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Ulam type stability of ψ-Riemann-Liouville fractional differential equations using (k, ψ)-generalized Laplace transform

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Abstract

The primary objective of this paper is to explore the Hyers-Ulam stability of the ψ-Riemann-Liouville fractional differential equations by employing the (k, ψ)-generalized Laplace transform method. The outcomes of our investigation represent advancements over certain existing results in the literature. Furthermore, we present illustrative examples to elucidate our primary findings.

Keywords: Hyers-Ulam stability, Riemann-Liouville fractional derivative, linear differential equation, (k, ψ)-generalized Laplace transform.

2020 MSC: 34K20, 26D10, 44A10, 26A33, 33B15.

1. Introduction

Fractional calculus has made significant strides in diverse scientific and engineering fields since 1695 [23]. The literature presents various definitions of fractional derivatives and integrals, prompting extensive efforts to generalize these concepts [1, 22, 26, 27, 29, 31]. For instance, Souza and Oliveira [38] introduced the ψ-Hilfer fractional derivative (ψ-HFD), which spurred further investigations into generalization of fractional differential equations [1, 4, 10, 11, 15, 20–22, 26–29, 31, 32, 38].

Fractional calculus has a broad range of applications in today's scientific landscape, covering mathematical physics, statistical mechanics, electrochemistry, electrical conductance in biological systems, astrophysics, computed tomography, control theory, modeling viscoelastic materials, thermodynamics, diffusion modeling, biophysics, fractional-order models for neurons, hydrology, geological surveying, signal and image processing, engineering, finance, and beyond. See for example [2, 3, 9, 13, 14, 35–37].

Various methodologies exist for investigating the stability of linear fractional differential equations that involve fractional derivatives. Recent research by different authors has contributed to this area. For

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example, Alqifiary et al. [5] demonstrated the generalized Hyers-Ulam stability (HUS) of linear differential equations, Rezaei et al. [30] established the HUS of linear differential equations, Wang et al. [39] provided proof of the HUS for two types of fractional linear differential equations, Shen et al. [33] focused on Ulam stability concerning linear fractional differential equations with constant coefficients, Liu et al. [24] proved the HUS of linear Caputo-Fabrizio fractional differential equations, Başcı et al. [7] demonstrated the HUS of linear Caputo-Fabrizio fractional differential equations using the Laplace transform, a widely utilized method for assessing HUS and Liu et al. [25] proved the stability of generalized linear Liouville-Caputo fractional equations using the ρ-Laplace transform introduced by Jarad et al. [17].

Analytically solving differential equations with fractional derivatives often faces challenges in finding suitable transformations. Integral transforms like Laplace, Mellin, and Fourier have been pivotal in generating solutions for such equations. To broaden the scope of function classes for these transformations, Başcı et al. [8] introduced the (k,ψ)-generalized Laplace transform ((k,ψ)-GLT), exploring its properties and establishing a convolution theorem.

Liu et al. [24] delved into the investigation of HUS for the following nonlinear Cauchy problem, utilizing the ρ-Laplace transform as a tool:

$$\left({}^{C}D_{0}^{\alpha,\rho}y\right)(t)=f\left(t,y\left(t\right)\right),\ y\left(0\right)=y_{0},\ \rho>0,\ 0<\alpha<1.$$

Zada et al. [41] examined the HUS of the following fractional differential equations by employing the ρ-Laplace transform as an instrumental tool:

$$\left({}^{C}D_{t_0}^{\alpha,\rho}W\right)(t)=AW\left(t\right)+q\left(t\right),\ W\left(0\right)=P,\ t\in\left[t_0,T\right],\ \rho>0,\ 0<\alpha<1,$$

where ${}^{C}D_{t_0}^{\alpha,\rho}$ denotes the left generalized α order Liouville-Caputo fractional derivative defined componentwise. A is n^{th} order matrix over the real field $\mathbb{R}$, $q(t)$ is an n-dimensional locally integrable column vector function on the closed interval $[t_0,T]$ and $W(t)=(w_1(t),w_2(t),\ldots,w_n(t))^T$ is an unknown vector function, while P is a specified vector.

In this paper, we consider with the following ψ-Riemann-Liouville fractional differential equations of the forms

$$\left(D_{a+}^{\alpha,\psi}y\right)(t)=f\left(t\right) \tag{1.1}$$

and

$$\left(D_{a+}^{\alpha,\psi}y\right)(t)-\lambda y\left(t\right)=f\left(t\right) \tag{1.2}$$

with the initial conditions

$$\left(I_{a^+}^{\alpha;\psi}y\right)^{[m]}(a)=c_m,\ m=0,1,\ldots,n-1, \tag{1.3}$$

where $0<T<+\infty$, λ is a constant in $\mathbb{C}$, $f\in C\left((0,T]\times\mathbb{C}\right)$, $\psi\in C^n\left[a,b\right]$ such that $\psi'\left(t\right)>0$ on $[a,b]$ and $D_{a+}^{\alpha,\psi}$ is ψ-Riemann-Liouville fractional derivative (ψ-RLFD) of order $\alpha>0$. The notation $\left(I_{a^+}^{\alpha;\psi}y\right)^{[m]}$ will be presented in the next section.

The primary objective of this paper is to investigate the HUS for fractional differential equations (1.1) and (1.2) when $\alpha>0$, utilizing the (k,ψ)-GLT.

The paper is structured as follows. Section 2 revisits crucial definitions, lemmas, and fundamental properties associated with the (k,ψ)-GLT. Section 3 focuses on examining the stability of problems (1.1) and (1.2) and (1.2) and (1.3). Additionally, two examples are provided to demonstrate the applications of the obtained results and the generalizations found in existing literature. The final section of this article presents the conclusion.

2. Preliminaries and basic notations

In this section, we introduce some basic definitions, notations, lemmas and theorems which are used throughout this paper. To simplify the notation and the proof of some results, we will introduce the following notation: $z_{\psi}^{[n]}=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}z$ and $\zeta_y\left(x\right)=\psi\left(x\right)-\psi\left(y\right).$

Definition 2.1 ([17]). Let $\psi \in C^{n}[a,b]$ such that $\psi'(t) > 0$ on $[a,b]$. Then for $n \in \mathbb{N}$

$$AC_{\psi}^{n} = \left\{ g : (a,b) \to \mathbb{R} \text{ and } g_{\psi}^{[n-1]} \in AC[a,b] \right\}.$$

Definition 2.2 ([8]). Let $I = [a,b]$ $(0 < a < b < \infty)$ be a finite interval. Also, let $\psi : I \to \mathbb{R}$ and $\psi'(t) > 0$ for all $t \in I$. Then for $0 \leqslant \sigma < 1$, the space $C_{\sigma;\psi}(I,\mathbb{R})$ of weighted functions g is defined on I as

$$C_{\sigma;\psi}[a,b] = \{ g : (a,b) \to \mathbb{R} \text{ and } (\psi(\cdot) - \psi(a))^{\sigma} g(\cdot) \in C[a,b] \}, \text{ where } C_{0,\psi}(I,\mathbb{R}) = C[a,b]$$

and

$$C_{\sigma;\psi}^{n}[a,b] = \left\{ g : g_{\psi}^{[n-1]} \in C[a,b] \text{ and } g_{\psi}^{[n]} \in C_{\sigma;\psi}[a,b] \right\}, n \in \mathbb{N}, \text{ where } C_{0;\psi}^{n}[a,b] = C^{n}[a,b].$$

Definition 2.3 ([22]). Let (a,b) $(-\infty \leqslant a < b \leqslant \infty)$ be a finite or infinite of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ on (a,b). Then, the ψ-Riemann-Liouville fractional integrals of order α for an integrable function g with respect to another function ψ on $[a,b]$ are defined by

$$I_{a+}^{\alpha;\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (\zeta_u(t))^{\alpha-1} g(u) \psi'(u)\, du, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function. When $\alpha = 0$ we set $I_{a+}^{0;\psi} g(t) = g(t)$.

Lemma 2.4 ([38]). *Let $\alpha, \beta > 0$. Then we have the following semigroup property given by*

$$I_{a+}^{\alpha;\psi} I_{a+}^{\beta;\psi} g(t) = I_{a+}^{\alpha+\beta;\psi} g(t).$$

Lemma 2.5 ([38]). *Let $\alpha > 0$ and $\delta > 0$. If $g(t) = (\psi(t) - \psi(a))^{\delta-1} = (\zeta_a(t))^{\delta-1}$, then*

$$I_{a+}^{\alpha;\psi} g(t) = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\zeta_a(t))^{\alpha+\delta-1}.$$

Definition 2.6 ([22]). Let $n-1 < \alpha \leqslant n \in \mathbb{N}$, $\psi \in C^{n}[a,b]$, $\psi'(t) \neq 0$, $t \in [a,b]$, and $g \in C[a,b]$. Then, the ψ-RLFD of a function g with respect to ψ of order α, is defined by

$$D_{a+}^{\alpha;\psi} g(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^{n} I_{a+}^{n-\alpha;\psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^{n} \int_{a}^{t} (\zeta_u(t))^{n-\alpha-1} g(u) \psi'(u)\, du. \tag{2.2}$$

Lemma 2.7 ([38]). *Let $\alpha > 0$ and $\delta > 0$. If $g(t) = (\zeta_a(t))^{\delta-1}$, then*

$$D_{a+}^{\alpha;\psi} g(t) = \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)} (\zeta_a(t))^{\delta-\alpha-1}.$$

Definition 2.8 ([8]). Let $g, \psi \in C[a,\infty)$ be real valued functions such that ψ is continuous and $\psi'(t) > 0$ on (a,b). Also, let $\rho, k > 0$. The (k,ψ)-GLT of g is defined as

$$\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s) = \int_{a}^{\infty} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\psi'(t)\, dt. \tag{2.3}$$

Then (2.3) is valid for all values of s.

Remark 2.9.

1) If we take $k = 1$ or $\rho = k$, $\psi(t) = t$ in (2.3), then we obtain the classical Laplace transform.
2) If we take $k = 1$ or $\rho = k$ in (2.3), then we obtain the the generalized Laplace transform in [17].
3) If we take $k = 1$ or $\rho = k$, $\psi(t) = \frac{t^\rho}{\rho}$ in (2.3), then we obtain the the generalized Laplace transform in [16].
4) If we take $\omega(t) = 1$ in [18], then we obtain the generalized Laplace transform in (2.3).
5) If we take $k = 1$ and $a = 0$ in (2.3), then we obtain the Definition 3.1 in [11].

Theorem 2.10 ([8])**.** *Let $g, \psi \in C[a, \infty)$ be real valued functions such that ψ is continuous and $\psi'(t) > 0$ on $[a, b)$. Also, let $\rho, k > 0$ and the (k, ψ)-GLT of g exists. Then*

$$\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s) = \frac{1}{k^{1-\frac{\rho}{k}}}\mathcal{L}\left\{g\left(\psi^{-1}\left(\frac{t}{k^{1-\frac{\rho}{k}}} + \psi(a)\right)\right)\right\}(s).$$

Note that $\mathcal{L}\{g\}$ is the classical Laplace transform of g.

Definition 2.11 ([8])**.** A function $g : [0, \infty) \to \mathbb{R}$ is said to be of ψ-exponential order if there exist non-negative constants M, c, T such that $|g(t)| \leqslant Me^{c\psi(t)}$ for $t \geqslant T$.

Theorem 2.12 ([8])**.** *If $g : [a, \infty) \to \mathbb{R}$ is a piecewise continuous function and is $\psi(t)$-exponential order, then (k, ψ)-GLT exists for $s > c$.*

Theorem 2.13 ([8])**.** *If the (k, ψ)-GLT of $g_1 : [a, \infty) \to \mathbb{R}$ exists for $s > d_1$, and the (k, ψ)-GLT of $g_2 : [a, \infty) \to \mathbb{R}$ exists for $s > d_2$, then for any constants c_1 and c_2, the (k, ψ)-GLT of $c_1g_1 + c_2g_2$ exists and*

$$\mathcal{L}_{k,a+}^{\rho;\psi}\{c_1g_1(t) + c_2g_2(t)\}(s) = c_1\mathcal{L}_{k,a+}^{\rho;\psi}\{g_1(t)\}(s) + c_2\mathcal{L}_{k,a+}^{\rho;\psi}\{g_2(t)\}(s)$$

for $s > \max\{d_1, d_2\}$.

Lemma 2.14 ([8])**.**

1) $\mathcal{L}_{k,a+}^{\rho;\psi}\{1\}(s) = \frac{1}{sk^{1-\frac{\rho}{k}}}\ (s > 0)$.
2) $\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\psi(t) - \psi(a))^\beta\right\}(s) = \frac{\Gamma_k((\beta+1)k)}{\left(sk^{1-\frac{\rho}{k}}\right)^{\beta+1}k^\beta} = \frac{\Gamma(\beta+1)}{\left(sk^{1-\frac{\rho}{k}}\right)^{\beta+1}}\ (s > 0)$, $\Re(\beta) > 0$, *where $\Gamma_k(u)$ is the k-gamma function in the half plane and is defined as $\Gamma_k(u) = \int\limits_0^\infty e^{-\frac{z^k}{k}}z^{u-1}du$ and has the equality $\Gamma_k(u) = k^{\frac{u}{k}-1}\Gamma\left(\frac{u}{k}\right)$.*
3) $\mathcal{L}_{k,a+}^{\rho;\psi}\left\{e^{\lambda\psi(t)}\right\}(s) = \frac{e^{\lambda\psi(a)}}{sk^{1-\frac{\rho}{k}}-\lambda}, \left(s > \frac{\lambda}{k^{1-\frac{\rho}{k}}}\right)$.

Definition 2.15 ([8])**.** Let g and h be two exponential order functions such that they are piecewise continuous on the interval $[a, T]$. Then, we define the generalized convolution of g and h as the following

$$\left(g *_\psi h\right)(t) = \int\limits_a^t g(u)\,h\left(\psi^{-1}\left(\zeta_u(t) + \psi(a)\right)\right)\psi'(u)\,du. \tag{2.4}$$

The following lemma gives that g and h are commutative.

Lemma 2.16 ([8])**.** *Let g and h be two exponential order functions such that they are piecewise continuous at each interval $[a, T]$. Then*

$$g *_\psi h = h *_\psi g.$$

Theorem 2.17 ([8])**.** *Let g and h be two exponential order functions such that they are piecewise continuous at each interval $[a, T]$. Then*

$$\mathcal{L}_{k,a+}^{\rho;\psi}\left\{g *_\psi h\right\}(s) = \mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s)\,\mathcal{L}_{k,a+}^{\rho;\psi}\{h(t)\}(s). \tag{2.5}$$

Definition 2.18 ([19]). Mittag-Leffler function of one parameter is denoted by $E_\xi(t)$ and it is defined as

$$E_\xi(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\xi k + 1)},$$

where $t, \xi \in \mathbb{C}$, and $\mathrm{Re}\xi > 0$. If we put $\xi = 1$, then the above equation becomes

$$E_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

The generalization of $E_\xi(t)$ is defined as a function

$$E_{\xi,\eta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\xi k + \eta)},$$

where $t, \xi, \eta \in \mathbb{C}$, $\mathrm{Re}\xi > 0$, and $\mathrm{Re}\eta > 0$.

3. Main results

In the following theorems, we will prove the HUS of the equations (1.1) and (1.2) with initial conditions (1.3) by using the (k, ψ)-GLT.

Theorem 3.1. *Let the function $g(t) \in C_\psi[a, T]$ and of ψ-exponential order such that $g_\psi^{[1]}$ is a piecewise continuous over every finite interval $[a, T]$. Then the (k, ψ)-GLT of $g_\psi^{[1]}$ exists and*

$$\mathcal{L}_{k,a+}^{\rho;\psi}\left\{g_\psi^{[1]}(t)\right\}(s) = sk^{1-\frac{\rho}{k}}\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s) - g(a).$$

Proof. Let $a < t_1 < t_2 < \cdots < t_n < T$ such that $g_\psi^{[1]}$ is discontinuous in these points in $[a, T]$. Then, we have

$$\begin{aligned}\int_a^T e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g_\psi^{[1]}(t)\,\psi'(t)\,dt &= \int_a^T e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g'(t)\,dt \\ &= \int_a^{t_1} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g'(t)\,dt + \sum_{j=1}^{n-1}\int_{t_j}^{t_{j+1}} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g'(t)\,dt \\ &\quad + \int_{t_n}^{T} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g'(t)\,dt.\end{aligned} \tag{3.1}$$

Integrating by parts in (3.1), we have

$$\begin{aligned}&\int_a^T e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g_\psi^{[1]}(t)\,\psi'(t)\,dt \\ &= e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\Big|_a^{t_1} + \sum_{j=1}^{n-1} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\Big|_{t_j}^{t_{j+1}} + e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\Big|_{t_n}^{T}\end{aligned} \tag{3.2}$$

$$+ sk^{1-\frac{\rho}{k}} \int_a^{t_1} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\psi'(t)\,dt + sk^{1-\frac{\rho}{k}} \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\psi'(t)\,dt$$
$$+ sk^{1-\frac{\rho}{k}} \int_{t_n}^{T} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\psi'(t)\,dt.$$

If we rewrite (3.2), we get

$$\int_a^T e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g_\psi^{[1]}(t)\psi'(t)\,dt = e^{-s\zeta_a(T)k^{1-\frac{\rho}{k}}} g(T) - g(a) + sk^{1-\frac{\rho}{k}} \int_{t_n}^{T} e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\psi'(t)\,dt. \tag{3.3}$$

Taking the limit as $T \to \infty$ of both sides of (3.3), we give

$$\begin{aligned}\int_a^\infty e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g_\psi^{[1]}(t)\psi'(t)\,dt &= sk^{1-\frac{\rho}{k}} \int_a^\infty e^{-s\zeta_a(t)k^{1-\frac{\rho}{k}}} g(t)\psi'(t)\,dt - g(a) \\ &= sk^{1-\frac{\rho}{k}} \mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s) - g(a).\end{aligned}$$

So, the proof is complete. □

Now we generalize Theorem 3.1.

Corollary 3.2. *Let $g(t) \in C_\psi^{n-1}[a,T]$ such that $g_\psi^{[j]}$ $(j = 0, 1, \ldots, n-1)$ are ψ-exponential order. Also, let $g_\psi^{[j]}$ be a piecewise continuous over every finite interval $[a,T]$. Then the (k,ψ)-GLT of $g_\psi^{[n]}$ exists and*

$$\mathcal{L}_{k,a+}^{\rho;\psi}\left\{g_\psi^{[n]}(t)\right\}(s) = \left(sk^{1-\frac{\rho}{k}}\right)^n \mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s) - \sum_{j=0}^{n-1}\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1} g_\psi^{[j]}(a).$$

Proof. The proof of the theorem is done by mathematical induction. □

Now we present the (k,ψ)-GLT of the Riemann-Liouville fractional integrals and derivatives of a function g with respect to ψ of order α.

Theorem 3.3. *Let $g(t)$ be piecewise continuous over every finite interval $[a,T]$ and of $\psi(t)$-exponential order. Also let $\alpha > 0$ and $\psi'(t) > 0$. Then*

$$\mathcal{L}_{k,a+}^{\rho;\psi}\left\{I_{a+}^{\alpha;\psi} g(t)\right\}(s) = \frac{\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s)}{\left(sk^{1-\frac{\rho}{k}}\right)^\alpha}.$$

Proof. If we use the change of variable $\tau = \psi^{-1}(\zeta_u(t) + \psi(a))$ in (2.1) we have

$$I_{a+}^{\alpha;\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\zeta_a(\tau))^{\alpha-1} g\left(\psi^{-1}(\zeta_\tau(t) + \psi(a))\right)\psi'(\tau)\,d\tau. \tag{3.4}$$

Taking the (k,ψ)-GLT on both of sides of (3.4), we have

$$\mathcal{L}_{k,a+}^{\rho;\psi}\left\{I_{a+}^{\alpha;\psi} g(t)\right\}(s) = \frac{1}{\Gamma(\alpha)} \mathcal{L}_{k,a+}^{\rho;\psi}\left\{\int_a^t (\zeta_a(\tau))^{\alpha-1} g\left(\psi^{-1}(\zeta_\tau(t) + \psi(a))\right)\psi'(\tau)\,d\tau\right\}(s).$$

Also, using (2.4), Theorem 2.17, and Lemma 2.14 in the last equation, we can write

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{I_{a+}^{\alpha;\psi}g(t)\right\}(s) &= \frac{1}{\Gamma(\alpha)}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha-1} *_{\psi} g(t)\right\}(s)\\ &= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}}\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s) = \frac{\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s)}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}}.\end{aligned}$$

So, the proof is complete. □

Corollary 3.4. *Let* $\alpha > 0$, $g \in AC_{\psi}^{n}[a,b]$ *for any* $b > a$, $\psi \in C_{\sigma;\psi}^{n}[a,b]$ *such that* $\psi'(t) > 0$ *and* $I_{a+}^{n-\alpha-j}g$ *be of* $\psi(t)$*-exponential order. Then*

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{D_{a+}^{\alpha;\psi}g(t)\right\}(s) &= \left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s)\\ &\quad-\sum_{j=0}^{n-1}\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}\left(I_{a+}^{n-\alpha;\psi}g(t)\right)^{[j]}(a).\end{aligned}\tag{3.5}$$

Proof. Taking the (k,ψ)-GLT both of side (2.2), we have

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{D_{a+}^{\alpha;\psi}g(t)\right\}(s) &= \mathcal{L}_{k,a+}^{\rho;\psi}\left\{\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}I_{a+}^{n-\alpha;\psi}g(t)\right\}(s)\\ &= \mathcal{L}_{k,a+}^{\rho;\psi}\left\{\left(I_{a+}^{n-\alpha;\psi}g(t)\right)^{[n]}\right\}(s).\end{aligned}\tag{3.6}$$

Also, using Theorem 3.3 and Corollary 3.2 in (3.6), we obtain

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{D_{a+}^{\alpha;\psi}g(t)\right\}(s) &= \left(sk^{1-\frac{\rho}{k}}\right)^{n}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{I_{a+}^{n-\alpha;\psi}g(t)\right\}(s)-\sum_{j=0}^{n-1}\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}\left(I_{a+}^{n-\alpha;\psi}g(t)\right)^{[j]}(a)\\ &= \left(sk^{1-\frac{\rho}{k}}\right)^{n}\frac{\mathcal{L}_{k,a+}^{\rho;\psi}\{g(t)\}(s)}{\left(sk^{1-\frac{\rho}{k}}\right)^{n-\alpha}}-\sum_{j=0}^{n-1}\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}\left(I_{a+}^{n-\alpha;\psi}g(t)\right)^{[j]}(a).\end{aligned}$$

So, the proof is complete. □

Lemma 3.5. *Let* $\Re(\alpha) > 0$, $\left|\frac{\lambda}{sk^{1-\frac{\rho}{k}}}\right| < 1$. *Then*

$$\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right\}(s) = \frac{1}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}-\lambda}.$$

Proof. From Definition 2.18 and Lemma 2.14, we obtain

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right\}(s) &= \sum_{m=0}^{\infty}\frac{\lambda^{m}}{\Gamma(m\alpha+\alpha)}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{m\alpha+\alpha-1}\right\}(s)\\ &= \sum_{m=0}^{\infty}\frac{\lambda^{m}}{\Gamma(m\alpha+\alpha)}\frac{\Gamma(m\alpha+\alpha)}{\left(sk^{1-\frac{\rho}{k}}\right)^{m\alpha+\alpha}}\\ &= \frac{1}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}}\sum_{m=0}^{\infty}\left(\frac{\lambda^{m}}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}}\right)^{m} = \frac{1}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}-\lambda}.\end{aligned}$$

So, the proof is complete. □

Definition 3.6. We say that equation (1.1) has the HUS with initial conditions (1.3) if there exists a positive constant $K > 0$ with the following property: if for given $\varepsilon > 0$ and a function y such that $\left|\left(D_{a+}^{\alpha,\psi}y\right)(t) - f(t)\right| \leqslant \varepsilon$, there exists a solution y_e of the differential equation (1.1) such that the inequality $|y(t) - y_e(t)| \leqslant \varepsilon K$ holds.

Definition 3.7. We say that equation (1.2) has the HUS with initial conditions (1.3) if there exists a positive constant $K > 0$ with the following property: if for given $\varepsilon > 0$ and a function y such that $\left|\left(D_{a+}^{\alpha,\psi}y\right)(t) - \lambda y(t) - f(t)\right| \leqslant \varepsilon$, there exists a solution y_e of the differential equation (1.2) such that $|y(t) - y_e(t)| \leqslant \varepsilon K$.

Theorem 3.8. *Let $\alpha > 0$, $0 < T < +\infty$, $\psi(t)$ be an increasing and positive function on $(a, b]$ $(-\infty \leqslant a < b \leqslant \infty)$. If a function $y : (0, T] \to \mathbb{C}$ satisfies the inequality*

$$\left|D_{a+}^{\alpha;\psi}y(t) - f(t)\right| \leqslant \varepsilon \tag{3.7}$$

with the initial conditions (1.3) for each $t \in (0, T]$ and some $\varepsilon > 0$, then there exists a solution $y_e : (0, T] \to \mathbb{C}$ of the differential equation (1.1) such that

$$|y(t) - y_e(t)| \leqslant \varepsilon \frac{(\zeta_a(T))^{\alpha}}{\Gamma(\alpha+1)}.$$

Proof. Let

$$Y_1(t) = \left(D_{a+}^{\alpha;\psi}y\right)(t) - f(t) \tag{3.8}$$

for $t \in (0, T]$. Taking the (k, ψ)-GLT on both sides of (3.8) via Corollary 3.4, we have

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\{Y_1(t)\}(s) &= \mathcal{L}_{k,a+}^{\rho;\psi}\left\{D_{a+}^{\alpha;\psi}y(t)\right\}(s) - \mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s)\\ &= \left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}\mathcal{L}_{k,a+}^{\rho;\psi}\{y(t)\}(s)\\ &\quad - \sum_{j=0}^{n-1}\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}\left(I_{a+}^{n-\alpha;\psi}y(t)\right)^{[j]}(a) - \mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s).\end{aligned} \tag{3.9}$$

If we rewrite (3.9) and use the initial conditions (1.3), we obtain

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\{y(t)\}(s) &= \left(sk^{1-\frac{\rho}{k}}\right)^{-\alpha}\mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s) + \sum_{j=0}^{n-1}c_j\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-\alpha-1}\\ &\quad + \left(sk^{1-\frac{\rho}{k}}\right)^{-\alpha}\mathcal{L}_{k,a+}^{\rho;\psi}\{Y_1(t)\}(s).\end{aligned} \tag{3.10}$$

By using the second part of Lemma 2.14 for $\beta = \alpha + j - n$, $\beta = \alpha - 1$, and $\beta = -n + j$, respectively, we have

$$\begin{aligned}\left(sk^{1-\frac{\rho}{k}}\right)^{n-\alpha-j-1} &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha+j-n}\right\}(s)}{\Gamma(\alpha+j-n+1)},\\ \left(sk^{1-\frac{\rho}{k}}\right)^{-\alpha} &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha-1}\right\}(s)}{\Gamma(\alpha)},\\ \left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1} &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{-n+j}\right\}(s)}{\Gamma(-n+j+1)}.\end{aligned} \tag{3.11}$$

If we use (3.11) in (3.10), we get

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\{y(t)\}(s) &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s)\,\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha-1}\right\}(s)}{\Gamma(\alpha)}\\ &+\sum_{j=0}^{n-1}\frac{c_j\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha+j-n}\right\}(s)}{\Gamma(\alpha+j-n+1)}+\frac{\mathcal{L}_{k,a+}^{\rho;\psi}\{Y_1(t)\}(s)\,\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha-1}\right\}(s)}{\Gamma(\alpha)}.\end{aligned}$$

Set

$$y_e(t)=\frac{1}{\Gamma(\alpha)}\left(\left((\zeta_a(t))^{\alpha-1}\right)*_\psi f(t)\right)+\sum_{j=0}^{n-1}\frac{c_j(\zeta_a(t))^{\alpha+j-n}}{\Gamma(\alpha+j-n+1)}. \tag{3.12}$$

Taking the (k,ψ)-GLT on both sides of (3.12) and using (2.5) and Lemma 2.14 one has

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\{y_e(t)\}(s) &= \frac{1}{\Gamma(\alpha)}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{\left((\zeta_a(t))^{\alpha-1}\right)\right\}(s)\,\mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s)\\ &+\sum_{j=0}^{n-1}\frac{c_j\mathcal{L}_{k,a+}^{\rho;\psi}\left\{\left((\zeta_a(t))^{\alpha+j-n}\right)\right\}(s)}{\Gamma(\alpha+j-n+1)}\\ &=\left(sk^{1-\frac{\rho}{k}}\right)^{-\alpha}\mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s)+\sum_{j=0}^{n-1}c_j\left(sk^{1-\frac{\rho}{k}}\right)^{n-\alpha-j-1}.\end{aligned} \tag{3.13}$$

By Corollary 3.4 and (3.11), we get

$$\begin{aligned}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{D_{a+}^{\alpha;\psi}y_e(t)\right\}(s) &= \left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}\mathcal{L}_{k,a+}^{\rho;\psi}\{y_e(t)\}(s)-\sum_{j=0}^{n-1}c_j\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}\\ &=\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}\left\{\frac{1}{\Gamma(\alpha)}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{(\zeta_a(t))^{\alpha-1}\right\}(s)\,\mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s)\right\}\\ &-\sum_{j=0}^{n-1}c_j\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}=\mathcal{L}_{k,a+}^{\rho;\psi}\{f(t)\}(s).\end{aligned}$$

The fact that $\mathcal{L}_{k,a+}^{\rho;\psi}$ is one-to-one infers that

$$\left(D_{a+}^{\alpha;\psi}y_e\right)(t)=f(t).$$

So, $y_e(t)$ is a solution of equation (1.1). If we use (3.13) in (3.10), we get

$$\mathcal{L}_{k,a+}^{\rho;\psi}\{y(t)-y_e(t)\}(s)=\frac{1}{\Gamma(\alpha)}\mathcal{L}_{k,a+}^{\rho;\psi}\left\{Y_1(t)*_\psi(\zeta_a(t))^{\alpha-1}\right\}(s).$$

If we use again the fact that $\mathcal{L}_{k,a+}^{\rho;\psi}$ is one-to-one, we get

$$y(t)-y_e(t)=\frac{1}{\Gamma(\alpha)}\left[Y_1(t)*_\psi(\zeta_a(t))^{\alpha-1}\right]. \tag{3.14}$$

Therefore, from (3.7) and (3.14), it follows that

$$|y(t)-y_e(t)|\leqslant\frac{1}{\Gamma(\alpha)}\left|\int_a^t Y_1(u)(\zeta_u(t))^{\alpha-1}\psi'(u)\,du\right|$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_a^t |Y_1(u)| \left|(\zeta_u(t))^{\alpha-1} \psi'(u)\right| du \\
&\leqslant \frac{\varepsilon}{\Gamma(\alpha)} \int_a^t \left|(\zeta_u(t))^{\alpha-1} \psi'(u)\right| du = \frac{\varepsilon(\zeta_a(t))^{\alpha}}{\Gamma(\alpha+1)} \leqslant \varepsilon \frac{(\zeta_a(T))^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}
$$

which completes the proof. □

Remark 3.9. If $T < \infty$, then (1.1) is Hyers-Ulam stable whith the constant $K = \frac{(\zeta_a(T))^{\alpha}}{\Gamma(\alpha+1)}$. Note that, (1.1) is not Hyers-Ulam stable, at $t = \infty$.

Theorem 3.10. *Let* $\alpha > 0$, $0 < T < +\infty$, λ *be a scaler,* $\psi(t)$ *be an increasing and positive function on* $(a, b]$ $(-\infty \leqslant a < b < +\infty)$ *and* $f(t)$ *be a given real continuous function on* $[0, \infty)$. *If a function* $y : (0, T] \to \mathbb{C}$ *satisfies the following inequality*

$$
\left|D_{a+}^{\alpha;\psi} y(t) - \lambda y(t) - f(t)\right| \leqslant \varepsilon \tag{3.15}
$$

with the initial conditions (1.3) *for each* $t \geqslant 0$ *and some* $\varepsilon > 0$, *then there exists a solution* $y_e : (0, T] \to \mathbb{C}$ *of* (1.2) *such that*

$$
|y(t) - y_e(t)| \leqslant \varepsilon (\zeta_a(T))^{\alpha} E_{\alpha,\alpha+1}\left(|\lambda| (\zeta_a(T))^{\alpha}\right).
$$

Proof. Let

$$
Y_2(t) = \left(D_{a+}^{\alpha;\psi} y\right)(t) - \lambda y(t) - f(t), \quad t \geqslant 0. \tag{3.16}
$$

Taking the (k, ψ)-GLT of (3.16) via Corollary 3.4, we have

$$
\begin{aligned}
\mathcal{L}_{k,a+}^{\rho;\psi} \{Y_2(t)\}(s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{D_{a+}^{\alpha;\psi} y(t)\right\}(s) - \lambda \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) - \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s) \\
&= \left[\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha} - \lambda\right] \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) \\
&\quad - \sum_{j=0}^{n-1} \left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1} \left(I_{a+}^{n-\alpha;\psi} y(t)\right)^{[j]}(a) - \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s).
\end{aligned}
$$

If we use the initial conditions (1.3), Lemma 3.5, and the second part of Lemma 2.14, we can write

$$
\begin{aligned}
\mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s)}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha} - \lambda} + \frac{1}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha} - \lambda} \sum_{j=0}^{n-1} c_j \left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1} + \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{Y_2(t)\}(s)}{\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha} - \lambda} \\
&= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{\left((\zeta_a(t))^{\alpha-1} E_{\alpha,\alpha}\left(\lambda (\zeta_a(t))^{\alpha}\right)\right) *_{\psi} f(t)\right\}(s) \\
&\quad + \mathcal{L}_{k,a+}^{\rho;\psi} \left\{\left((\zeta_a(t))^{\alpha-1} E_{\alpha,\alpha}\left(\lambda (\zeta_a(t))^{\alpha}\right)\right) *_{\psi} \sum_{j=0}^{n-1} \frac{c_j (\zeta_a(t))^{-n+j}}{\Gamma(-n+j+1)}\right\}(s) \\
&\quad + \mathcal{L}_{k,a+}^{\rho;\psi} \left\{\left((\zeta_a(t))^{\alpha-1} E_{\alpha,\alpha}\left(\lambda (\zeta_a(t))^{\alpha}\right)\right) *_{\psi} Y_2(t)\right\}(s).
\end{aligned} \tag{3.17}
$$

Set

$$
\begin{aligned}
y_e(t) &= \left((\zeta_a(t))^{\alpha-1} E_{\alpha,\alpha}\left(\lambda (\zeta_a(t))^{\alpha}\right)\right) *_{\psi} f(t) \\
&\quad + \left((\zeta_a(t))^{\alpha-1} E_{\alpha,\alpha}\left(\lambda (\zeta_a(t))^{\alpha}\right)\right) *_{\psi} \sum_{j=0}^{n-1} \frac{c_j (\zeta_a(t))^{-n+j}}{\Gamma(-n+j+1)}.
\end{aligned} \tag{3.18}
$$

Taking the (k,ψ)-GLT on both sides of (3.18), one has

$$\begin{aligned}\mathcal{L}^{\rho;\psi}_{k,a+}\{y_e(t)\}(s) &= \mathcal{L}^{\rho;\psi}_{k,a+}\left\{\left((\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right) *_\psi \sum_{j=0}^{n-1}\frac{c_j(\zeta_a(t))^{-n+j}}{\Gamma(-n+j+1)}\right\}(s)\\ &\quad + \mathcal{L}^{\rho;\psi}_{k,a+}\left\{\left((\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right) *_\psi f(t)\right\}(s).\end{aligned} \tag{3.19}$$

By Lemma 3.5 and (3.11), we get

$$\begin{aligned}&\mathcal{L}^{\rho;\psi}_{k,a+}\left\{D^{\alpha;\psi}_{a+}y_e(t)-\lambda y_e(t)\right\}(s)\\ &= \mathcal{L}^{\rho;\psi}_{k,a+}\left\{D^{\alpha;\psi}_{a+}y_e(t)\right\}(s)-\lambda\mathcal{L}^{\rho;\psi}_{k,a+}\{y_e(t)\}(s)\\ &= \left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}\mathcal{L}^{\rho;\psi}_{k,a+}\{y_e(t)\}(s)-\sum_{j=0}^{n-1}c_j\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}-\lambda\mathcal{L}^{\rho;\psi}_{k,a+}\{y_e(t)\}(s)\\ &= \left[\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}-\lambda\right]\mathcal{L}^{\rho;\psi}_{k,a+}\{y_e(t)\}(s)-\sum_{j=0}^{n-1}c_j\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}\\ &= \left[\left(sk^{1-\frac{\rho}{k}}\right)^{\alpha}-\lambda\right]\left[\left\{\mathcal{L}^{\rho;\psi}_{k,a+}\left\{\left((\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right)\right\}(s)\sum_{j=0}^{n-1}\frac{c_j^{\alpha}\mathcal{L}^{\rho;\psi}_{k,a+}\left\{(\zeta_a(t))^{-n+j}\right\}(s)}{\Gamma(-n+j+1)}\right.\right.\\ &\quad\left.+\mathcal{L}^{\rho;\psi}_{k,a+}\left\{\left((\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right)\right\}(s)\,\mathcal{L}^{\rho;\psi}_{k,a+}\{f(t)\}(s)\right]-\sum_{j=0}^{n-1}c_j\left(sk^{1-\frac{\rho}{k}}\right)^{n-j-1}\\ &= \mathcal{L}^{\rho;\psi}_{k,a+}\{f(t)\}(s).\end{aligned}$$

The fact that $\mathcal{L}^{\rho;\psi}_{k,a+}$ is one-to-one infers

$$\left(D^{\alpha;\psi}_{a+}y_e\right)(t)-\lambda y_e(t)=f(t).$$

So, $y_e(t)$ is a solution of equation (1.2). By using (3.17) and (3.19), we have

$$\mathcal{L}^{\rho;\psi}_{k,a+}\{y(t)-y_e(t)\}(s)=\mathcal{L}^{\rho;\psi}_{k,a+}\left\{\left((\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right) *_\psi Y_2(t)\right\}(s).$$

If we use again the fact that $\mathcal{L}^{\rho;\psi}_{k,a+}$ is one-to-one, we get

$$y(t)-y_e(t)=Y_2(t) *_\psi \left((\zeta_a(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_a(t))^{\alpha}\right)\right).$$

If we take the absolute value on both sides of last equation, we can write

$$\begin{aligned}|y(t)-y_e(t)| &= \left|\int_a^t Y_2(u)(\zeta_u(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_u(t))^{\alpha}\right)\psi'(u)\,du\right|\\ &\leqslant \int_a^t |Y_2(u)|\left|(\zeta_u(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_u(t))^{\alpha}\right)\psi'(u)\right|du.\end{aligned}$$

Therefore, from (3.15) it follows that

$$\begin{aligned}|y(t)-y_e(t)| &\leqslant \varepsilon\int_a^t\left|(\zeta_u(t))^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\zeta_u(t))^{\alpha}\right)\psi'(u)\right|du\\ &= \varepsilon\sum_{m=0}^{\infty}\frac{|\lambda|^m}{\Gamma(m\alpha+\alpha)}\int_a^t\left|(\zeta_u(t))^{m\alpha+\alpha-1}\psi'(u)\right|du\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon \sum_{m=0}^{\infty} \frac{|\lambda|^m}{\Gamma(m\alpha+\alpha+1)} (\zeta_a(t))^{m\alpha+\alpha} \\
&= \varepsilon (\zeta_a(t))^{\alpha} \sum_{m=0}^{\infty} \frac{(|\lambda| (\zeta_a(t))^{\alpha})^m}{\Gamma(m\alpha+\alpha+1)} \\
&= \varepsilon (\zeta_a(t))^{\alpha} E_{\alpha,\alpha+1}(|\lambda| (\zeta_a(t))^{\alpha}) \leqslant \varepsilon (\zeta_a(T))^{\alpha} E_{\alpha,\alpha+1}(|\lambda| (\zeta_a(T))^{\alpha}),
\end{aligned}
$$

which completes the proof. □

Remark 3.11. If $T < \infty$, then equation (1.2) is Hyers-Ulam stable whith the constant K, where $K = (\zeta_a(T))^{\alpha} E_{\alpha,\alpha+1}(|\lambda| (\zeta_a(T))^{\alpha})$. Note that, (1.2) is not Hyers-Ulam stable, at $t = \infty$.

Remark 3.12. Since the classical Laplace transform is a special case of the (k,ψ)-GLT, the results obtained in Theorem 3.8 include the results of [40] in Theorem 3.2 if we take $k = 1$ or $\rho = k$, $\psi(t) = t$ in (2.3).

Remark 3.13. The results obtained in Theorem 3.8 include the results of [41, Theorem 3.1] if we take $k = 1$ or $\rho = k$, $\psi(t) = \frac{t^{\rho}}{\rho}$ in (2.3) and $y(t) = W(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T$ is an unknown vector function, λ is n^{th} order matrix over the real field $\mathbb{R}$, and $f(t)$ is an n-dimensional locally integrable column vector function on the closed interval $[a, T]$.

Example 3.14. Consider the following initial value problem:

$$
D_{a+}^{\frac{1}{2};\psi} y(t) = \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} + \frac{1}{20}, \quad \left(I_{a^+}^{\frac{1}{2};\psi} y(t)\right)^{[0]}(a) = 0, \tag{3.20}
$$

where $\alpha = \frac{1}{2}$ and $f(t) = \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} + \frac{1}{20}$. For $\frac{1}{20} < \varepsilon \leqslant 1$, we show that by using Lemma 2.7 the function $y_1(t) = (\zeta_a(t))^3$ satisfies

$$
\left| D_{a+}^{\frac{1}{2};\psi} y_1(t) - f(t) \right| = \left| \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} - \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} - \frac{1}{20} \right| = \left| \frac{1}{20} \right| < \varepsilon.
$$

Also, the initial value of $y_1(t)$ is $\left(I_{a^+}^{\frac{1}{2};\psi} y_1(t)\right)^{[0]}(a) = 0$. If we take $f(t) = \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} + \frac{1}{20}$ and $\alpha = \frac{1}{2}$ in (3.12), we write

$$
y_e(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_a^t (\zeta_a(u))^{-\frac{1}{2}} \left[\frac{16}{5\sqrt{\pi}} (\zeta_u(t))^{\frac{5}{2}} + \frac{1}{20} \right] \psi'(u)\, du + \sum_{j=0}^{n-1} \frac{c_j (\zeta_a(t))^{\frac{1}{2}+j-n}}{\Gamma\left(\frac{1}{2}+j-n+1\right)}.
$$

Because of $\alpha = \frac{1}{2}$, we obtain $n = 1$ and $c_0 = 0$. So using (3.12), we have

$$
y_e(t) = \frac{16}{5\sqrt{\pi}} \int_a^t (\zeta_a(u))^{-\frac{1}{2}} \; (\zeta_u(t))^{\frac{5}{2}} \psi'(u)\, du + \frac{1}{20\sqrt{\pi}} \int_a^t (\zeta_a(u))^{-\frac{1}{2}} \; \psi'(u)\, du. \tag{3.21}
$$

If we use the change of variable $u = \psi^{-1}(\psi a) + \xi(\psi(t) - \psi(a)))$ in the first integral of (3.21), we have

$$
y_e(t) = (\zeta_a(t))^3 + \frac{1}{10\sqrt{\pi}} (\zeta_a(t))^{\frac{1}{2}}.
$$

Then

$$
\begin{aligned}
|y_1(t) - y_e(t)| &= \left| (\zeta_a(t))^3 - \left((\zeta_a(t))^3 + \frac{1}{10\sqrt{\pi}} (\zeta_a(t))^{\frac{1}{2}} \right) \right| \\
&= \frac{1}{10\sqrt{\pi}} |\zeta_a(t)|^{\frac{1}{2}}
\end{aligned}
$$

$$= \frac{1}{10} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sqrt{\zeta_a(t)}$$
$$= \frac{2}{20} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sqrt{\zeta_a(t)} = \frac{1}{20} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \sqrt{\zeta_a(t)} < \varepsilon \frac{1}{\Gamma\left(\frac{3}{2}\right)} \sqrt{\zeta_a(t)} < \varepsilon \frac{1}{\Gamma\left(\frac{3}{2}\right)} \sqrt{\zeta_a(T)}.$$

Therefore (3.20) satisfies Theorem 3.8 for $\varepsilon = \frac{1}{20}$ and $K = \frac{\sqrt{\zeta_a(T)}}{\Gamma\left(\frac{3}{2}\right)}$. Thus (3.20) is Hyers-Ulam stable.

Example 3.15. Consider the following initial value problem:

$$D_{a+}^{\frac{1}{2};\psi} y(t) + 7y(t) = \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} + 7(\zeta_a(t))^3 + \frac{1}{20}, \quad \left(I_{a+}^{\frac{1}{2};\psi} y(t)\right)^{[0]}(a) = 0, \tag{3.22}$$

where $\lambda = -7$, $\alpha = \frac{1}{2}$, and $f(t) = \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} + 7(\zeta_a(t))^3 + \frac{1}{20}$. For $\frac{1}{20} < \varepsilon \leqslant 1$ we show that using Lemma 2.7, function $y_1(t) = (\zeta_a(t))^3$ satisfies

$$\left| D_{a+}^{\frac{1}{2};\psi} y_1(t) + 7y_1(t) - f(t) \right| = \left| \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} + 7(\zeta_a(t))^3 - \frac{16}{5\sqrt{\pi}} (\zeta_a(t))^{\frac{5}{2}} - 7(\zeta_a(t))^3 - \frac{1}{20} \right| = \left| \frac{1}{20} \right| < \varepsilon.$$

Also, the initial value of $y_1(t)$ is $\left(I_{a+}^{\frac{1}{2};\psi} y_1(t)\right)^{[0]}(a) = 0$. Because of $\alpha = \frac{1}{2}$, we obtain $n = 1$ and $c_0 = 0$. So, using (3.18), we have

$$y_e(t) = \int_a^t (\zeta_a(u))^{-\frac{1}{2}} \; E_{\frac{1}{2},\frac{1}{2}}\left(-7(\zeta_a(u))^{\frac{1}{2}}\right) \left[\frac{16}{5\sqrt{\pi}} (\zeta_u(t))^{\frac{5}{2}} + 7(\zeta_u(t))^3 + \frac{1}{20}\right] \psi'(u)\, du$$
$$= (\zeta_a(t))^3 + \frac{1}{20} \sum_{m=0}^{\infty} \frac{(-1)^m 7^m (\zeta_a(t))^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2} + 1\right)}.$$

Then

$$|y_1(t) - y_e(t)| = \left| (\zeta_a(t))^3 - (\zeta_a(t))^3 - \frac{1}{20} \sum_{m=0}^{\infty} \frac{(-1)^m 7^m (\zeta_a(t))^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2} + 1\right)} \right|$$
$$= \frac{1}{20} (\zeta_a(t))^{\frac{1}{2}} \; E_{\frac{1}{2},\frac{3}{2}}\left(|-7| (\zeta_a(t))^{\frac{1}{2}}\right)$$
$$< \varepsilon (\zeta_a(t))^{\frac{1}{2}} \; E_{\frac{1}{2},\frac{3}{2}}\left(|-7| (\zeta_a(t))^{\frac{1}{2}}\right) < \varepsilon \sqrt{\zeta_a(T)} \; E_{\frac{1}{2},\frac{3}{2}}\left(|-7| \sqrt{\zeta_a(T)}\right).$$

Therefore equation (3.22) satisfies the conditions of Theorem 3.10 for $\frac{1}{20} < \varepsilon \leqslant 1$ and $K = (\zeta_a(T))^{\frac{1}{2}} \; E_{\frac{1}{2},\frac{3}{2}}\left(|-7| (\zeta_a(T))^{\frac{1}{2}}\right)$. Thus (3.22) is Hyers-Ulam stable.

Remark 3.16. If we take $k = 1$ or $\rho = k$, $\psi(t) = t$, $a = 0$, and $\varepsilon = \frac{1}{10}$ in Example 3.15, it reduces to results of [40, Example 3.5].

4. Conclusion

In this paper, we proved the Hyers-Ulam stability of linear ψ-Riemann-Liouville fractional differential equations using the (k, ψ)-generalized Laplace transform method. In other words, we established sufficient criteria for the Hyers-Ulam stability of linear ψ-Riemann-Liouville fractional linear differential equations using the (k, ψ)-generalized Laplace transform method.

Moreover, we provided a new method to investigate the Hyers-Ulam stability of differential equations. This is the first attempt to use the (k, ψ)-generalized Laplace transform to prove the Hyers-Ulam stability for linear ψ-Riemann-Liouville fractional differential equations.

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