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# Strong convergence of perturbed Mann iteration for systems of variational inequality problems over the set of common fixed points of a finite family of demicontractive mappings in Banach spaces

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**Abstract**

In this paper, we propose an iterative algorithm, which is based on the Mann iterative method for solving simultaneously common fixed point problem with a finite family of demicontractive mappings and systems of variational inequalities involving an infinite family of strongly accretive operators. Under suitable assumptions, we prove the strong convergence of this algorithm in Banach spaces. Application to systems of constrained convex minimization problem is provided to support our main results. The results of this paper improve and extend results of [M. Eslamian, C. R. Math. Acad. Sci. Paris, **355** (2017), 1168–1177], and of many others.

**Keywords:** Perturbed Mann iteration, systems of variational inequalities, demicontractive operators, strongly accretive operators.

**2020 MSC:** 47H05, 47J05, 47J25.



## 1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$ and $C$ is a nonempty closed convex subset of $H$. A mapping $A : H \to H$ is said to be $k$-*strongly monotone* if there exists $k \in (0, 1)$ such that for all $x, y \in D(A)$,

$$\langle Ax - Ay, x - y \rangle_H \geqslant k\|x - y\|^2.$$

Recall that the mapping $T : C \to C$ is said to be Lipschitz if there exists an $L \geqslant 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \quad \forall\, x, y \in C,$$

if $L < 1$, $T$ is called *contraction* and if $L = 1$, $T$ is called nonexpansive. We denote by $\mathrm{Fix}(T)$ the set of fixed points of the mapping $T$, that is $\mathrm{Fix}(T) := \{x \in D(T) : x = Tx\}$. We assume that $\mathrm{Fix}(T)$ is nonempty.

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If $T$ is nonexpansive mapping, it is well known $Fix(T)$ is closed and convex. A map $T$ is called quasi-nonexpansive if $\|Tx-p\| \leqslant \|x-p\|$ holds for all $x$ in $C$ and $p \in Fix(T)$. The mapping $T : C \to C$ is said to be firmly nonexpansive, if

$$\|Tx-Ty\|^2 \leqslant \|x-y\|^2 - \|(x-y)-(Tx-Ty)\|^2, \ \forall x, y \in C.$$

A mapping $T : C \to H$ is called k-strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx-Ty\|^2 \leqslant \|x-y\|^2 + k\|x-y-(Tx-Ty)\|^2, \ \forall x, y \in C.$$

A map $T$ is called k-demi-contractive if $Fix(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

$$\|Tx-p\| \leqslant \|x-p\| + k\|x-Tx\|^2, \ \forall x \in C, \ \ p \in Fix(T). \tag{1.1}$$

We note that the following inclusions hold for the classes of the mappings: firmly nonexpansive $\subset$ nonexpansive $\subset$ quasi-nonexpansive $\subset$ k-strictly pseudo-contractive $\subset$ k-demi-contractive. Demicontractive mappings constitute one of the most general classes of nonexpansive type mappings with important applications for which the fixed points can be obtained by iterative schemes, there was and still is a great interest in studying their properties, see [15, 19, 20] and most of the references therein. Construction of fixed points of nonlinear mappings is an important subject in the theory of nonlinear mappings and find applications in a variety of applied areas, in particular, in inverse problems, partial differential equations, image recovery, and signal processing (see, [1, 3]). Mann iteration algorithm [18] is widely used for solving a fixed point equation of the form $Tx = x$. This algorithm is a sequence $\{x_n\}$, which is generated by the following recursive way:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. But Mann's iteration process has only weak convergence, even in Hilbert space setting. Therefore, many authors try to modify Mann's iteration to have strong convergence for nonlinear operators; see, for example, [9, 20, 21] and the references therein. The varational inequility problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, y-x^* \rangle \geqslant 0, \ \forall y \in C, \tag{1.2}$$

we denote the set of solutions of variational inequality problem of the mapping $A$ on set $C$ in (1.2) by $VI(C, A)$. The important problems of the $VI(C, A)$ are existence and uniqueness of solutions. It is known that, if $A$ is a strongly monotone and Lipschitzian mapping on $C$, then the $VI(C, A)$ has a unique solution. One of the interesting problems is how to find a solution of the $VI(C, A)$ if $A$ is others. In recent years, variational inequalities have been used to study a large variety of problems arising in structural analysis, economics, optimization, operations research, and engineering sciences (see, e.g., [7, 23, 24] and the references therein). Observe that the feasible set $C$ of the variational inequality problem can always be represented as the fixed point set of some operator, say, $C = Fix(P_C)$. Following this idea, Yamada [23] considered the variational inequality problem $VI(A, Fix(T))$, which calls for finding a point $x^* \in Fix(T)$ such that

$$\langle Ax^*, y-x^* \rangle \geqslant 0, \ \forall y \in Fix(T).$$

Yamada [23] considered the following hybrid steepest-descent iterative method:

$$x_{n+1} = (I - \mu\alpha_n A)Tx_n,$$

where $A$ is a Lipschitzian continuous and strongly monotone operator and $T$ is a nonexpansive operator. Under some appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique point in $VI(A, Fix(T))$. The literature on variational inequalities is vast, and the hybrid steepest-descent method has received great attention from many authors, who improved it in various ways; see, e.g., [13] and references therein. Recently, in 2017, Eslamian proposed an explicit parallel algorithm for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of demi-contractive operators, as follows.

**Theorem 1.1** ([13]). *Let $H$ be a real Hilbert space. Let for each $i \in \{1, 2, \ldots, m\}, F_i : H \to H$ be a $k_i$-inverse monotone operator and $T_i : H \to H$ be a $\lambda_i$-demicontractive operator such that $I - T_i$ is demiclosed at $0$. Assume that $\bigcap_{i=1}^{m} A_i^{-1}0 \bigcap \bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0, v \in H$ and by*

$$\begin{cases} y_n{}^{(i)} = (I - \mu^{(i)}\beta_n^{(i)} F_i) T_i x_n, \ i = 1, 2, \ldots, n, \\ x_{n+1} = \gamma_n^{(0)} v + \sum_{i=1}^{m} \gamma_n^{(i)} y_n{}^{(i)}, \ \forall n \geqslant 0, \end{cases}$$

*where $T_i{}^n = \alpha_n^{(i)} I + (1 - \alpha_n^{(i)}) T_i$. Let the sequences $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:*

(i) $y_n{}^{(i)} \subset [a_i, b_i] \subset (0, 1)$ *and* $\sum_{i=0}^{m} \gamma_n^{(i)} = 1$;
(ii) $\lim_{n\to\infty} \gamma_n^{(0)} = 0$ *and* $\sum_{i=0}^{m} \gamma_n^{(0)} = \infty$;
(iii) $\{\mu^{(i)}\beta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2k_i)$;
(iv) $\lambda_i < \alpha_n^{(i)} \leqslant e_i < 1$.

*Then the sequence $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^{m} A_i^{-1}0 \bigcap \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$.*

Note that, most of the algorithms proposed and studied for solving (1.2) are largely confined to real Hilbert spaces. This is understandable because, as is well known, among all infinite dimensional Banach spaces, Hilbert spaces have the nicest geometric properties, most of which characterize inner product spaces and make problems posed in real Hilbert spaces more manageable than those posed in more general Banach spaces. This leads to this important natural question.

**Question 1.2.** *Can we construct a new iterative method based on the Mann iteration for solving simultaneously common fixed point problem involving a finite family of demicontractive mappings and systems of variational inequalities in Banach spaces?*

The purpose of this paper is to answer the above question in the affirmative. Thus, we introduce a perturbed Mann iteration for solving simultaneously common fixed point problem with a finite family of demicontractive mappings and systems of variational inequalities involving an infinite family of strongly accretive operators. We prove the strong convergence of this algorithm in Banach spaces without any compactness assumption.

## 2. Preliminaries

Let $E$ be a Banach space with norm $\|\cdot\|$ and dual $E^*$. Let $\varphi : [0, +\infty) \to [0, \infty)$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function $\varphi$ is called gauge. Associated to a gauge, a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\}.$$

If the gauge is defined by $\varphi(t) = t^{q-1}$, $q > 1$, then the corresponding duality map is called the *generalized duality mapping* from $E$ to $2^{E^*}$ defined by

$$J_q(x) := \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q \text{ and } \|x^*\| = \|x\|^{q-1}\right\}.$$

$J_2$ is called the *normalized duality mapping* and is denoted by J. Notice that

$$J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|}J(x), \ x \neq 0.$$

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if $\lim\limits_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S$ (see, e.g., [11] for more details on duality maps).

*Remark* 2.1. Note also that a duality mapping exists in each Banach space. We recall from [2] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.

(i) $l_p : Jx = \|x\|_{l_p}^{2-p} y \in l_q, \ x = (x_1, x_2, \ldots, x_n, \ldots); \ y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \ldots, x_n|x_n|^{p-2}, \ldots)$;

(ii) $L_p : Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u \in L_q$;

(iii) $W^{m,p} : Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leqslant m|}(-1)^{|\alpha|}D^\alpha\Big(|D^\alpha u|^{p-2}D^\alpha u\Big) \in W^{-m,q}$,

where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

Recall that a real Banach space E that has a weakly continuous duality map satisfies Opial's property, (see, e.g., [12]). Let $C \subseteq E$ be a nonempty set. An operator $A : C \to E$ is said to be accretive if there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geqslant 0, \ \forall x, y \in C.$$

An operator $A : C \to E$ is said to be $\alpha$-inverse strongly accretive if, for some $\alpha > 0$,

$$\langle Ax - Ay, j_q(x-y)\rangle \geqslant \alpha\|Ax - Ay\|^q, \ \forall x, y \in C.$$

In [10], Chidume extended the condition (1.1) to arbitrary real Banach space X. If X is q-uniformly smooth, then the condition (1.1) becomes

$$\langle x - Tx, j_q(x-p)\rangle \geqslant \frac{(1-k)^{q-1}}{2^{q-1}}\|x - Tx\|^q, \ x \in X, \ p \in Fix(T). \tag{2.1}$$

Let C be a nonempty subset of real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_Cx + t(x - Q_Cx)) = Q_Cx$$

for each $x \in E$ and $t \geqslant 0$. A mapping $Q_C : E \to C$ is said to be a retraction if $Q_Cx = x$ for each $x \in C$.

**Lemma 2.2** ([14])**.** *Let* C *and* D *be nonempty subsets of a real Banach space* E *with* $D \subset C$ *and* $Q_D : C \to D$ *a retraction from* C *into* D*. Then* $Q_D$ *is sunny and nonexpansive if and only if*

$$\langle z - Q_Dz, j(y - Q_Dz)\rangle \leqslant 0$$

*for all* $z \in C$ *and* $y \in D$.

It is noted that Lemma 2.2 still holds if the normalized duality map is replaced by the general duality map $J_\varphi$, where $\varphi$ is gauge function.

*Remark* 2.3. If K is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection $P_K$ from H to K is the sunny nonexpansive retraction.

**Lemma 2.4** (Demiclosedness principle, Browder [6])**.** *Let* E *be a Banach space satisfying Opial's property,* K *be a closed convex subset of* E*, and* $T : K \to K$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$*. Then* $I - T$ *is demiclosed; that is,*

$$\{x_n\} \subset K, \ x_n \rightharpoonup x \in K, \ \textit{and} \ (I-T)x_n \to y \ \textit{implies that} \ (I-T)x = y.$$

**Lemma 2.5** ([16]). *Assume that a Banach space $E$ has a weakly continous duality mapping $J_\varphi$ with jauge $\varphi$.*

$$\Phi(\|x+y\|) \leqslant \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle \tag{2.2}$$

*for all x,y $\in E$, where $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma t \geqslant 0$. In particular, for the normilized duality mapping, we have the important special version of (2.2):*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

**Theorem 2.6** ([11]). *Let $q > 1$ be a fixed real number and $E$ be a smooth Banach space. Then the following statements are equivalent:*

(i) *$E$ is q-uniformly smooth;*

(ii) *there is a constant $d_q > 0$ such that for all $x, y \in E$,*

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y\,,\, J_q(x)\rangle + d_q\|y\|^q;$$

(iii) *there is a constant $c_1 > 0$ such that*

$$\langle x-y\,,\, J_q(x) - J_q(y)\rangle \leqslant c_1\|x-y\|^q, \ \ \forall\ x, y \in E.$$

**Lemma 2.7** (Xu, [22]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leqslant (1-\alpha_n)a_n + \sigma_n$ for all $n \geqslant 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

(a) $\sum_{n=0}^\infty \alpha_n = \infty$,

(b) $\limsup_{n\to\infty} \dfrac{\sigma_n}{\alpha_n} \leqslant 0$ *or* $\sum_{n=0}^\infty |\sigma_n| < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.8** ([17]). *Let $t_n$ be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence $t_{n_i}$ of $t_n$ such that $t_{n_i} \leqslant t_{n_{i+1}}$ for all $i \geqslant 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leqslant n : t_k \leqslant t_{k+1}\}.$$

*Then, $\tau(n) \to \infty$ as $n \to \infty$ and*

$$\max\{t_{\tau(n)},\ t_n\} \leqslant t_{\tau(n)+1}.$$

**Lemma 2.9** (Chang et al., [8]). *Let $E$ be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : \|x\| \leqslant r\}$, a closed ball with center 0 and radius $r > 0$. For any given sequence $\{u_1, u_2, \ldots, u_n, \ldots\} \subset B(0)_r$ and any positive real numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ with $\sum_{k=1}^\infty \lambda_k = 1$, then there exists a continuous, strictly increasing, and convex function*

$$g : [0, 2r] \to \mathbb{R}^+,\ g(0) = 0,$$

*such that for any integer $i, j$ with $i < j$,*

$$\Big\|\sum_{k=1}^\infty \lambda_k u_k\Big\|^2 \leqslant \sum_{k=1}^\infty \lambda_k\|u_k\|^2 - \lambda_i\lambda_j g(\|u_i - u_j\|).$$

**Lemma 2.10** ([5]). *For any $r > 0$,*

(i) *$A$ is accretive if and only if the resolvent $J_r^A$ of $A$ is single-valued and firmly nonexpansive;*

(ii) *$A$ is m-accretive if and only if $J_r^A$ of $A$ is single-valued and firmly nonexpansive and its domain is the entire $E$;*

(iii) $0 \in A(x^*)$ *if and only if* $x^* \in \mathrm{Fix}(J_r^A)$, *where* $\mathrm{Fix}(J_r^A)$ *denotes the fixed-point set of* $J_r^A$.

**Lemma 2.11** ([15]). *Let $C$ be a nonempty closed convex subset of $q$-uniformly smooth Banach space $E$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$ and let $A : C \to E$ be a mapping. Then $VI(C,A) = Fix(Q_C(I-\lambda A))$, for all $\lambda > 0$.*

**Lemma 2.12.** *Let $C$ be a nonempty closed convex subset of a $q$-uniformly smooth Banach space $E$. Let $\lambda > 0$ and let $A$ be an $\alpha$-inverse strongly accretive operator of $C$ into $E$. If $0 < \lambda < \left(\frac{q\alpha}{d_q}\right)^{\frac{1}{q-1}}$, where $d_q$ is the $q$-uniformly smooth constant of $E$, then $Q_C(I-\lambda A)$ is a nonexpansive mapping.*

*Proof.* Let $x, y \in C$, we have

$$\begin{aligned}\|Q_C(I-\lambda A)x - Q_C(I-\lambda A)y\|^q &= \|(I-\lambda A)x - (I-\lambda A)y\|^q \\ &\leqslant \|x-y\|^q - q\lambda\langle Ax - Ay, J_q(x-y)\rangle + d_q\lambda^q\|Ax-Ay\|^q \\ &\leqslant \|x-y\|^q - q\lambda\alpha\|Ax-Ay\|^q + d_q\lambda^q\|Ax-Ay\|^q \\ &\leqslant \|x-y\|^q - \lambda(q\alpha - d_q\lambda^{q-1})\|Ax-Ay\|^q \leqslant \|x-y\|^q.\end{aligned}$$

Then $Q_C(I-\lambda A)$ is a nonexpansive mapping. □

## 3. Main results

We now prove our main results.

**Theorem 3.1.** *For $q > 1$, let $E$ be a $q$-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map $J_\varphi$. Let $C$ be a nonempty, closed convex cone of $E$, and $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Let $A_j : C \to E$ be $\alpha_j$-inverse strongly accretive and $\eta_j \in \left[a, \left(\frac{q\alpha_j}{d_q}\right)^{\frac{1}{q-1}}\right]$ for some $a > 0$. Let $T_i : C \to C$ be a $k_i$-demicontractive mapping such that $\Gamma := \bigcap_{j=1}^{\infty} VI(C, A_j) \bigcap \bigcap_{i=1}^{m} Fix(T_i) \neq \emptyset$ and $I - T_i$ is demiclosed at $0$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \ \text{chosen arbitrarily}, \\ z_n = \theta_0 x_n + \sum_{i=1}^m \theta_i T_i x_n, \\ y_n = \beta_0 z_n + \sum_{j=1}^\infty \beta_j Q_C(I-\eta_j A_j)z_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \end{cases} \tag{3.1}$$

*where $\theta_i \in (0,\gamma)$,*

$$\gamma := \min_{1\leqslant i\leqslant m}\left\{1, \left(\frac{q\mu_i^{q-1}}{2^{(m-1)q}d_q}\right)^{\frac{1}{q-1}}\right\}, \ \text{with } \mu_i = \frac{1-k_i}{2},$$

*$\sum_{i=0}^m \theta_i = 1$, $\sum_{j=0}^\infty \beta_i = 1$, $\{\alpha_n\} \subset (0,1)$, and $\{\lambda_n\} \subset (0,1)$. Assume that the above control sequences satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^\infty (1-\lambda_n)\alpha_n = \infty$.

*Then, the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^* \in \Gamma$, where $x^* = Q_\Gamma(0)$.*

*Proof.* Let $p \in \Gamma$. Using (3.1), inequality (ii) of Theorem 2.6, and inequality (2.1), we have

$$\|z_n - p\|^q = \left\|\theta_0(x_n - p) + \sum_{i=1}^m \theta_i(T_i x_n - p)\right\|^q$$

$$
\begin{aligned}
&= \Big\|\theta_0(x_n - p) + \sum_{i=1}^{m} \theta_i(T_i x_n - x_n) + \sum_{i=1}^{m} \theta_i(x_n - p)\Big\|^q \\
&= \Big\|x_n - p + \sum_{i=1}^{m} \theta_i(T_i x_n - x_n)\Big\|^q \\
&\leqslant \|x_n - p\|^q - q\sum_{i=1}^{m} \theta_i \langle x_n - T_i x_n, J_q(x_n - p)\rangle + d_q\Big\|\sum_{i=1}^{m} \theta_i(x_n - T_i x_n)\Big\|^q.
\end{aligned}
$$

Hence,

$$
\|z_n - p\|^q \leqslant \|x_n - p\|^q - q\sum_{i=1}^{m} \lambda_i \mu_i^{q-1}\|x_n - T_i x_n\|^q + d_q\Big\|\sum_{i=1}^{m} \theta_i(T_i x_n - x_n)\Big\|^q. \tag{3.2}
$$

We have,

$$
\Big\|\sum_{i=1}^{m} \theta_i(T_i x_n - x_n)\Big\|^q \leqslant 2^{(m-1)q}\sum_{i=1}^{m} \theta_i^q\Big\|T_i x_n - x_n\Big\|^q. \tag{3.3}
$$

Combining inequalities (3.2) and (3.3), it then follows that

$$
\begin{aligned}
\Big\|z_n - p\Big\|^q &\leqslant \Big\|x_n - p\Big\|^q - q\sum_{i=1}^{m} \theta_i \mu_i^{q-1}\Big\|x_n - T_i x_n\Big\|^q + d_q 2^{(m-1)q}\sum_{i=1}^{m} \theta_i^q\Big\|T_i x_n - x_n\Big\|^q. \\
&= \Big\|x_n - p\Big\|^q - \sum_{i=1}^{m} \theta_i\Big[q\mu_i^{q-1} - 2^{(m-1)q} d_q \theta_i^{q-1}\Big]\Big\|x_n - T_i x_n\Big\|^q.
\end{aligned} \tag{3.4}
$$

Since $q\mu_i^{q-1} - 2^{(m-1)q} d_q \theta_i^{q-1} > 0,\ \forall\, i = 1, \dots, m$, we obtain,

$$
\|z_n - p\Big\| \leqslant \|x_n - p\Big\|.
$$

From (3.1) and Lemmas 2.11 and 2.12, it follows that

$$
\begin{aligned}
\|y_n - p\| &= \|\beta_0 z_n + \sum_{j=1}^{\infty} \beta_j Q_C(I - \eta_j A_j) z_n - p\| \\
&\leqslant \beta_0\|z_n - p\| + \sum_{j=1}^{\infty} \beta_i\|Q_C(I - \eta_j A_j) z_n - p\| \leqslant \|z_n - p\|.
\end{aligned}
$$

Therefore, we have

$$
\|y_n - p\| \leqslant \|z_n - p\| \leqslant \|x_n - p\|. \tag{3.5}
$$

Hence,

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1 - \alpha_n) y_n - p\| \\
&\leqslant \alpha_n \lambda_n\|x_n - p\| + (1 - \alpha_n)\|y_n - p\| + (1 - \lambda_n)\alpha_n\|p\| \\
&\leqslant \alpha_n \lambda_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + (1 - \lambda_n)\alpha_n\|p\| \\
&\leqslant [1 - (1 - \lambda_n)\alpha_n]\|x_n - p\| + (1 - \lambda_n)\alpha_n\|p\| \leqslant \max\{\|x_n - p\|,\ \|p\|\}.
\end{aligned}
$$

By induction, it is easy to see that

$$
\|x_n - p\| \leqslant \max\{\|x_0 - p\|, \|p\|\},\ n \geqslant 1.
$$

Consequently, using inequality (3.4) and the fact that $\|y_n - p\| \leqslant \|x_n - p\|$, we obtain,

$$
\|x_{n+1} - p\|^q = \|\alpha_n(\lambda_n x_n) + (1 - \alpha_n) y_n - p\|^q
$$

$$
\begin{aligned}
&= \|y_n - p + \alpha_n((\lambda_n x_n) - y_n)\|^q \\
&\leqslant \|y_n - p\|^q + q\alpha_n\langle(\lambda_n x_n) - y_n, J_q(y_n - p)\rangle + d_q\Big\|\alpha_n((\lambda_n x_n) - y_n)\Big\|^q \\
&\leqslant \|y_n - p\|^q + q\alpha_n\|(\lambda_n x_n) - y_n\|\|y_n - p\|^{q-1} + d_q\alpha_n^q\Big\|(\lambda_n x_n) - y_n\Big\|^q \\
&\leqslant \Big\|x_n - p\Big\|^q - \sum_{i=1}^{m}\theta_i\Big[q\mu_i^{q-1} - 2^{(m-1)q}d_q\theta_i^{q-1}\Big]\Big\|x_n - T_i x_n\Big\|^q \\
&\quad + q\alpha_n\|(\lambda_n x_n) - y_n\|\|y_n - p\|^{q-1} + d_q\alpha_n^q\Big\|(\lambda_n x_n) - y_n\Big\|^q.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\sum_{i=1}^{m}\theta_i\Big[q\mu_i^{q-1} - 2^{(m-1)q}d_q\theta_i^{q-1}\Big]\Big\|x_n - T_i x_n\Big\|^q \\
&\quad \leqslant \Big\|x_n - p\Big\|^q - \Big\|x_{n+1} - p\Big\|^q + q\alpha_n\|(\lambda_n x_n) - y_n\|\|y_n - p\|^{q-1} + d_q\alpha_n^q\|(\lambda_n x_n) - y_n\|^q.
\end{aligned}
$$

Since $\{y_n\}$ and $\{(\lambda_n x_n)\}$ are bounded, then there exists a constant $K > 0$ such that for every $i, 1 \leqslant i \leqslant m$,

$$
\sum_{i=1}^{m}\theta_i\Big[q\mu_i^{q-1} - 2^{(m-1)q}d_q\theta_i^{q-1}\Big]\|x_n - T_i x_n\|^q \leqslant \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n K. \tag{3.6}
$$

Now we prove that $\{x_n\}$ converges strongly to $x^*$. We divide the proof into two cases.

**Case 1**. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$
\|x_n - p\|^q - \|x_{n+1} - p\|^q \to 0.
$$

It then implies from (3.6) that

$$
\lim_{n\to\infty}\sum_{i=1}^{m}\theta_i\Big[q\mu_i^{q-1} - 2^{(m-1)q}d_q\theta_i^{q-1}\Big]\Big\|x_n - T_i x_n\Big\|^q = 0.
$$

Since $q\mu_i^{q-1} - 2^{(m-1)q}d_q\theta_i^{q-1} > 0,\ \forall i = 1, \dots, m$, we have

$$
\lim_{n\to\infty}\Big\|x_n - T_i x_n\Big\| = 0. \tag{3.7}
$$

Now, using the fact that $\sum_{i=0}^{m}\theta_i = 1$, we have,

$$
\|z_n - x_n\| = \|\theta_0 x_n + \sum_{i=1}^{m}\theta_i T_i x_n - x_n\| \leqslant \|T_i x_n - x_n\|.
$$

Therefore, from (3.7) we have

$$
\lim_{n\to\infty}\|z_n - x_n\| = 0.
$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_n)\rangle \leqslant 0$. Since $E$ is reflexive and $\{x_n\}_{n\geqslant 0}$ is bounded there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}$ converges weakly to $a$ in $C$ and

$$
\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_n)\rangle = \lim_{j\to+\infty}\langle x^*, J_\varphi(x^* - x_{n_j})\rangle.
$$

From (3.7) and $I-T_i$ being demiclosed, we obtain $a \in \bigcap_{i=1}^{m} Fix(T_i)$. Let $j \geqslant 0$, by using Lemma 2.9, the fact that $Q_C(I-\eta_j A_j)$ is nonexpansive, and (3.5), we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_0 z_n + \sum_{j=1}^{\infty} \beta_i Q_C(I-\eta_j A_j)z_n - p\|^2\\ &\leqslant \beta_0\|z_n - p\|^2 + \sum_{j=1}^{\infty} \beta_j \|Q_C(I-\eta_j A_j)z_n - p\|^2 - \beta_0\beta_j g_1(\|Q_C(I-\eta_j A_j)z_n - z_n\|)\\ &\leqslant \|x_n - p\|^2 - \beta_0\beta_j g_1(\|Q_C(I-\eta_j A_j)z_n - z_n\|).\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2\\ &= \|\alpha_n\lambda_n(x_n - p) + (1-\alpha_n)(y_n - p) - (1-\lambda_n)\alpha_n p\|^2\\ &\leqslant \|\alpha_n(\lambda_n x_n - \lambda_n p) + (1-\alpha_n)(y_n - p)\|^2 + 2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle\\ &\leqslant \alpha_n\lambda_n^2\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 + 2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle\\ &\leqslant \alpha_n\lambda_n\|x_n - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\|^2 - \beta_0\beta_j g_1(\|Q_C(I-\eta_j A_j)z_n - z_n\|)]\\ &\quad + 2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle\\ &\leqslant [1-(1-\lambda_n)\alpha_n]\|x_n - p\|^2 - (1-\alpha_n)\beta_0\beta_j g_1(\|Q_C(I-\eta_j A_j)z_n - z_n\|)\\ &\quad + 2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle.\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $D > 0$ sucht that

$$(1-\alpha_n)\beta_0\beta_j g_1(\|Q_C(I-\eta_j A_j)z_n - z_n\|) \leqslant \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n D.$$

Thus we have

$$\lim_{n\to\infty} g_1(\|Q_C(I-\eta_j A_j)z_n - z_n\|) = 0.$$

Using property of $g_1$, we have

$$\lim_{n\to\infty} \|Q_C(I-\eta_j A_j)z_n - z_n\| = 0. \tag{3.8}$$

From (3.8) and Lemma 2.4, we obtain $a \in \bigcap_{i=1}^{\infty} Fix(Q_C(I-\eta_j A_j))$. Using Lemma 2.11, we have $a \in \bigcap_{i=1}^{\infty} VI(C, A_j)$. Therefore, $a \in \Gamma$. On the other hand, using $x^* = Q_\Gamma(0)$ and the assumption that the duality mapping $J_\varphi$ is weakly continuous, we have,

$$\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_n)\rangle = \lim_{j\to+\infty}\langle x^*, J_\varphi(x^* - x_{n_j})\rangle = \langle x^*, J_\varphi(x^* - a)\rangle \leqslant 0.$$

Finally, we show that $x_n \to x^*$. In fact, since $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma$, $\forall t \geqslant 0$, and $\varphi$ is a gauge function, then for $1 \geqslant k \geqslant 0$, $\Phi(kt) \leqslant k\Phi(t)$. From (3.1) and Lemma 2.5, we get that

$$\begin{aligned}\Phi(\|x_{n+1} - x^*\|) &= \Phi(\|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - x^*\|)\\ &\leqslant \Phi(\|\alpha_n\lambda_n(x_n - x^*) + (1-\alpha_n)(y_n - x^*)\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leqslant \Phi(\alpha_n\lambda_n\|x_n - x^*\| + \|(1-\alpha_n)(y_n - x^*)\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leqslant \Phi(\alpha_n\lambda_n\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leqslant \Phi((1-(1-\lambda_n)\alpha_n)\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leqslant [1-(1-\lambda_n)\alpha_n]\Phi(\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle.\end{aligned}$$

From Lemma 2.7, its follows that $x_n \to x^*$.

**Case 2**. Suppose that Case 1 fails. Set $B_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geqslant n_0$ (for some $n_0$ large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leqslant n,\ B_k \leqslant B_{k+1}\}$. We have $\tau$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leqslant B_{\tau(n)+1}$ for $n \geqslant n_0$. Let $i \in \mathbb{N}^*$, from (3.6), we have

$$\sum_{i=1}^{m} \theta_i \left[q\mu_i^{q-1} - 2^{(m-1)q} d_q \theta_i^{q-1}\right] \left\|x_{\tau(n)} - T_i x_{\tau(n)}\right\|^q \leqslant \alpha_{\tau(n)} K.$$

The last inequality implies

$$\lim_{n\to\infty} \sum_{i=1}^{m} \lambda_i \left[q\mu_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1}\right] \left\|x_{\tau(n)} - T_i \tau(n)\right\|^q = 0.$$

Since $q\mu_i^{q-1} - 2^{(m-1)q} d_q \lambda_i^{q-1} > 0,\ \forall i = 1, \dots, m$, we have

$$\lim_{n\to\infty} \left\|x_{\tau(n)} - T_i \tau(n)\right\|^q = 0.$$

By same argument as in Case 1, we can show that $\limsup_{\tau(n)\to+\infty} \langle x^*, J_\varphi(x^* - x_{\tau(n)})\rangle \leqslant 0$. We have for all $n \geqslant n_0$,

$$0 \leqslant \Phi(\|x_{\tau(n)+1} - x^*\|) - \Phi(\|x_{\tau(n)} - x^*\|) \leqslant \left(1 - \lambda_{\tau(n)}\right)\alpha_{\tau(n)}[-\Phi(\|x_{\tau(n)} - x^*\|) + \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle],$$

which implies that

$$\Phi(\|x_{\tau(n)} - x^*\|) \leqslant \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty} \Phi(\|x_{\tau(n)} - x^*\|) = 0. \tag{3.9}$$

Using properties of $\Phi$, (3.9), and the fact that $B_{\tau(n)} = \|x_{\tau(n)} - x^*\|$, we have

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

We have for all $n \geqslant n_0$,

$$0 \leqslant B_n \leqslant \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to $x^*$. This completes the proof. □

We now apply Theorem 3.1 and Lemma 2.10 for solving system of variational inequalities coupled with inclusion problems involving a finite family of accretive operators.

**Theorem 3.2.** *For $q > 1$, let $E$ be a $q$-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map $J_\varphi$. Let $C$ be a nonempty, closed convex cone of $E$ and $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Let $A_j : C \to E$ being $\alpha_j$-inverse strongly accretive and $\eta_j \in \left[a, \left(\frac{q\alpha_j}{d_q}\right)^{\frac{1}{q-1}}\right]$ for some $a > 0$. Let $B_i : D(B_i) \subset C \to 2^E$ be an accretive operator such that $\overline{D(B_i)} \subset C \subset \bigcap_{r>0} R(I + rB_i)$ such that $\Gamma := \bigcap_{j=1}^{\infty} VI(C, A_j) \bigcap \bigcap_{i=1}^{m} {B_i}^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ z_n = \theta_0 x_n + \sum_{i=1}^{m} \theta_i J_r^{B_i} x_n, \\ y_n = \beta_0 z_n + \sum_{j=1}^{\infty} \beta_j Q_C(I - \eta_j A_j) z_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n) y_n, \end{cases} \tag{3.10}$$

*$\sum_{i=0}^{m} \theta_i = 1$, $\sum_{j=0}^{\infty} \beta_i = 1$, $\{\alpha_n\} \subset (0,1)$, and $\{\lambda_n\} \subset (0,1)$. Assume that the above control sequences satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty} (1 - \lambda_n)\alpha_n = \infty$.

*Then, the sequence $\{x_n\}$ generated by (3.10) converges strongly to $x^* \in \Gamma$, where $x^* = Q_\Gamma(0)$.*

## 4. Application to systems of constrained convex minimization problems

In this section, we study the problem of finding a common solution of an infinite family of convex minimization problems coupled with fixed point problem involving finite family of demicontractive mappings in real Hilbert spaces. Precisely, find an $x^*$ with the property:

$$x^* \in \Big(\bigcap_{j=1}^{\infty} \operatorname{argmin}_{x\in C}\, g_j(x)\Big) \bigcap \bigcap_{i=1}^{m} \operatorname{Fix}(T_i). \tag{4.1}$$

**Lemma 4.1** (Baillon and Haddad, [4])**.** *Let $H$ be a real Hilbert space, $g$ a continuously Fréchet differentiable convex functional on $H$, and $\nabla g$ the gradient of $g$. If $\nabla g$ is $\frac{1}{\alpha}$-Lipschitz continuous, then $\nabla g$ is $\alpha$-inverse strongly monotone.*

*Remark* 4.2. A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (4.1) is that $x^* \in \Big(\bigcap_{j=1}^{\infty} VI(\nabla g_j, C)\Big) \cap \bigcap_{i=1}^{m} \operatorname{Fix}(T_i)$.

Hence, one has the following result.

**Theorem 4.3.** *Let $H$ be a real Hilbert space. Let $C$ be a nonempty closed convex cone of $H$. Let $g_j : C \to \mathbb{R}$ be a continuously Fréchet differentiable convex functional on $C$ with a $\dfrac{1}{\alpha_j}$-Lipschitz continuous $\nabla g_j$. Let $T_i : C \to C$ be a $k_i$-demicontractive and $I - T_i$ is demiclosed at $0$. Assume that the Problem (4.1) is consistent. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \ \textit{choosen arbitrarily},\\ z_n = \theta_0 x_n + \sum_{i=1}^{m} \theta_i T_i x_n,\\ y_n = \beta_0 z_n + \sum_{j=1}^{\infty} \beta_j P_C(I - \eta_j \nabla g_j) z_n,\\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n) y_n. \end{cases} \tag{4.2}$$

*Assume that the above control sequences satisfy the following conditions:*

(i) $\sum_{i=0}^{m} \theta_i = 1,\ \sum_{j=0}^{\infty} \beta_i = 1,\ \eta_j \in ]0, 2\alpha_j]$;
(ii) $\theta_i \in [a, b] \subset (0,\ k_i)$, $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty} (1-\lambda_n)\alpha_n = \infty$.

*Then, the sequence $\{x_n\}$ generated by (4.2) converges strongly to a solution of Problem (4.1).*

*Proof.* We set $H = E$, $P_C = Q_C$, and $\nabla g_j = A_j$, into Theorem 3.1. Then, the proof follows from Theorem 3.1 and Remark 4.2. □

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