}{(1 + \gamma)(1 + \delta)}.$$

Therefore

$$\zeta(d_J(u, v), d_J(x, y)) = \frac{1}{2} d_J(x, y) - d_J(u, v) \geqslant 0.$$

Case 2: If $\gamma = 1, 0 < \delta \leqslant 1$, then using the relations $d_J(u, Tx) = d_J(A, B)$ and $d_J(v, Ty) = d_J(A, B)$ we get $\alpha = \frac{1}{6}$ and $\beta = \frac{\delta^2}{3(1+\delta)}$. Thus $d_J(u, v) = |\alpha - \beta| = \frac{1}{3}(\frac{1}{2} - \frac{\delta^2}{(1+\delta)})$ and hence we get

$$\zeta(d_J(u, v), d_J(x, y)) = \frac{1}{2} d_J(x, y) - d_J(u, v) \geqslant 0.$$

Case 3: If $\gamma = 0, 0 \leqslant \delta < 1$, then using the relations $d_J(u, Tx) = d_J(A, B)$ and $d_J(v, Ty) = d_J(A, B)$ we get $\alpha = 0$ and $\beta = \frac{\delta^2}{3(1+\delta)}$. Thus $d_J(u, v) = |\alpha - \beta| = \frac{\delta^2}{3(1+\delta)}$ and therefore we get

$$\zeta(d_J(u, v), d_J(x, y)) = \frac{1}{2} d_J(x, y) - d_J(u, v) \geqslant 0.$$

Case 4: If $\gamma = 0$ and $\delta = 1$, then we get $\alpha = 0$ and $\beta = \frac{1}{6}$ from the relations $d_J(u, Tx) = d_J(A, B)$ and $d_J(v, Ty) = d_J(A, B)$. So $d_J(u, v) = \frac{1}{6}$, $d_J(x, y) = 1$, and therefore we get

$$\zeta(d_J(u, v), d_J(x, y)) = d_J(x, y) - \frac{1}{3} - d_J(u, v) \geqslant 0.$$

Hence $d_J(u, Tx) = d_J(A, B)$, $d_J(v, Ty) = d_J(A, B)$, and $d_J(u, v) = \infty \Rightarrow d_J(x, y) = \infty$ and for the simulation function $\zeta \in \mathcal{Z}$ we have $x \sqsubseteq y$ or $y \sqsubseteq x$, $d_J(x, y) < \infty$, $d_J(u, Tx) = d_J(A, B)$, and $d_J(v, Ty) = d_J(A, B) \Rightarrow \zeta(d_J(u, v), d_J(x, y)) \geqslant 0$ for all $x, y, u, v \in A$. For $x_0 = (\frac{1}{3}, 0)$ and $x_1 = (\frac{1}{36}, 0)$ we see that $x_1 \sqsubseteq x_0$ with $d_J(x_1, Tx_0) = d_J(A, B)$. Also clearly $\delta(J, T, x_0, x_1) \leqslant \frac{1}{3} < \infty$ and another conditions of Theorem 9.10 are also satisfied. Here we see that $(0, 0)$ is the unique best proximity point of T.

Next we present an application of above results to a variational inequality problem.

Let $\mathbb{H}$ be a real Hilbert space, with the inner product $\langle ., . \rangle$ and induced norm $\|.\|$. Let $\mathbb{K}$ be a nonempty closed and convex subset of $\mathbb{H}$ and $\Upsilon : \mathbb{H} \to \mathbb{H}$ be a mapping. Consider the variational inequality problem:

$$\text{Find } \nu \in \mathbb{K} \text{ such that } \langle \Upsilon\nu, w - \nu \rangle \geqslant 0 \text{ for all } w \in \mathbb{K}.$$

Let us consider the metric projection operator $P_{\mathbb{K}} : \mathbb{H} \to \mathbb{K}$. Then for all $w \in \mathbb{K}$ the following inequality holds:

$$\|\nu - P_{\mathbb{K}}\nu\| \leqslant \|\nu - w\|.$$

Lemma 9.21 ([35]). *Let $y \in \mathbb{H}$. Then $w \in \mathbb{K}$ satisfies the inequality $\langle w - y, z - w \rangle \geqslant 0$ for all $z \in \mathbb{K}$, if and only if $P_{\mathbb{K}}(y) = w$.*

Lemma 9.22 ([35]). *Let $\Upsilon : \mathbb{H} \to \mathbb{H}$ be a mapping. Then $\nu \in \mathbb{K}$ is a solution of $\langle \Upsilon\nu, w - \nu \rangle \geqslant 0$ for all $w \in \mathbb{K}$, if and only if $P_{\mathbb{K}}(\nu - \mu\Upsilon\nu) = \nu$, with $\mu > 0$.*

Definition 9.23. Let $(X, \sqsubseteq)$ a nonempty partially ordered set and (X, J) be a complete symmetric S^{JS}-metric space. Also let $T : X \to X$ be a mapping. A sequence $\{x_n\}$ is said to be ordered sequence corresponding to the mapping T and some pair $(x_0, x_1) \in X^2$ if either $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ and $x_{n+1} = Tx_n$ for all $n \geqslant 0$ or $x_0 \sqsupseteq x_1 \sqsupseteq x_2 \sqsupseteq \cdots$ and $x_{n+1} = Tx_n$ for all $n \geqslant 0$.

If $\mathbb{H}$ is a real Hilbert space, with the inner product $\langle ., . \rangle$, then it is an S^{JS}-metric space with the S^{JS}-metric defined by $J_{\langle .,. \rangle}(x, y, z) = \|x - z\| + \|y - z\|$, where $\|.\|$ is the induced norm. Clearly all the topological properties of $(\mathbb{H}, \langle ., . \rangle)$ and $(\mathbb{H}, d_{J_{\langle .,. \rangle}})$ are same.

Let us denote $Fix(\Upsilon) = \{\nu \in \mathbb{H} : \Upsilon\nu = \nu\}$. Now we consider the following hypotheses:

(a_1) $(\mathbb{H}, \sqsubseteq)$ is a partially ordered set with the S^{JS}-metric defined as above;

(b_1) $P_{\mathbb{K}}(I_{\mathbb{K}} - \mu\Upsilon)(= \Psi) : \mathbb{K} \to \mathbb{K}$, with $\mu > 0$, satisfies for all $x, y \in \mathbb{K}$ with either $x \sqsubseteq y$ or $y \sqsubseteq x$,

$$\|\Psi x - \Psi y\| \leqslant k \max\{\|x - y\|, \|x - \Psi x\|, \|y - \Psi y\|, \frac{\|x - \Psi y\| + \|y - \Psi x\|}{2}\}, \quad k \in [0, 1);$$

(c_1) $x \sqsubseteq y$ implies $\Psi x \sqsubseteq \Psi y$ for all $(x, y) \in \mathbb{K}^2$;

(d_1) for any ordered convergent sequence $\{x_n\}$ converging to u, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Psi x_{n_k} \to \Psi u$ as $k \to \infty$;

(e_1) there exists $h_0, h_1 \in \mathbb{K}$ such that either $h_0 \sqsubseteq h_1$ or $h_1 \sqsubseteq h_0$ and $h_1 = \Psi h_0$.

Theorem 9.24. *Assume that conditions* (a_1)-(e_1) *hold, then problem* (8.1) *admits atleast one solution that is* $\mathrm{Fix}(\Psi) \neq \emptyset$. *Moreover, there exists an ordered sequence* $\{h_n\} \subset \mathbb{K}$ *such that* $h_{n+1} = P_{\mathbb{K}}(h_n - \mu\Upsilon h_n)$ *for every* $n \geqslant 0$ *and* $\lim_{n\to\infty} h_n = h^* \in \mathrm{Fix}(\Psi)$.

Proof. By Lemma 9.22, $e^* \in \mathbb{K}$ is a solution of $\langle \Upsilon e^*, w - e^*\rangle \geqslant 0$ for all $w \in \mathbb{K}$ if and only if $\Psi e^* = e^*$. Now by setting $A = B = \mathbb{K}$ we see that Ψ and $d_{J_{\langle\cdot,\cdot\rangle}}$ satisfies all the hypotheses (a) to (e) of Theorem 9.8. Also for any ordered sequence $\{x_n\}$ we have for all $n \geqslant 1$,

$$\begin{aligned}\|x_n - x_{n+1}\| &= \|\Psi x_{n-1} - \Psi x_n\| \\ &\leqslant k\max\{\|x_{n-1} - x_n\|, \|x_{n-1} - \Psi x_{n-1}\|, \|x_n - \Psi x_n\|, \frac{\|x_{n-1} - \Psi x_n\| + \|x_n - \Psi x_{n-1}\|}{2}\}.\end{aligned}$$

Therefore we get $\|x_n - x_{n+1}\| \leqslant k\|x_{n-1} - x_n\|$ for all $n \in \mathbb{N}$. Therefore we obtain that for all $1 \leqslant n < m$, $d_{J_{\langle\cdot,\cdot\rangle}}(x_n, x_m) = 2\|x_n - x_m\| \leqslant \frac{2}{1-k}\|x_0 - x_1\|$ that is $\delta(J_{\langle\cdot,\cdot\rangle}, \Psi, x_0, x_1) = \sup\{d_{J_{\langle\cdot,\cdot\rangle}}(x_i, x_j) : i, j \geqslant 0\} \leqslant \frac{2}{1-k}\|x_0 - x_1\| < \infty$. Therefore Ψ satisfies all the conditions of Theorem 9.8 and thus (8.1) has at least one solution h^* in $\mathbb{K}$. □

In a similar way assuming the following conditions.

(a_2) $(\mathbb{H}, \sqsubseteq)$ is a partially ordered set with the S^{JS}-metric defined as above;

(b_2) $P_{\mathbb{K}}(I_{\mathbb{K}} - \mu\Upsilon)(= \Psi) : \mathbb{K} \to \mathbb{K}$, with $\mu > 0$, satisfies for all $x, y \in \mathbb{K}$ with either $x \sqsubseteq y$ or $y \sqsubseteq x$,

$$\|\Psi^2 x - \Psi^2 y\| \leqslant k\max\{\|\Psi x - \Psi y\|, \|\Psi x - \Psi^2 x\|, \|\Psi y - \Psi^2 y\|, \frac{\|\Psi x - \Psi^2 y\| + \|\Psi y - \Psi^2 x\|}{2}\}, k \in [0, 1);$$

(c_2) $x \sqsubseteq y$ implies $\Psi x \sqsubseteq \Psi y$ for all $(x, y) \in \mathbb{K}^2$;

(d_2) for any ordered sequence $\{x_n\}$ if $\{\Psi x_n\}$ converges to some $y \in \mathbb{K}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z$ as $k \to \infty$ with $y = Tz$;

(e_2) there exists $h_0, h_1 \in \mathbb{K}$ such that either $h_0 \sqsubseteq h_1$ or $h_1 \sqsubseteq h_0$ and $h_1 = \Psi h_0$.

We obtain the following Theorem.

Theorem 9.25. *Assume that conditions* (a_2)-(e_2) *hold, then problem* (8.1) *admits atleast one solution that is* $\mathrm{Fix}(\Psi) \neq \emptyset$. *Moreover, there exists an ordered sequence* $\{h_n\} \subset \mathbb{K}$ *such that* $h_{n+1} = P_{\mathbb{K}}(h_n - \mu\Upsilon h_n)$ *for every* $n \geqslant 0$ *and* $\lim_{n\to\infty} \Psi h_n = h^* \in \mathrm{Fix}(\Psi)$.

10. Discussion and conclusion

In this article we presented recently published resuts on the study of S^{JS}-metric, related topological spaces and sequentially compact S^{JS}-metric spaces with several classical theorems like Cantor's intersection theorem, Ekeland's varitional priciple, Caristi's fixed point theorem, best S^{JS}-proximity point theorem, etc. Proving Baire's Category Theorem in S^{JS}-metric spaces is still an open challenging problem. This is a very challenging area of research and there are vast opportunities for future research work on Bolzano Weierstrass Theorem, Lebesgue's Covering Theorem, notion of totally bounded and some thing like Ascoli's theorem on sequentially compact S^{JS}-metric spaces, some notion of convexity. We also expect further applications of these spaces in approximation theory, variational problems, nonconvex minimization problems, fixed point theory, optimization theory, and controll theory.

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