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Equivalence between best proximity point and fixed point for some class of multi-valued mappings

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Abstract

In the year 2016, Pragadeeswarar et al. [V. Pragadeeswarar, M. Marudai, P. Kumam, J. Nonlinear Sci. Appl., **9** (2016), 1911–1921] considered a special class of multi-valued mappings and investigated the existence of best proximity points for such class of mappings which generalizes the fixed point result established by Choudhury et al. [B. S. Choudhury, N. Metiya, Arab J. Math. Sci., **17** (2011), 135–151] in the context of partially ordered metric spaces. In this note, we have showed that the best proximity point result is a direct consequence of the corresponding fixed point result. In the last part of this note, we applied our result in [V. Pragadeeswarar, M. Marudai, P. Kumam, J. Nonlinear Sci. Appl., **9** (2016), 1911–1921, Example 2.2] to validate our claim.

Keywords: Best proximity point, multi-valued mapping, partially ordered metric space, fixed point.

2020 MSC: 47H10, 54H25.

1. Introduction and preliminary notions

Let (W, d) be a metric space and $f : H \to K$ be a mapping, where H, K are non-empty subsets of the metric space W. If $f(H) \cap H \neq \phi$, then one search for a necessary and sufficient condition under which the mapping f has a fixed point. One of the fundamental results in metric fixed point theory is the Banach contraction principle. In the year 1922, Banach [1] proved that if W is a complete metric space and $f : W \to W$ is a contraction mapping (i.e., f satisfies the condition $d(f(x), f(y)) \leqslant \alpha d(x, y)$ for all $x, y \in W$ and for some $\alpha \in (0, 1)$), then f has a unique fixed point. For a mapping $f : H \to K$ if $f(H) \cap H = \phi$, then the mapping f has no fixed points. In this case, one interesting problem is to search for an element $x \in H$ such that $d(x, f(x)) = D(H, K)$, where $D(H, K) = \inf\{d(x, y) : x \in H, y \in K\}$. This element $x \in H$ is called best proximity point of the mapping f. Best proximity points are the generalization of fixed points in case of non-self mappings. Authors often prove best proximity point result to generalize the corresponding fixed point result for self mappings. Pragadeeswarar et al. [4] considered a class of multi-valued non-self mappings and presented a best proximity point result to generalize the fixed point result established by Choudhury et al. [2] in the context of partially ordered metric spaces. In this note, we show that

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the best proximity point result can be proved by the fixed point result and also, we apply our result to [4, Example 2.2] to validate our claim. Recently, Jain et al. [3] developed best proximity point results for generalized multi-valued contractions on partially ordered metric spaces. In the article [5] we firstly proved the corresponding fixed point result and showed that the best proximity point result follows from the fixed point result. So, best proximity point result is a real generalization of the corresponding fixed point result, when the best proximity point result cannot be obtained from the corresponding fixed point result. So, it is interesting to develop best proximity point results which cannot be obtained from the corresponding fixed point theorem.

Let $(X, d, \preceq)$ be a partially ordered metric space and $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of X. Throughout this paper, the class of all non-empty closed and bounded subsets of X are denoted by $CB(X)$ and the class of non-empty bounded subsets of X are denoted as $B(X)$. The following notations are used in this paper:

$$\begin{aligned}
D(\mathcal{A},\mathcal{B}) &= \inf\{d(\alpha,\beta) : \alpha \in \mathcal{A}, \beta \in \mathcal{B}\},\\
\delta(\mathcal{A},\mathcal{B}) &= \sup\{d(\alpha,\beta) : \alpha \in \mathcal{A}, \beta \in \mathcal{B}\},\\
\mathcal{A}' &= \{\alpha \in \mathcal{A} : d(\alpha,\beta) = D(\mathcal{A},\mathcal{B}) \text{ for some } \beta \in \mathcal{B}\},\\
\mathcal{B}' &= \{\beta \in \mathcal{B} : d(\alpha,\beta) = D(\mathcal{A},\mathcal{B}) \text{ for some } \alpha \in \mathcal{A}\}.
\end{aligned}$$

Definition 1.1 ([4]). A function $\psi : [0,\infty) \longrightarrow [0,\infty)$ is said to be an altering distance function if it satisfies the following conditions:

i) ψ is monotonically non-decreasing and continuous;
ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.2 ([4]). Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of a partially ordered set $(X, \preceq)$. The relation between $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \prec_1 \mathcal{B}$ and defined as: for every $\alpha \in \mathcal{A}$ there exists $\beta \in \mathcal{B}$ such that $\alpha \preceq \beta$.

Definition 1.3 ([4]). Let $(\mathcal{A}, \mathcal{B})$ be a pair of non-empty subsets of a metric space X with $\mathcal{A}' \neq \phi$. The pair $(\mathcal{A}, \mathcal{B})$ is said to have the P-property if and only if

$$\left.\begin{aligned} d(\alpha_1,\beta_1) = D(\mathcal{A},\mathcal{B})\\ d(\alpha_2,\beta_2) = D(\mathcal{A},\mathcal{B}) \end{aligned}\right\} \quad \Rightarrow \quad d(\alpha_1,\alpha_2) = d(\beta_1,\beta_2),$$

where $\alpha_1, \alpha_2 \in \mathcal{A}'$ and $\beta_1, \beta_2 \in \mathcal{B}'$.

Let us recall the notion of proximal relation between two non-empty subsets of a partially ordered metric space $(X, d, \preceq)$ from [4].

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of a partially ordered metric space $(X, d, \preceq)$ such that $\mathcal{A}' \neq \phi$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two non-empty subsets of $\mathcal{B}'$. Then proximal relation between $\mathcal{B}_1$ and $\mathcal{B}_2$ denoted by $\mathcal{B}_1 \prec_{(1)} \mathcal{B}_2$ and is defined as: for every $\beta_1 \in \mathcal{B}_1$ with $d(\alpha_1, \beta_1) = D(\mathcal{A}, \mathcal{B})$ there exists $\beta_2 \in \mathcal{B}_2$ with $d(\alpha_2, \beta_2) = D(\mathcal{A}, \mathcal{B})$ such that $\alpha_1 \preceq \alpha_2$.

It is obvious that, for $\mathcal{A} = \mathcal{B}$, $\mathcal{B}_1 \prec_{(1)} \mathcal{B}_2$ reduces to $\mathcal{B}_1 \prec_1 \mathcal{B}_2$. In 2011, Choudhury et al. [2] established the following fixed point result for multivariate mapping in partially ordered metric spaces.

Theorem 1.5. *Let $(X, \preceq)$ be a poset and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $\Psi : X \to B(X)$ be a multi-valued mapping such that the following conditions are satisfied:*

(i) *there exists $\alpha_0 \in X$ such that $\{\alpha_0\} \prec_1 \Psi\alpha_0$;*
(ii) *for $\alpha, \beta \in X$, $\alpha \preceq \beta$ implies $\Psi\alpha \prec_1 \Psi\beta$;*
(iii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$, $\forall n$;*

(iv) $\psi(\delta(\Psi\alpha, \Psi\beta)) \leqslant k\psi(\max\{d(\alpha, \beta), D(\alpha, \Psi\alpha), D(\beta, \Psi\beta), \frac{D(\alpha,\Psi\beta)+D(\beta,\Psi\alpha)}{2}\})$, *for all comparable* $\alpha, \beta \in X$, *where* $0 < k < 1$ *and* ψ *is an altering distance function.*

Then Ψ *has a fixed point.*

To generalize the above fixed point result, Pragadeeswarar et al. [4] established the following best proximity point in partially ordered metric space.

Theorem 1.6. *Let* $(X, \preceq, d)$ *be a partially ordered complete metric space. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be a non-empty closed subsets of the metric space* (X, d) *such that* $\mathcal{A}' \neq \phi$ *and* $(\mathcal{A}, \mathcal{B})$ *satisfies the* P*-property. Let* $\Psi : \mathcal{A} \to CB(\mathcal{B})$ *be a multi-valued mapping such that the following conditions are satisfied:*

(i) *there exists two elements* $\alpha_0, \alpha_1 \in \mathcal{A}'$ *and* $\beta_0 \in \Psi\alpha_0$ *such that* $d(\alpha_1, \beta_0) = D(\mathcal{A}, \mathcal{B})$ *and* $\alpha_0 \preceq \alpha_1$*;*
(ii) $\Psi\alpha_0$ *is included in* $\mathcal{B}'$ *for all* $\alpha_0 \in \mathcal{A}'$ *and* $\psi(\delta(\Psi\alpha, \Psi\beta)) \leqslant k\psi(M(\alpha, \beta)) - k\psi(D(\mathcal{A}, \mathcal{B}))$ *for all* $\alpha \preceq \beta$ *in* $\mathcal{A}$*, where* $k \in (0, 1)$, $M(\alpha, \beta) = \max\{d(\alpha, \beta), D(\alpha, \Psi\alpha), D(\beta, \Psi\beta), \frac{D(\alpha,\Psi\beta)+D(\beta,\Psi\alpha)}{2}\}$ *and* ψ *is an altering distance function also it satisfies* $\psi(s + t) \leqslant \psi(s) + \psi(t)$, *for all* $s, t \in [0, \infty)$*;*
(iii) *for* $\alpha, \beta \in \mathcal{A}', \alpha \preceq \beta$ *implies* $\Psi\alpha \prec_{(1)} \Psi\beta$*;*
(iv) *if* $\{x_n\}$ *is a non-decreasing sequence in* A *such that* $x_n \to x$*, then* $x_n \preceq x$ *for all* n*.*

Then, there exists an element κ *in* $\mathcal{A}$ *such that* $D(\alpha, \Psi\alpha) = D(\mathcal{A}, \mathcal{B})$*. That is,* κ *is a best proximity point of the mapping* Ψ*.*

2. Main result

Now we give our main result.

Theorem 2.1. *Theorem 1.6 is a straightforward consequence of Theorem 1.5.*

Proof. We consider an element α in $\mathcal{A}' = \{a \in \mathcal{A} : d(a, b) = D(\mathcal{A}, \mathcal{B})$, for some $b \in \mathcal{B}\}$ and we define a set $\Omega\alpha = \{\gamma \in \mathcal{A}' : \exists\, \gamma' \in \Psi\alpha$ such that $d(\gamma, \gamma') = D(\mathcal{A}, \mathcal{B})\}$. Since $\Psi\alpha \subseteq \mathcal{B}'$, so, it follows that $\Omega\alpha \neq \phi$. Let $\xi, \eta \in \Omega\alpha$. Then $\exists\, \xi', \eta' \in \Psi\alpha$ such that $d(\xi, \xi') = D(\mathcal{A}, \mathcal{B})$ and $d(\eta, \eta') = D(\mathcal{A}, \mathcal{B})$. So, by the P-property we have $d(\xi, \eta) = d(\xi', \eta')$. Since $\xi', \eta' \in \Psi\alpha$, where $\Psi\alpha \subseteq CB(\mathcal{B})$, which implies $d(\xi', \eta') < r$, where $r > 0$, therefore $d(\xi, \eta) = d(\xi', \eta') < r$, where $\xi, \eta \in \Omega\alpha$. So, $\Omega\alpha$ is a bounded subset of $\mathcal{A}'$.

Let us define a mapping $\chi : \mathcal{A}' \to B(\mathcal{A}')$ by $\chi\alpha = \Omega\alpha$, $\forall \alpha \in \mathcal{A}'$. Now, we have to show that χ has a fixed point. Firstly, let us consider the first assumption in Theorem 1.6. From here, we conclude that $\xi_1 \in \chi\xi_0 \Rightarrow \xi_1 \in \Omega\xi_0$. Also, $\xi_0 \preceq \xi_1$. So, there exists $\xi_0 \in \mathcal{A}'$ such that $\xi_0 \prec_1 \chi\xi_0$, which implies the first condition of Theorem 1.5. Let $\alpha, \beta \in \mathcal{A}'$ with $\alpha \preceq \beta$. Let us consider an element $\xi \in \chi\alpha$ and $\eta \in \chi\beta$. Then, there exists $\xi' \in \Psi\alpha$ and $\eta' \in \Psi\beta$ such that

$$d(\xi, \xi') = D(\mathcal{A}, \mathcal{B}) \text{ and } d(\eta, \eta') = D(\mathcal{A}, \mathcal{B}). \tag{2.1}$$

So,

$$D(\alpha, \Psi\alpha) \leqslant d(\alpha, \xi') \Rightarrow D(\alpha, \Psi\alpha) \leqslant d(\alpha, \xi) + D(\mathcal{A}, \mathcal{B}) \Rightarrow D(\alpha, \Psi\alpha) \leqslant D(\alpha, \chi\alpha) + D(\mathcal{A}, \mathcal{B}) \ (\because \xi \in \chi\alpha). \tag{2.2}$$

Similarly we have

$$D(\beta, \Psi\beta) \leqslant D(\beta, \chi\beta) + D(\mathcal{A}, \mathcal{B}). \tag{2.3}$$

So,

$$\frac{D(\alpha, \Psi\beta) + D(\beta, \Psi\alpha)}{2} \leqslant \frac{D(\alpha, \chi\beta) + D(\beta, \chi\alpha)}{2} + D(\mathcal{A}, \mathcal{B}). \tag{2.4}$$

we denote $M'(\alpha,\beta) = \max\{d(\alpha,\beta), D(\alpha,\chi\alpha), D(\beta,\chi\beta), \frac{D(\alpha,\chi\beta)+D(\beta,\chi\alpha)}{2}\}$, for all $\alpha \preceq \beta$ in $\mathcal{A}'$. So, from (2.1)-(2.4), we have

$$M(\alpha,\beta) \leqslant M'(\alpha,\beta) + D(\mathcal{A},\mathcal{B}). \tag{2.5}$$

Now, using P-property from (2.1) and (2.2), we get

$$d(\xi,\eta) = d(\xi',\eta') \leqslant \delta(\Psi\alpha,\Psi\beta) \quad \Rightarrow \quad \delta(\chi\alpha,\chi\beta) \leqslant \delta(\Psi\alpha,\Psi\beta).$$

Now applying the condition of the *altering distance function* ψ, we have

$$\begin{aligned} \psi(\delta(\chi\alpha,\chi\beta)) \leqslant \psi(\delta(\Psi\alpha,\Psi\beta)) &\Rightarrow \psi(\delta(\chi\alpha,\chi\beta) \leqslant k\psi(M(\alpha,\beta)) - k\psi(D(\mathcal{A},\mathcal{B})) \\ &\Rightarrow \psi(\delta(\chi\alpha,\chi\beta) \leqslant k\psi(M(\alpha',\beta') + D(\mathcal{A},\mathcal{B})) - k\psi(D(\mathcal{A},\mathcal{B})) \\ \text{(from (2.5))} &\Rightarrow \psi(\delta(\chi\alpha,\chi\beta) \leqslant k\psi(M(\alpha',\beta') + k\psi(D(\mathcal{A},\mathcal{B})) - k\psi(D(\mathcal{A},\mathcal{B})) \\ &\Rightarrow \psi(\delta(\chi\alpha,\chi\beta) \leqslant k\psi(M(\alpha,\beta)), \text{ for all } \alpha,\beta \in \mathcal{A}' \text{ with } \alpha \preceq \beta. \end{aligned}$$

Let $\alpha,\beta \in \mathcal{A}'$ with $\alpha \preceq \beta$. Let $\xi \in \Omega\alpha$. Then $\exists\eta \in \Psi\alpha$ such that $d(\xi,\eta) = D(\mathcal{A},\mathcal{B})$. Now, from assumption (iii) of Theorem 1.6, we have $\Psi\alpha \prec_{(1)} \Psi\beta$, so there exists $\eta_1 \in \Psi\beta$ with $d(\xi_1,\eta_1) = D(\mathcal{A},\mathcal{B})$ such that $\xi \preceq \xi_1$. Now, $\xi_1 \in \Omega\beta$ and $\xi \preceq \xi_1$. This shows that $\Omega\alpha \prec_1 \Omega\beta$. Also, third condition of Theorem 1.5 is also satisfied. The fourth criteria holds easily. So, χ has a fixed point. Now we have to show that, the fixed point of χ is the best proximity point of Ψ. For that, let us consider an element κ be a fixed point of χ. So, $\kappa \in \chi\kappa$, where $\kappa \in \mathcal{A}' \subseteq A$. So, $\kappa \in \Omega\kappa$. Therefore $\exists\ \kappa' \in \Psi\kappa$ such that

$$d(\kappa,\kappa') = D(\mathcal{A},\mathcal{B}) \quad \Rightarrow \quad D(\kappa,\Psi\kappa) \leqslant D(\mathcal{A},\mathcal{B}). \tag{2.6}$$

Also,

$$D(\mathcal{A},\mathcal{B}) \leqslant D(\kappa,\Psi\kappa). \tag{2.7}$$

So, from (2.6) and (2.7), we get

$$D(\kappa,\Psi\kappa) = D(\mathcal{A},\mathcal{B}).$$

So, we can say that κ is a best proximity point of Ψ, which is a fixed point of χ. □

Now, we apply our result to [4, Example 2.2] to validate our claim. Let $X = \mathbb{R}^2$ and consider the order $(\alpha,\beta) \preceq (\xi,\eta) \Leftrightarrow \alpha \leqslant \xi$ and $\beta \leqslant \eta$, where $\leqslant$ is usual order in $\mathbb{R}$. Thus, $(X,\preceq)$ is a partially ordered set. Besides, (X,d') is a complete metric space where the metric is defined as

$$d'((\alpha_1,\beta_1),(\alpha_2,\beta_2)) = |\alpha_1-\alpha_2| + |\beta_1-\beta_2|.$$

Let $\mathcal{A} = \{(-6,0),(0,-6),(0,5)\}$ and $\mathcal{B} = \{(-1,0),(0,-1),(0,0),(-1,1),(1,1)\}$ be a closed subset of X. Then $D(\mathcal{A},\mathcal{B}) = 5$, $\mathcal{A} = \mathcal{A}'$, and $\mathcal{B} = \mathcal{B}'$. Let $\Psi : \mathcal{A} \to CB(\mathcal{B})$ be defined by,

$$\Psi(\alpha,\beta) = \begin{cases} \{(0,-1),(0,0)\}, & \text{if } (\alpha,\beta) = (-6,0), \\ \{(-1,1),(0,0),(-1,0)\}, & \text{if } (\alpha,\beta) = (0,-6), \\ \{(1,1),(-1,1)\}, & \text{if } (\alpha,\beta) = (0,5). \end{cases}$$

Pragadeeswarar et al. [4] showed that $(0,5)$ is the best proximity point of Ψ. Let us define a mapping $\chi : \mathcal{A}' \to B(\mathcal{A}')$ such that $\chi(\alpha,\beta) = \Omega(\alpha,\beta)$, where

$$\Omega(\alpha,\beta) = \{(\xi,\eta) \in \mathcal{A}' : \exists\ (\xi,\eta) \in \Psi(\alpha,\beta) \text{ such that } d'((\alpha,\beta),(\xi,\eta)) = D(\mathcal{A},\mathcal{B})\}.$$

So, the mapping χ is defined as

$$\chi(\alpha,\beta)=\begin{cases}\{(0,-6),(0,5)\}, & \text{if } (\alpha,\beta)=(-6,0),\\ \{(-6,0),(0,5)\}, & \text{if } (\alpha,\beta)=(0,-6),\\ \{(0,5)\}, & \text{if } (\alpha,\beta)=(0,5).\end{cases}$$

So, we see that $(0,5)$ is the fixed point in $\mathcal{A}'$ of χ, which is also the best proximity point of Ψ. We can also take [3, Example 2.1] to validate our claim. Here $\mathcal{L}=\{(-7,0),(0,-7),(0,5)\}$ and $\mathcal{M}=\{(-2,0),(0,-2),(0,0),(-2,2),(2,2)\}$ be a closed subset of X. Then $D(\mathcal{L},\mathcal{M})=5$, $\mathcal{L}=\mathcal{L}'$, and $\mathcal{M}=\mathcal{M}'$. Let, $T:\mathcal{L}\to CB(\mathcal{M})$ be defined by

$$T(\alpha,\beta)=\begin{cases}\{(0,-2),(0,0)\}, & \text{if } (\alpha,\beta)=(-7,0),\\ \{(2,2),(-2,2)\}, & \text{if } (\alpha,\beta)=(0,-7),\\ \{(-2,2),(0,0),(0,-2),(2,2)\}, & \text{if } (\alpha,\beta)=(0,5).\end{cases}$$

Now, it can be shown that T satisfies all the conditions of [4, Example 2.2] for $\psi(t)=\frac{t}{2}, t\in[0,\infty)$ and for $k\in[\frac{6}{7},1)$. Here, the mapping χ is defined as

$$\chi(\alpha,\beta)=\begin{cases}\{(0,-7),(0,5)\}, & \text{if } (\alpha,\beta)=(-7,0),\\ \{(0,5)\}, & \text{if } (\alpha,\beta)=(0,-7),\\ \{(0,-7),(0,5)\}, & \text{if } (\alpha,\beta)=(0,5).\end{cases}$$

So, we see that $(0,5)$ is the fixed point in $\mathcal{L}'$ of χ, which is also the best proximity point of T.

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