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Existence fixed point in convex extended s-metric spaces with applications

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Abstract

In this paper, we introduce the definition of a convex extended s-metric space and establish the existence of fixed points for some contraction mappings in convex extended s-metric spaces. Additionally, we provide several examples to validate our findings. Furthermore, we apply the main results to approximate solutions of the Fredholm integral equation.

Keywords: Extended s-metric space, convex structure, convex extended s-metric space, fixed point, Fredholm integral equation.

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1. Introduction

Bakhitin introduced the concept of a s-metric in their work [3], while Czerwik [7] extended it to generalize Banach's fixed point theorem in s-metric spaces. This extension involves replacing the triangle inequality with a more generalized inequality incorporating a coefficient $s \geqslant 1$. Subsequently, numerous mathematicians have been studying the fixed point theory in s-metric spaces (see [1, 14–16, 18, 22, 23]). Recently, Kamaran et al. [17] introduced the concept of extended s-metric space. Many other researchers have subsequently demonstrated various existence and uniqueness results regarding fixed points in extended s-metric spaces (see [9, 13, 24, 25]).

Takahashi [26] established a notion of a convex structure and created a term "convex metric space" to a metric space with convex structure in 1970. Kirk [19] and Goebel and Kirk [11] described the convex metric space by term "hyperbolic type space". After that, many researchers have investigated various properties of convex metric space with a fixed point (see [4, 5, 8, 12, 20, 21]).

In this study, we introduce convex extended s-metric spaces and prove strong convergence theorems for certain contraction mappings within these spaces. We present illustrative examples to validate our findings. Furthermore, we utilize the primary outcomes to approximate solutions for the Fredholm integral equation.

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Definition 1.1 ([7]). Let C be a nonempty set. A function $\hbar : C \times C \longrightarrow [0, \infty)$ is called a b-metric with $s \geqslant 1$, if the following conditions hold for $v, r, t \in C$:

(i) $\hbar(v, r) = 0$ if and only if $v = r$;
(ii) $q(v, r) = q(r, v)$ for all r,v $\in C$;
(iii) $\hbar(v, r) \leqslant s[\hbar(v, t) + \hbar(t, r)]$.

A pair $(C, \hbar)$ is said to be a b-metric space.

Definition 1.2 ([17]). A mapping $\hbar_\mu : C \times C \to [0, \infty)$ is called an extended b-metric on a nonempty set C if for any $n, m, v \in C$ and a mapping $\mu : C \times C \to [1, \infty)$, it satisfies the following conditions:

(i) $\hbar_\mu(n, m) = 0$ and $\hbar_\mu(m, n) = 0$ if and only if $n = m$;
(ii) $\hbar_\mu(n, m) = \hbar_\mu(m, n)$ for all n,m $\in C$;
(iii) $\hbar_\mu(n, m) \leqslant \mu(n, m)[\hbar_\mu(n, v) + \hbar_\mu(v, m)]$.

The function $\hbar_\mu$ is known as an extended b-metric and the pair $(C, \hbar_\mu)$ is called an extended b-metric space.

Example 1.3. Let $S = [0, \infty)$. Define functions $\mu : S \times S \to [1, \infty)$ with $\hbar_\mu : S \times S \to [0, \infty)$ as follows: $\mu(u, v) = 1 + u + v$ for all $u, v \in S$ and

$$\hbar_\mu(u, v) = \begin{cases} u + v, & \text{for all } u, v \in S, u \neq v, \\ 0, & \text{for all } u = v. \end{cases}$$

Then $(S, \hbar_\mu)$ is an extended b-metric space.

Definition 1.4 ([2, 10]). Any sequence $\{J_n\}$ in C converges to some $\chi \in C$ if and only if

$$\lim_{n \longrightarrow \infty} \hbar_\mu(J_n, \chi) = \lim_{n \longrightarrow \infty} \hbar_\mu(\chi, J_n).$$

Definition 1.5 ([6]). A sequence $\{J_n\}$ in C is called right-Cauchy (left-Cauchy) if for every $\epsilon > 0$, there exists a positive integer $N = N(\epsilon) > 0$ such that $\hbar_\mu(J_{n_1}, J_{n_2}) < \epsilon$ for all $n_2 \geqslant n_1 > N$ ($\hbar_\mu(J_{n_1}, l_{n_2}) < \epsilon$ for all $n_1 \geqslant n_2 > N$).

Definition 1.6 ([6]). Any sequence $\{J_n\}$ in C is called Cauchy if for each $\epsilon > 0$, there is an integer number $N = N(\epsilon) > 0$ such that $\hbar_\mu(J_{n_1}, J_{n_2}) < \epsilon$ for all $n_1, n_2 > N$.

Definition 1.7 ([8]). Let $(C, \hbar)$ be an s-metric space and $I = [0, 1]$. Let $F : C \times C \times I \to C$ be a continuous function. Then F is said to be the convex structure on C if, for all $v, n, m \in C$, $\tau \in I$, the following holds:

$$\hbar(v, F(n, m; \tau)) \leqslant \tau\hbar(v, n) + (1 - \tau)\hbar(v, m)$$

2. Main results

In this section, we begin by defining convex extended b-metric space.

Definition 2.1. Let $(C, \hbar_\mu)$ be an extended b-metric space with a function $\mu : C \times C \to [1, \infty)$ and $I = [0, 1]$. Let $F : C \times C \times I \to C$ be a continuous mapping. Then F is said to be the convex structure on C if, for all $r, n, m \in C$, $\tau \in I$, the following holds:

$$\hbar_\mu(r, F(n, m; \tau)) \leqslant \tau\hbar_\mu(r, n) + (1 - \tau)\hbar_\mu(r, m).$$

Then, we say that $(C, \hbar_\mu, F)$ is a convex extended s-metric space.

Example 2.2. Consider the extended s-metric space $(S, \hbar_\mu)$ that was defined in Example 1.3. Define the mapping $F : S \times S \times I \to S$ such that $F(u, v; \zeta) = \zeta u + (1 - \zeta)v$ for all $u, v \in S, \zeta \in I$. To prove that F is a convex structure on s, let $u, v \in S, \zeta \in I$. Then

$$\begin{aligned}\hbar_\mu(o, F(u, v, \zeta)) = o + F(u, v, \zeta) &= o + \zeta u + (1 - \zeta)v \\ &= o - \zeta o + \zeta u + \zeta o + (1 - \zeta)v \\ &= \zeta(o + u) + (1 - \zeta)(o + v) = \zeta\hbar_\mu(o, u) + (1 - \zeta)\hbar_\mu(o, v).\end{aligned}$$

Hence, $(S, \hbar_\mu, F)$ is a convex extended s-metric space.

Example 2.3. Let $S = [0, 1]$, define the function $\mu : S \times S \to [1, \infty)$ such that $\mu(u, v) = 1 + |u| + |v|$. We define the function $\hbar_\mu : S \times S \to S$ as

$$\hbar_\mu(u, v) = \begin{cases} u^2 + v^2, & \text{for all } u, v \in S, u \neq v, \\ 0, & \text{for all } u = v. \end{cases}$$

For $I = [0, 1]$, we define the function $F : S \times S \times I \to S$ such that $F(u, v; \zeta) = \zeta u^2 + (1 - \zeta)v^2$. Then, $(S, \hbar_\mu, F)$ is a convex extended s-metric space.

Proof. Indeed,it is clear that the conditions (i) and (ii) in Definition 1.2 hold. We will prove that (iii) in Definition 1.2 holds.

(1) If $u = v$, then (ii) holds.

(2) If $u \neq v$, $u = a$, then

$$\mu(u, v)[\hbar_\mu(u, a) + \hbar_\mu(a, v)] = (1 + |u| + |v|)[0 + a^2 + v^2] = (1 + |u| + |v|)(u^2 + v^2) \geqslant u^2 + v^2 = \hbar_\mu(u, v).$$

(3) If $u \neq v$, $v = a$, then

$$\mu(u, v)[\hbar_\mu(u, a) + \hbar_\mu(a, v)] = (1 + |u| + |v|)[(u^2 + a^2) + 0] = (1 + |u| + |v|)(u^2 + v^2) \geqslant u^2 + v^2 = \hbar_\mu(u, v).$$

(4) If $u \neq v$, $v \neq a$,$u \neq a$, then

$$\begin{aligned}\mu(u, v)[\hbar_\mu(u, a) + \hbar_\mu(a, v)] &= (1 + |u| + |v|)[(u^2 + a^2) + (a^2 + v^2)] \\ &= (1 + |u| + |v|)(u^2 + v^2) \geqslant u^2 + v^2 = \hbar_\mu(u, v).\end{aligned}$$

Hence, the condition (iii) in Definition 1.2 holds, and $(S, \hbar_\mu)$ is an extended b-metric space. Now, to prove that F is a convex structure on s, for each $o, u, v \in S$, and $\zeta \in I$, we have

$$\begin{aligned}\hbar_\mu(o, F(u, v; \zeta)) &= o^2 + (\zeta u^2 + (1 - \zeta)v^2)^2 \\ &\leqslant o^2 + (\zeta u^2 + (1 - \zeta)v^2) \\ &\leqslant o^2 + \zeta o^2 - \zeta o^2 + \zeta u^2 + (1 - \zeta)v^2 \\ &\leqslant \zeta(o^2 + v^2) + (1 - \zeta)(o^2 + v^2) \leqslant \zeta\hbar_\mu(u, o) + (1 - \zeta)\hbar(o, v).\end{aligned}$$

As a result, the claim is valid. □

Remark 2.4. Let $(S, \hbar_\mu, F)$ be a convex extended s-metric space and $H : S \to S$ be a mapping. Then a sequence $\{x_n\}$, where

$$x_{n+1} = F(x_n, Hx_n; \zeta_n), \quad n \in N,$$

is called Maan's iteration sequence for H.

Theorem 2.5. *Let* $(C, \hbar_\mu, F)$ *be a complete convex extended* s*-metric space with a mapping* $\mu : C \times C \to (1, \infty)$ *and* $H : C \to C$ *be a contraction mapping, that is, there exists* $\alpha \in [0, 1)$ *such that*

$$\hbar_\mu(Hs, Hr) \leqslant \alpha \hbar_\mu(s, r),$$

for all $s, r \in C$. *Let us choose* $\nu_0 \in C$ *such that* $\hbar_\mu(\nu_0, H\nu_0) = N < \infty$ *and define* $\nu_n = F(\nu_{n-1}, H\nu_{n-1}; \tau_{n-1})$, *where* $0 \leqslant \tau_{n-1} < 1$ *and* $n \in \mathbb{N}$. *If* $[\tau_{n-1} + \alpha(1 - \tau_{n-1})] \leqslant \frac{1}{(\mu(\nu_{n-1}, \nu_n))^2}$, *then* H *has a unique fixed point in* C.

Proof. We have for any $n \in \mathbb{N}$, $\hbar_\mu(\nu_n, \nu_{n+1}) = \hbar_\mu(\nu_n, F(\nu_n, H\nu_n; \tau_n)) \leqslant (1 - \tau_n)\hbar_\mu(\nu_n, H\nu_n)$ and

$$\begin{aligned}
\hbar_\mu(\nu_n, H\nu_n) &\leqslant \mu(\nu_n, H\nu_n)\hbar_\mu(\nu_n, H\nu_{n-1}) + \mu(\nu_n, H\nu_n)\hbar_\mu(H\nu_{n-1}, H\nu_n)\\
&\leqslant \mu(\nu_n, H\nu_n)\hbar_\mu(F(\nu_{n-1}, H\nu_{n-1}; \tau_{n-1}), H\nu_{n-1}) + \mu(\nu_n, H\nu_n)\alpha\hbar_\mu(\nu_{n-1}, \nu_n)\\
&\leqslant \mu(\nu_n, H\nu_n)[\tau_{n-1}\hbar_\mu(\nu_{n-1}, H\nu_{n-1}) + \alpha(1 - \tau_{n-1})\hbar_\mu(\nu_{n-1}, H\nu_{n-1})]\\
&= \mu(\nu_{n-1}, \nu_n)[\tau_{n-1} + \alpha(1 - \tau_{n-1})]\hbar_\mu(\nu_{n-1}, H\nu_{n-1}).
\end{aligned}$$

Since $[\tau_{n-1} + \alpha(1 - \tau_{n-1})] \leqslant \frac{1}{(\mu(\nu_{n-1}, \nu_n))^2}$ for any $n \in \mathbb{N}$, we get

$$\hbar_\mu(\nu_n, H\nu_n) \leqslant \mu(\nu_{n-1}, \nu_n)[\tau_{n-1} + \alpha(1 - \tau_{n-1})]\hbar_\mu(\nu_{n-1}, H\nu_{n-1}) < \frac{1}{[\mu(\nu_{n-1}, \nu_n)]}\hbar_\mu(\nu_{n-1}, H\nu_{n-1}). \quad (2.1)$$

This means that, the sequence $\{\hbar_\mu(\nu_n, H\nu_n)\}$ of non-negative reals is decreasing. Hence, there exists $\eta \geqslant 0$ such that

$$\lim_{n \longrightarrow \infty} \hbar_\mu(\nu_n, H\nu_n) = \eta.$$

Suppose that $\eta > 0$. Letting $n \to \infty$ in (2.1), we get $\eta \leqslant \frac{1}{[\mu(\nu_{n-1}, \nu_n)]}\eta$. Since $[\mu(\nu_{n-1}, \nu_n)] > 1$, we have

$$0 \leqslant \eta \leqslant \frac{1}{[\mu(\nu_{n-1}, \nu_n)]}\eta < \eta,$$

which is a contradiction. Thus $\eta = 0$ and we have

$$\hbar_\mu(\nu_n, \nu_{n+1}) \leqslant (1 - \tau_n)\hbar_\mu(\nu_n, H\nu_n) < \hbar_\mu(\nu_n, H\nu_n),$$

which means that

$$\lim_{n \longrightarrow \infty} \hbar_\mu(\nu_n, \nu_{n+1}) = 0.$$

Now, we show that $\{\nu_n\}$ is a Cauchy sequence. For any $n, k \in \mathbb{N}$, we have

$$\begin{aligned}
&\hbar_\mu(\nu_n, \nu_{n+k+1})\\
&\quad = \hbar_\mu(\nu_n, F(\nu_{n+k}, H\nu_{n+k}, \tau_{n+k}))\\
&\quad \leqslant \tau_{n+k}\hbar_\mu(\nu_n, \nu_{n+k}) + (1 - \tau_{n+k})\hbar_\mu(\nu_n, H\nu_{n+k})\\
&\quad \leqslant \tau_{n+k}\mu(\nu_n, \nu_{n+k}[\hbar_\mu(\nu_n, H\nu_{n+k}) + \hbar_\mu(H\nu_{n+k}, \nu_{n+k})] + (1 - \tau_{n+k})\hbar_\mu(\nu_n, H\nu_{n+k})\\
&\quad = [\tau_{n+k}\mu(\nu_n, \nu_{n+k}) + 1 - \tau_{n+k}]\hbar_\mu(\nu_n, H\nu_{n+k}) + \tau_{n+k}\mu(\nu_n, \nu_{n+k})\hbar_\mu(H\nu_{n+k}, \nu_{n+k})\\
&\quad \leqslant [\tau_{n+k}\mu(\nu_n, \nu_{n+k}) + 1]\hbar_\mu(\nu_n, H\nu_{n+k}) + \tau_{n+k}\mu(\nu_n, \nu_{n+k})\hbar_\mu(H\nu_{n+k}, \nu_{n+k})\\
&\quad \leqslant \mu(\nu_n, H\nu_{n+k})[\tau_{n+k}\mu(\nu_n, \nu_{n+k}) + 1][\hbar_\mu(\nu_n, H\nu_n) + \hbar_\mu(H\nu_n, H\nu_{n+k})]\\
&\quad \leqslant \mu(\nu_n, H\nu_{n+k})[\tau_{n+k}\mu(\nu_n, \nu_{n+k} + 1]\alpha\hbar_\mu(\nu_n, \nu_{n+k})\\
&\quad \leqslant \mu(\nu_n, H\nu_{n+k})[\tau_{n+k}\mu(\nu_n, \nu_{n+k}) + 1]\mu(\nu_n, H\nu_{n+k-1})[\tau_{n+k-1}\mu(\nu_n, \nu_{n+k-1}) + 1]\alpha^2\hbar_\mu(\nu_n, \nu_{n+k-1})\\
&\qquad \vdots\\
&\quad \leqslant \mu(\nu_n, H\nu_{n+k})[\tau_{n+k}\mu(\nu_n, \nu_{n+k}) + 1]\mu(\nu_n, H\nu_{n+k-1})[\tau_{n+k-1}\mu(\nu_n, \nu_{n+k-1}) + 1]\\
&\qquad \times \cdots \times \mu(\nu_n, H\nu_{n+k-(k-1)})[\tau_{n+k-(k-1)}\mu(\nu_n, \nu_{n+k-(k-1)}) + 1]\alpha^k\hbar_\mu(\nu_n, \nu_n).
\end{aligned}$$

So, $\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, v_{n+k+1}) = 0$ for all k. Hence, $\{v_n\}$ is a Cauchy sequence in the convex extended s-metric space C. Since $(C, \hbar_\mu)$ is complete, the sequence $\{v_n\}$ converges to some $v^* \in C$, that is, $\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, v^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(v^*, v_n) = 0$. The continuity of H yields,

$$\lim_{n\longrightarrow\infty} \hbar_\mu(Hv_n, Hv^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(Hv_{n+1}, Hv^*) = 0.$$

By uniqueness of the limit, we get $Hv^* = v^*$. Therefore, v^* is a fixed point of H. □

Example 2.6. Let $S = [0, 1]$. Define the function $\mu : S \times S \to [1, \infty)$ such that $\mu(u, v) = 1 + u + v$. We define the function $\hbar_\mu : S \times S \to S$ as

$$\hbar_\mu(u, v) = \begin{cases} u^2 + v^2, & \text{for all } u, v \in S, u \neq v, \\ 0, & \text{for all } u = v. \end{cases}$$

For $I = [0, 1]$, we define the function $F : S \times S \times I \to S$ such that $F(u, v; \zeta) = \zeta u^2 + (1 - \zeta)v^2$. Let $H : S \to S$ be a mapping such that $Hv = \frac{v}{5}$ for all $v \in S$. Set $\alpha = \frac{1}{16}$ and $v_n = F(v_{n-1}, Hv_{n-1}; \zeta_{n-1})$, where $v_0 = 1$ and $\zeta_{n-1} \leqslant \frac{16}{15(\mu(u,v))^2} - \frac{1}{15}$. Then H has a unique fixed point in s.

Proof. From Example 2.3, we have $(S, \hbar_\mu, F)$ is a convex extended s-metric space. Let $s, r \in S$. Since $\hbar_\mu(Hs, Hr) = \hbar_\mu(\frac{s}{5}, \frac{r}{5}) = \frac{1}{25}(s^2 - r^2) \leqslant \alpha\hbar_\mu(s, r)$, H satisfies the inequality $\hbar_\mu(Hs, Hr) \leqslant \alpha\hbar_\mu(s, r)$, for all $s, r \in S$. Next, we will prove that H has a unique fixed point in s. In order to do it, we have

$$v_n = \zeta_{n-1}v_{n-1} + (1 - \zeta_{n-1})Hv_{n-1} = (\frac{1}{5} + \frac{4}{5}\zeta_{n-1})v_{n-1},$$

and $v_{n-1} = (\frac{1}{5} + \frac{4}{5}\zeta_{n-2})v_{n-2}$, $v_{n-2} = (\frac{1}{5} + \frac{4}{5}\zeta_{n-3})v_{n-3}$, ..., $v_1 = (\frac{1}{5} + \frac{4}{5}\zeta_0)v_0$. Since $\zeta_n < 1$ for all n, for $M = \max\{\zeta_n\} < 1$, $v_n < (\frac{1}{5} + \frac{4}{5}M)^n v_0$, and $Hv_n = \frac{(\frac{1}{5}+\frac{4}{5}M)^{2n}}{5}(v_0)^2$, we get that $v_n \to 0$ and $Hv_n \to 0$ when $n \to \infty$. Hence, the fixed point of H is $0 \in S$. Now, to prove that the fixed point is unique, suppose that $v^*, q^* \in S$ are distinct fixed points of H. Then,

$$0 < \hbar_\mu(q^*, v^*) = \hbar_\mu(\frac{q^*}{5}, \frac{v^*}{5}) = \frac{1}{25}\hbar_\mu(q^*, v^*),$$

which is a contradiction. Therefore, 0 is the unique fixed point of $H \in S$. □

Theorem 2.7. *Let $(C, \hbar_\mu, F)$ be a complete convex extended s-metric space with a mapping $\mu : C \times C \to [1, \infty)$ and $H : C \to C$ be a contraction mapping, that is, there exists $k \in (0, \frac{1}{2})$ such that*

$$\hbar_\mu(Hv, Hu) \leqslant k[\hbar_\mu(u, Hu) + \hbar_\mu(v, Hv)],$$

for all $u, v \in C$. Let us choose $v_0 \in C$ such that $\hbar_\mu(v_0, Hv_0) = N < \infty$ and define $v_n = F(v_{n-1}, Hv_{n-1}; \tau_{n-1})$, where $0 < \tau_{n-1} \leqslant \frac{1}{4[\mu(r,s)]^2}$ and $n \in \mathbb{N}$. If for any $r, s \in C$, $k \in [0, \frac{1}{4[\mu(r,s)]^2}]$, then H has a unique fixed point in C.

Proof. We have for any $n \in \mathbb{N}$, $\hbar_\mu(v_n, v_{n+1}) = \hbar_\mu(v_n, F(v_n, Hv_n; \tau_n)) \leqslant (1-\tau_n)\hbar_\mu(v_n, Hv_n) \leqslant \hbar_\mu(v_n, Hv_n)$,

$$\begin{aligned}
\hbar_\mu(v_n, Hv_n) &= \hbar_\mu(F(v_{n-1}, Hv_{n-1}; \tau_{n-1}), Hv_n) \\
&\leqslant \tau_{n-1}\hbar_\mu(v_{n-1}, Hv_n) + (1 - \tau_{n-1})\hbar_\mu(Hv_{n-1}, Hv_n) \\
&\leqslant \tau_{n-1}\hbar_\mu(v_{n-1}, Hv_n) + \hbar_\mu(Hv_{n-1}, Hv_n) \\
&\leqslant \tau_{n-1}\hbar_\mu(v_{n-1}, Hv_n) + k\hbar_\mu(v_{n-1}, Hv_{n-1}) + k\hbar_\mu(v_n, Hv_n) \\
&\leqslant \tau_{n-1}\mu(v_{n-1}, Hv_n)[\hbar_\mu(v_{n-1}, Hv_{n-1}) + \hbar_\mu(v_n, Hv_n)] + k\hbar_\mu(v_{n-1}, Hv_{n-1}) + k\hbar_\mu(v_n, Hv_n) \\
&\leqslant \tau_{n-1}\mu(v_{n-1}, Hv_n)\hbar_\mu(v_{n-1}, Hv_{n-1}) + k\tau_{n-1}\mu(v_{n-1}, Hv_n)\hbar_\mu(v_{n-1}, Hv_{n-1})
\end{aligned}$$

$$+ k\tau_{n-1}\mu(\nu_{n-1}, H\nu_n)\hbar_\mu(\nu_n, H\nu_n) + k\hbar_\mu(\nu_{n-1}, H\nu_{n-1}) + k\hbar_\mu(\nu_n, H\nu_n),$$

and so we get

$$\begin{aligned}&[1 - k\tau_{n-1}\mu(\nu_{n-1}, H\nu_n) - k]\hbar_\mu(\nu_n, H\nu_n)\\ &\qquad \leqslant [k\tau_{n-1}\mu(\nu_{n-1}, H\nu_n) + \tau_{n-1}\mu(\nu_{n-1}, H\nu_n) + k]\hbar_\mu(\nu_{n-1}, H\nu_{n-1}),\end{aligned}$$

and $\hbar_\mu(\nu_n, H\nu_n) \leqslant \frac{[k\tau_{n-1}\mu(\nu_{n-1},H\nu_n)+\tau_{n-1}\mu(\nu_{n-1},H\nu_n)+k]}{[1-k\tau_{n-1}\mu(\nu_{n-1},H\nu_n)-k]}\hbar_\mu(\nu_{n-1}, H\nu_{n-1})$. Since

$$\frac{[k\tau_{n-1}\mu(\nu_{n-1}, H\nu_n) + \tau_{n-1}\mu(\nu_{n-1}, H\nu_n) + k]}{[1 - k\tau_{n-1}\mu(\nu_{n-1}, H\nu_n) - k]} \leqslant \frac{9}{11},$$

$$\hbar_\mu(\nu_n, H\nu_n) < \frac{9}{11}\hbar_\mu(\nu_{n-1}, H\nu_{n-1}), \tag{2.2}$$

which enables us to deduce that $\{\hbar_\mu(\nu_n, H\nu_n)\}$ is a decreasing sequence of non-negative reals, hence, there exists $\rho \geqslant 0$ such that $\lim_{n\longrightarrow\infty} \hbar_\mu(\nu_n, H\nu_n) = \rho$. Letting $n \to \infty$ in (2.2), we get that $\rho \leqslant \frac{9}{11}\rho < \rho$, a contradiction. Hence, we obtain that $\rho = 0$, i.e.,

$$\lim_{n\longrightarrow\infty} \hbar_\mu(\nu_n, H\nu_n) = 0.$$

Now, we show that $\{\nu_n\}$ is a Cauchy sequence. For any $n, k \in \mathbb{N}$, we have

$$\begin{aligned}\hbar_\mu(\nu_n, \nu_{n+k+1}) &= \hbar_\mu(\nu_n, F(\nu_{n+k}, H\nu_{n+k}, \tau_{n+k}))\\ &\leqslant \tau_{n+k}\hbar_\mu(\nu_n, \nu_{n+k}) + (1 - \tau_{n+k})\hbar_\mu(\nu_n, H\nu_{n+k})\\ &\leqslant \tau_{n+k}\mu(\nu_n, \nu_{n+k}[\hbar_\mu(\nu_n, H\nu_{n+k}) + \hbar_\mu(H\nu_{n+k}, \nu_{n+k})] + (1 - \tau_{n+k})\hbar_\mu(\nu_n, H\nu_{n+k})\\ &= [\tau_{n+k}\mu(\nu_n, \nu_{n+k}) + 1 - \tau_{n+k}]\hbar_\mu(\nu_n, H\nu_{n+k}) + \tau_{n+k}\mu(\nu_n, \nu_{n+k})\hbar_\mu(H\nu_{n+k}, \nu_{n+k})\\ &\leqslant [\tau_{n+k}\mu(\nu_n, \nu_{n+k}) + 1]\hbar_\mu(\nu_n, H\nu_{n+k}) + \tau_{n+k}\mu(\nu_n, \nu_{n+k})\hbar_\mu(H\nu_{n+k}, \nu_{n+k}).\end{aligned}$$

Hence, $\lim_{n\longrightarrow\infty} \hbar_\mu(\nu_n, \nu_{n+k+1}) = 0$ for all k. It follows that $\{\nu_n\}$ is a Cauchy sequence in the convex extended s-metric space C. Since $(C, \hbar_\mu)$ is complete, the sequence ν_n converges to some $\nu^* \in C$, that is $\lim_{n\longrightarrow\infty} \hbar_\mu(\nu_n, \nu^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(\nu^*, \nu_n) = 0$. The continuity of H yields

$$\lim_{n\longrightarrow\infty} \hbar_\mu(H\nu_n, H\nu^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(H\nu_{n+1}, H\nu^*) = 0.$$

By uniqueness of the limit, we get $H\nu^* = \nu^*$. Therefore, ν^* is a fixed point of H. □

Example 2.8. Let $S = [0, 1]$. For any $\nu, u \in S$ define the following mappings. $H : S \to S$ as $H\nu = \frac{\nu}{4}$, $\mu : S \times S \to [1, \infty)$ such that $\mu(u, \nu) = 1 + u + \nu$, and $\hbar_\mu : S \times S \to S$ as

$$\hbar_\mu(u, \nu) = \begin{cases} u^2 + \nu^2, & \text{for all } u, \nu \in S, u \neq \nu,\\ 0, & \text{for all } u = \nu.\end{cases}$$

And for $I = [0, 1]$, we define the function $F : S \times S \times I \to S$ such that $F(u, \nu; \zeta) = \zeta u^2 + (1 - \zeta)\nu^2$. Set $\zeta_n = \frac{1}{4}$, $K = \frac{1}{16}$, and let $\nu_0 = 1$ be the initial value and $\nu_n = F(\nu_{n-1}, H\nu_{n-1}; \zeta_{n-1})$. Then H has a unique fixed point in s.

Proof. From Example 2.3, we have $(S, \hbar_\mu, F)$ is convex extended s-metric space. In order to prove that H satisfies the following inequality

$$\hbar_\mu(Hu, H\nu) \leqslant K[\hbar_{mu}(u, Hu) + \hbar_{mu}(\nu, H\nu)] \tag{2.3}$$

for any $u, v \in S$, let $u, v \in S$. Then

$$\hbar_{mu}(Hu, Hv) - \frac{1}{16}[\hbar_{mu}(u, Hu) + \hbar_{mu}(v, Hv)] = \frac{1}{16}(u^2 + v^2) - \frac{17}{(16)^2}[u^2 + v^2] = \frac{-1}{(16)^2}(u^2 + v^2) \leqslant 0.$$

Hence, the inequality (2.3) holds. Next, we will prove that H has a unique fixed point. Consider the following sequence; $v_0 = 1$, $Hv_0 = \frac{v_0}{4} = \frac{1}{4}$, $v_1 = F(v_0, Hv_0; \zeta_0) = \zeta_0(v_0)^2 + (1-\zeta_0)(\frac{v_0}{4})^2 = \frac{4^2+3}{4^3}$, $Hv_1 = \frac{v_1}{4} = \frac{4^2+3}{4^4}$, $v_2 = F(v_1, Hv_1; \zeta_1) = \zeta_1(v_1)^2 + (1-\zeta_1)(\frac{v_1}{4})^2 = \frac{(4^2+3)^3}{4^9}$, $Hv_2 = \frac{v_2}{4} = \frac{(4^2+3)^3}{4^{10}}, \ldots$. We can conclude that when $n \to \infty$, $v_n \to 0$ and $Hv_n \to 0$, where 0 is a fixed point of H. Now, suppose that $q^* \neq 0$ is also a fixed point of H in s. Then

$$0 < \hbar_\mu(o, q^*) = \hbar_\mu(Ho, Hq^*) = \frac{1}{16}\hbar_\mu(o, q^*),$$

which is a contradiction. Thus, 0 is the unique fixed point of H in s. □

3. Application

In this section we apply Theorem 2.5 to find the existence and uniqueness solutions for a type of following integral equation:

$$u(t) = f(t) + \int_a^b K(t, \tau, u(\tau))d\tau. \tag{3.1}$$

Theorem 3.1. *Consider the integral equation (3.1) with a continuous function* $K(t, \tau, u(\tau))$, *where* $a \leqslant t, \tau \leqslant b$, *and* $f \in C[a, b]$. *Suppose that for all* $u, v \in C[a, b]$:

[i] *there exist a continuous function* $\Psi : [a, b] \times [a, b] \to \boldsymbol{R}$ *and* $L < \frac{1}{\sqrt{b-a}}$ *such that*

$$|K(t, \tau, u(\tau)) - K(t, \tau, v(\tau))| \leqslant L\,|\Psi(t, \tau)(u(\tau) - v(\tau))|$$

for all $t, \tau \in [a, b]$;

[ii] $\max_{t\in[ab]} \int_{[a,b]} \Psi(t, \tau)d\tau \leqslant 1$,

then the integral equation (3.1) has a unique solution in $C[a, b]$.

Proof. Let $C = C[a, b]$ and define $\hbar_\mu : C \times C \to [0, \infty)$ as

$$\hbar_\mu(u, v) = \max_{a\leqslant t,\tau\leqslant b} |u(t) - v(t)|^2. \tag{3.2}$$

Define $H : C \to C$ by

$$Hu(t) = f(t) + \int_a^b K(t, \tau, u(\tau))d\tau, \tag{3.3}$$

for all $u \in C$. Set

$$u_n = F(u_{n-1}, Hu_{n-1}; \zeta_{n-1}) = \zeta_{n-1}u_{n-1} + (1 - \zeta_{n-1})Hu_{n-1}.$$

Suppose that $(C, \hbar_\mu, F)$ is a complete convex extended s-metric space. From (3.2) and (3.3), we get

$$\begin{aligned}
\hbar_\mu(Hu, Hv) &= \max_{a\leqslant t,\tau\leqslant b} |Hu - Hv|^2 \\
&= \max_{a\leqslant t,\tau\leqslant b} \left|\int_a^b K(t, \tau, u(\tau))d\tau - \int_a^b K(t, \tau, v(\tau))d\tau\right|^2 \\
&\leqslant \max_{a\leqslant t,\tau\leqslant b} \left|\int_a^b |K(t, \tau, u(\tau))d\tau - K(t, \tau, v(\tau))|\, d\tau\right|^2
\end{aligned}$$

$$\leqslant \max_{a\leqslant t,\tau\leqslant b} \left| \int_a^b L\,|\Psi(t,\tau)(u(\tau)-v(\tau))|\,d\tau \right|^2$$
$$\leqslant L^2(b-a)\max_{a\leqslant t,\tau\leqslant b} |(u(\tau)-v(\tau))|^2 = L^2(b-a)\hbar_\mu(u,v).$$

Since $L^2(b-a) < 1$, the mapping H has a unique fixed point, that means the integral equation (3.1) has a unique solution. □

References

[1] M. U. Ali, T. Kamran, M. Postolache, *Solution of Volterra integral inclusion in* b*-metric spaces via new fixed point theorem*, Nonlinear Anal. Model. Control, **22** (2017), 17–30. 1

[2] H. Aydi, N. Bilgili, E. Karapınar, *Common fixed point results from quasi-metric spaces to* G*-metric spaces*, J. Egyptian Math. Soc., **23** (2015), 356–361. 1.4

[3] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., Gos. Ped. Inst. Unianowsk, **30** (1989), 26–37. 1

[4] I. Beg, A. Azam, *Fixed point on starshaped subset of convex metric spaces*, Indian J. Pure Appl. Math., **18** (1987), 594–596. 1

[5] I. Beg, N. Shahzad, M. Iqbal, *Fixed point theorems and best approximation in convex metric space*, Approx. Theory Appl., **8** (1992), 97–105. 1

[6] M. Bota, A. Molnár, C. Varga, *On Ekeland's variational principle in* b*-metric spaces*, Fixed Point Theory, **12** (2012), 21–28. 1.5, 1.6

[7] S. Czerwik, *Contraction mappings in* b*-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11. 1, 1.1

[8] X. P. Ding, *Iteration processes for nonlinear mappings in convex metric spaces*, J. Math. Anal. Appl., **132** (1988), 114–122. 1, 1.7

[9] N. V. Dung, V. T. L. Hang, *Remarks on partial* b*-metric spaces and fixed point theorems*, Mat. Vesnik, **69** (2017), 231–240. 1

[10] A. Felhi, S. Sahmim, H. Aydi, *Ulam-Hyers stability and well-posedness of fixed point problems for* α*-*λ*-contractions on quasi* b*-metric spaces*, Fixed Point Theory Appl., **2016** (2016), 20 pages. 1.4

[11] K. Goebel, W. A Kirk, *Iteration processes for nonexpansive mappings*, In: Topological methods in nonlinear functional analysis (Toronto, Ont., 1982), Amer. Math. Soc., Providence, RI, **21** (1983), 115–123. 1

[12] M. D. Guay, K. L. Singh, J. H. M. Whitfield, *Fixed point Theorems for nonexpansive mappings in convex metric spaces*, In: Nonlinear analysis and applications (St. Johns, Nfld.,1981), Dekker, New York, **80** (1982), 179–189. 1

[13] M. Imdad, M. Asim, R. Gubran, *Common fixed point theorems for* g*-generalized contractive mappings in* b*-metric spaces*, Indian J. Math., **60** (2018), 85–105. 1

[14] A. Kari, A. Al-Rawashdeh, *New fixed point theorems for* $\theta-\omega$*-contraction on* (λ,μ)*-generalized metric spaces*, J. Funct. Space, **2023** (2023), 14 pages. 1

[15] A. Kari, H. A. Hammad, A. Baiz, M. Jamal, *Best proximity point of generalized* $(F-\tau)$ *proximal non-self contractions in generalized metric spaces*, Appl. Math. Inf. Sci., **16** (2022), 853–861.

[16] A. Kari, M. Rossafi, J. R. Lee, *FIXED POINT THEOREMS IN QUASI-METRIC SPACES*, Nonlinear Funct. Anal. Appl., **28** (2023), 311–335. 1

[17] T. Kamran, M. Samreen, Q. UL Ain, *A generalization of b-metric space and some fixed point theorems*, Mathematics, **5** (2017), 7 pages. 1, 1.2

[18] T. Kamran, M. Postolache, M. U. Ali, Q. Kiran, *Feng and Liu type* F*-contraction in* b*-metric spaces with an application to integral equations*, J. Math. Anal., **7** (2016), 18–27. 1

[19] W. A. Kirk, *Krasnoselskiĭ's iteration process in hyperbolic space*, Numer. Funct. Anal. Optim., **4** (1981/82), 371–381. 1

[20] A. Kumar, S. Rathee, *Some common fixed point and invariant approximation results for nonexpansive mappings in convex metric space*, Fixed Point Theory Appl., **2014** (2014), 14 pages. 1

[21] A. Kumar, A. Tas, *Note on common fixed point theorems in convex metric spaces*, Axioms, **10** (2021), 1–7. 1

[22] M. Rossafi, A. Kari, *New fixed point theorems for* (ϕ, F)*-contraction on rectangular b-metric spaces*, Afr. Mat., **34** (2023), 26 pages. 1

[23] M. Rossafi, A. Kari, C. Park, J. R. Lee, *New fixed point theorems for* $\theta-\phi$ *- contraction on b-metric spaces*, J. Math. Comput. Sci., **29** (2023), 12–27. 1

[24] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Some common fixed point results in ordered partial* b*-metric spaces*, J. Inequal. Appl., **2013** (2013), 26 pages. 1

[25] M. Samreen, T. Kamran, M. Postolache, *Extended* b*-metric space, extended* b*-comparison function and nonlinear contractions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **80** (2018), 21–28. 1

[26] W. Takahashi, *A convexity in metric space and nonexpansive mappings. I*, Kodai Math. Sem. Rep., **22** (1970), 142–149. 1