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Duality for non-smooth semidefinite multiobjective programming problems with equilibrium constraints using convexificators

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Abstract

In this article, we investigate the duality theorems for a class of non-smooth semidefinite multiobjective programming problems with equilibrium constraints (in short, NSMPEC) via convexificators. Utilizing the properties of convexificators, we present Wolfe-type (in short, WMPEC) and Mond-Weir-type (in short, MWMPEC) dual models for the problem NSMPEC. Furthermore, we establish various duality theorems, such as weak, strong, and strict converse duality theorems relating to the primal problem NSMPEC and the corresponding dual models, in terms of convexificators. Numerous illustrative examples are furnished to demonstrate the importance of the established results. Furthermore, we discuss an application of semidefinite multiobjective programming problems in approximating K-means-type clustering problems. To the best of our knowledge, duality results presented in this paper for NSMPEC using convexificators have not been explored before.

Keywords: Semidefinite programming, multiobjective optimization, duality, equilibrium constraints, convexificators.

2020 MSC: 90C22, 90C46, 90C29.

1. Introduction

Multiobjective programming problems (in short, MOP) hold significant importance in practical optimization scenarios, such as business, economics, and various scientific and engineering fields (see, for instance, [6, 36] and the references mentioned therein). It plays a crucial role in making optimal decisions when multiple conflicting objectives must be simultaneously optimized. Several authors have established various results for MOP in various settings (see, for instance, [19, 38, 58, 63, 64] and the references mentioned therein). Nonlinear semidefinite programming problems are essentially a generalization of nonlinear programming problems. In this case, vector variables are substituted with symmetric positive semidefinite matrices. In the past few years, the study of semidefinite programming problems has emerged as a very significant area of modern research, for instance, see [1, 5, 8, 20, 21, 30, 47, 65, 66] and

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numerous authors have been studied comprehensively this problem, for instance, see [16, 50, 52, 53, 68] and the references mentioned therein.

In mathematical programming problems, convexity plays a crucial role as it ensures that a stationary point is a global minimum and first-order necessary optimality conditions are also sufficient for a point to be a global minimum. To deal with the nonconvex nature of many real-world optimization problems, several generalizations of convex functions have been introduced (see, for instance, [2, 34, 57], and the references mentioned therein).

In numerous real-world optimization problems in the field of science, engineering, and various other fields of modern research, non-smooth phenomena occur naturally. To deal with the non-smooth nature of mathematical programming problems, concepts of generalized derivatives and subdifferentials have been developed and studied extensively, for instance, see [9, 11, 35]. Convexificators are a weaker version of the various well-known subdifferentials, such as Clarke [9], Michel-Penot [35], Ioffe-Mordukhovich [25, 40], and Treiman [55]. In general, many of the familiar subdifferentials, such as [9, 25, 35, 40, 55] for a locally Lipschitz function can be considered as convexificators, and these established subdifferentials may include the convex hull of a convexificator, for instance, see [10, 12, 13, 26, 35]. Convexificators have been used to extend various results in non-smooth analysis, for instance, see [28, 32, 33, 46, 48, 59, 70]. In non-smooth semidefinite programming problems (in short, NSDP), Golestani and Nobakhtian [22] introduced constraint qualifications for NSDP and derived necessary and sufficient optimality conditions. Mishra et al. [37] presented optimality and duality results for multiobjective NSDP. Lai et al. [29] used convexificators to establish optimality conditions for multiobjective NSDP incorporated with vanishing constraints. Upadhyay and Singh [61] established optimality and duality for non-smooth semidefinite multiobjective fractional programming problems using convexificators.

In optimization theory, a mathematical programming problem incorporated with some complementarity constraints or variational inequality constraints is commonly referred to as a mathematical programming problem with equilibrium constraints (in short, MPEC). Such problems frequently arise in certain equilibrium applications in engineering and economics that are modelled by variational inequalities, such as chemical engineering [44], telecommunication [45], hydro-economic river basin model [7], etc. In the past few years, numerous authors have studied MPEC extensively, for instance, see [14, 15, 17, 24, 31, 41, 49, 69] and the references cited therein. Using convexificators, Ardali et al. [3] established necessary and sufficient optimality conditions for non-smooth single-objective MPEC. Later, Ardali et al. [4] established optimality criteria for non-smooth multiobjective MPEC with equilibrium constraints.

Duality is the principle through which the same optimization problem can be viewed differently. Duality plays a crucial role in mathematical programming problems as sometimes it is easier to solve the dual problem rather than the primal problem; for instance, see [34, 39, 67]. Wolfe [67] introduced Wolfe duality while Mond and Weir [39] introduced Mond-Weir type duality for differentiable scalar functions, two very popular dual models. Under various assumptions of generalized convexity and subdifferentials, these models were further extended for non-smooth functions in both scalar and multiobjective mathematical programming problems. Guo et al. [23] discussed the Wolfe-type dual model for MPEC and established various duality results for MPEC. Singh et al. [51] established various duality results for multiobjective MPEC. Utilizing convexificators, Pandey and Mishra [42] established several duality results in the context of non-smooth MPEC. Later, Joshi et al. [27] derived sufficient optimality criteria and duality theorems for non-smooth MPEC using generalized convexity. In the past few years, numerous authors have established various results with equilibrium constraints in various settings; for instance, see [18, 54, 56, 60] and the references mentioned therein.

It is worth mentioning that the duality for MPEC using convexificators in Euclidean space has been studied by numerous researchers, for instance, see [27, 42] and the references mentioned therein. However, the duality theorems for NSMPEC have not been explored yet. The main objective of the present article is to address this research gap by formulating WMPEC and MWMPEC dual models and deriving weak, strong, and strict converse duality theorems that relate the corresponding dual models with the primal problem NSMPEC.

Motivated by the works of [4, 27, 42, 51], a class of NSMPEC using convexificators is investigated in the present article. We formulate WMPEC and MWMPEC dual models and derive weak, strong, and strict converse duality theorems that relate the corresponding dual models to the primal problem NSMPEC. Several illustrative non-trivial examples are furnished to demonstrate the significance of the various results established throughout the article.

The primary contributions and novel aspects of the current article are twofold. In the first fold, we extend and generalize the duality theorems derived in [27, 42] from Euclidean space to a more general space, namely, the space of symmetric positive semidefinite matrices. In the second fold, we generalize the duality theorems established in [37] for a more general programming problem NSMPEC. To the best of our knowledge, this is for the first time that the weak, strong, and strict converse duality theorems for NSMPEC are explored using convexificators.

The present article is structured as follows. In Section 2, we revisit some basic definitions and preliminary concepts related to semidefinite matrices and convexificators that will be utilized in the subsequent sections of the article. In Section 3, we recall the optimality criteria developed for NSMPEC. In Section 4, we formulate WMPEC for NSMPEC and derive various duality results. In Section 5, we formulate MWMPEC for NSMPEC and derive the duality results for NSMPEC. In Section 6, an application of semidefinite programming problems with equilibrium in approximating K-Means-type clustering problem is discussed. In Section 7, conclusions are drawn and some future research directions are discussed.

2. Mathematical preliminaries and definitions

In this article, the symbols $\mathbb{R}^n$ and $\mathbb{N}$ are used to denote the n-dimensional Euclidean space and the set consisting of all natural numbers, respectively. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The space of $n \times n$ symmetric matrices, symmetric positive semidefinite matrices, and symmetric positive definite matrices are denoted by $\mathcal{S}^n$, $\mathcal{S}^n_+$, and $\mathcal{S}^n_{++}$, respectively. Let $p, q \in \mathbb{R}^n$. Then the following notation is used in the article:

$$p \prec q \iff p_j < q_j,\ \forall j \in \{1, \ldots, m\}, \qquad p \preceq q \iff \begin{cases} p_j \leqslant q_j, & \forall j \in \{1, \ldots, m\}, \\ p_r < q_r, & \text{for at least one } r \in \{1, \ldots, m\}. \end{cases}$$

For $\mathscr{A}, \mathscr{Z} \in \mathcal{S}^n$, we define the inner product between $\mathscr{A}$ and $\mathscr{Z}$ as $\langle \mathscr{A}, \mathscr{Z} \rangle = \operatorname{trace}(\mathscr{A}\mathscr{Z})$. The norm related to the inner product is referred to as the Frobenius norm, denoted by

$$\| \mathscr{A} \|_{\mathcal{F}} = \operatorname{tr}(\mathscr{A}\mathscr{A})^{1/2} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

Let $\mathcal{B}$ be a nonempty subset of $\mathcal{S}^n$. We use the symbols $\operatorname{cl}\mathcal{B}$, $\operatorname{co}\mathcal{B}$, and $\operatorname{cone}\mathcal{B}$, to denote the closure of $\mathcal{B}$, convex hull of $\mathcal{B}$, and the convex cone (including the origin) generated by $\mathcal{B}$, respectively. Now, we define the following sets that will be utilized in the subsequent sections:

$$\mathcal{B}^- := \{\mathscr{A} \in \mathcal{S}^n : \langle \mathscr{A}, \mathscr{Z} \rangle \leqslant 0,\ \forall \mathscr{Z} \in \mathcal{B}\}, \qquad \mathcal{B}^s := \{\mathscr{A} \in \mathcal{S}^n : \langle \mathscr{A}, \mathscr{Z} \rangle < 0,\ \forall \mathscr{Z} \in \mathcal{B}\}.$$

We recall the following definitions from [22, 37].

Definition 2.1. Let $\mathcal{B}$ be a nonempty subset of $\mathcal{S}^n$ and $\mathscr{A} \in \operatorname{cl}\mathcal{B}$. Then, the contingent cone $T(\mathcal{B}, \mathscr{A})$ at $\mathscr{A}$ is defined as

$$T(\mathcal{B}, \mathscr{A}) := \{\mathbb{V} \in \mathcal{S}^n : \exists \delta_n \downarrow 0 \ \&\ \mathbb{V}_n \to \mathbb{V} \text{ such that } \mathscr{A} + \delta_n \mathbb{V}_n \in \mathcal{B},\ \forall n \in \mathbb{N}\}.$$

Definition 2.2. Let $\Phi : \mathcal{S}^n \to \overline{\mathbb{R}}$ be a function and $\mathscr{A} \in \operatorname{dom}(\Phi)$, where $\operatorname{dom}(\Phi) := \{\mathscr{A} \in \mathcal{S}^n : \ \Phi(\mathscr{A}) \neq \infty\}$. Then we define the lower and upper Dini derivatives of Φ at $\mathscr{A}$ in the direction $\mathbb{V} \in \mathcal{S}^n$ as:

$$\Phi^-(\mathscr{A}; \mathbb{V}) := \liminf_{\lambda \downarrow 0} \frac{\Phi(\mathscr{A} + \lambda \mathbb{V}) - \Phi(\mathscr{A})}{\lambda}, \qquad \Phi^+(\mathscr{A}; \mathbb{V}) := \limsup_{\lambda \downarrow 0} \frac{\Phi(\mathscr{A} + \lambda \mathbb{V}) - \Phi(\mathscr{A})}{\lambda}.$$

Definition 2.3. Consider a function $\Phi : \mathbb{S}^n \to \overline{\mathbb{R}}$. We say that Φ has a upper semi-regular convexificator (USRC), $\partial^*\Phi(\mathscr{A}) \subset \mathbb{S}^n$ at $\mathscr{A} \in \mathrm{dom}(\Phi)$ if $\partial^*\Phi(\mathscr{A})$ is a closed set and for every $\mathbb{V} \in \mathbb{S}^n$ we have

$$\Phi^+(\mathscr{A};\mathbb{V}) \leqslant \sup_{\zeta\in\partial^*\Phi(\mathscr{A})} \langle\zeta,\ \mathbb{V}\rangle.$$

Definition 2.4. Consider a function $\Phi : \mathbb{S}^n \to \overline{\mathbb{R}}$. We say that Φ has a lower semi-regular convexificator (LSRC), $\partial_*\Phi(\mathscr{A}) \subset \mathbb{S}^n$ at $\mathscr{A} \in \mathrm{dom}(\Phi)$ if the set $\partial_*\Phi(\mathscr{A})$ is a closed set and for every $\mathbb{V} \in \mathbb{S}^n$ we have

$$\Phi^-(\mathscr{A};\mathbb{V}) \geqslant \inf_{\zeta\in\partial_*\Phi(\mathscr{A})} \langle\zeta,\ \mathbb{V}\rangle.$$

Definition 2.5. Consider a function $\Phi : \mathbb{S}^n \to \overline{\mathbb{R}}$. Let $\mathscr{A} \in \mathbb{S}^n$ such that $\Phi(\mathscr{A})$ is finite and Φ admits a convexificator $\partial^*\Phi(\mathscr{A})$ at $\mathscr{A}$. Then,

- Φ is ∂^*-convex at $\mathscr{A}$ if and only if $\forall\mathscr{V} \in \mathbb{S}^n$, $\Phi(\mathscr{V}) - \Phi(\mathscr{A}) \geqslant \langle\xi, \mathscr{V} - \mathscr{A}\rangle$, $\forall\xi \in \partial^*\Phi(\mathscr{A})$;
- Φ is strictly ∂^*-convex at $\mathscr{A}$ if and only if $\forall\mathscr{V} \in \mathbb{S}^n$, $\Phi(\mathscr{V}) - \Phi(\mathscr{A}) > \langle\xi, \mathscr{V} - \mathscr{A}\rangle$, $\forall\xi \in \partial^*\Phi(\mathscr{A})$;
- Φ is ∂^*-pseudoconvex at $\mathscr{A}$ if and only if $\forall\mathscr{V} \in \mathbb{S}^n$, $\Phi(\mathscr{V}) < \Phi(\mathscr{A}) \implies \langle\xi, \mathscr{V} - \mathscr{A}\rangle < 0$, $\forall\xi \in \partial^*\Phi(\mathscr{A})$;
- Φ is strictly ∂^*-pseudoconvex at $\mathscr{A}$ if and only if $\forall\mathscr{V}(\neq \mathscr{A}) \in \mathbb{S}^n$, $\Phi(\mathscr{V}) \leqslant \Phi(\mathscr{A}) \implies \langle\xi, \mathscr{V} - \mathscr{A}\rangle < 0$, $\forall\ \xi \in \partial^*\Phi(\mathscr{A})$;
- Φ is ∂^*-quasiconvex at $\mathscr{A}$ if and only if $\forall\mathscr{V} \in \mathbb{S}^n$, $\Phi(\mathscr{V}) \leqslant \Phi(\mathscr{A}) \implies \langle\xi, \mathscr{V} - \mathscr{A}\rangle \leqslant 0$, $\forall\xi \in \partial^*\Phi(\mathscr{A})$.

The subsequent Lemma from [22] will be utilized in the sequel.

Lemma 2.6. *Let $\mathscr{A} \in \mathbb{S}^n$ such that $\langle\mathscr{A},\ \mathscr{Z}\rangle \geqslant 0$, $\forall\mathscr{Z} \in \mathbb{S}^n_+$. Then $\mathscr{A} \in \mathbb{S}^n_+$.*

3. Optimality conditions

In this section, we recall the NSMPEC-tailored ACQ, GS-stationary point and necessary as well as sufficient optimality conditions established for non-smooth semidefinite multiobjective programming problems with equilibrium constraints (NSMPEC) by Upadhyay et al. [62].

Consider the following non-smooth semidefinite multiobjective programming problem with equilibrium constraints:

$$\begin{aligned}
\textbf{NSMPEC}\ \text{Minimize}\ \ & \Phi(\mathscr{A}) = (\Phi_1(\mathscr{A}),\ldots,\Phi_r(\mathscr{A})),\\
\text{subject to}\ \ & \Psi(\mathscr{A}) = (\Psi_1(\mathscr{A}),\ldots,\Psi_p(\mathscr{A})) \leqslant 0,\\
& \Theta(\mathscr{A}) = (\Theta_1(\mathscr{A}),\ldots,\Theta_q(\mathscr{A})) = 0,\\
& \mathtt{K}(\mathscr{A}) = (\mathtt{K}_1(\mathscr{A}),\ldots,\mathtt{K}_m(\mathscr{A})) \geqslant 0,\\
& \mathtt{L}(\mathscr{A}) = (\mathtt{L}_1(\mathscr{A}),\ldots,\mathtt{L}_m(\mathscr{A})) \geqslant 0,
\end{aligned}$$

where $\Phi_i : \mathbb{S}^n \to \overline{\mathbb{R}}$, $i \in \mathbb{I} = \{1,\ldots,r\}$, and $\Psi_i : \mathbb{S}^n \to \overline{\mathbb{R}}$, $i \in \mathbb{I}^* = \{1,\ldots,p\}$, $\Theta_i : \mathbb{S}^n \to \overline{\mathbb{R}}$, $i \in \mathbb{J} = \{1,\ldots,q\}$, $\mathtt{K}_i : \mathbb{S}^n \to \overline{\mathbb{R}}$, $i \in \mathbb{M} = \{1,\ldots,m\}$, and $\mathtt{L}_i : \mathbb{S}^n \to \overline{\mathbb{R}}$, $i \in \mathbb{M} = \{1,\ldots,m\}$ are extended real valued functions. We assume that each function admits USRC. Let $\mathbb{I}_\Psi := \{i|\Psi_i(\overline{\mathscr{A}}) = 0\}$. We define the set of all feasible solutions $\mathscr{F}$ of NSMPEC as

$$\mathscr{F} := \{\mathscr{A} \in \mathbb{S}^n_+|\Psi(\mathscr{A}) \leqslant 0,\ \Theta(\mathscr{A}) = 0,\ \mathtt{K}(\mathscr{A}) \geqslant 0,\ \mathtt{L}(\mathscr{A}) \geqslant 0,\ \mathtt{K}_i(\mathscr{A})\mathtt{L}_i(\mathscr{A}) = 0,\ i \in \mathbb{M}\}.$$

We recall the following definitions of Pareto efficient solutions, local Pareto efficient solutions, and weak Pareto efficient solutions for NSMPEC from [62].

Definition 3.1. Let $\overline{\mathscr{A}} \in \mathscr{F}$. Then $\overline{\mathscr{A}}$ is referred to as a Pareto efficient solution of NSMPEC if there does not exist any other $\mathscr{A} \in \mathscr{F}$ such that $\Phi(\mathscr{A}) \preceq \Phi(\overline{\mathscr{A}})$.

Definition 3.2. Let $\overline{\mathscr{A}} \in \mathscr{F}$. Then $\overline{\mathscr{A}}$ is referred to as a local Pareto efficient solution of NSMPEC if for any neighbourhood $\mathscr{N}$ of $\overline{\mathscr{A}}$ there does not exist any other $\mathscr{A} \in \mathscr{N} \cap \mathscr{F}$ such that $\Phi(\mathscr{A}) \preceq \Phi(\overline{\mathscr{A}})$.

Definition 3.3. Let $\overline{\mathscr{A}} \in \mathscr{F}$. Then $\overline{\mathscr{A}}$ is referred to as a weak Pareto efficient solution of NSMPEC if there does not exist any other $\mathscr{A} \in \mathscr{F}$ such that $\Phi(\mathscr{A}) \prec \Phi(\overline{\mathscr{A}})$.

For convenience, we define the subsequent index sets which will be utilized in the upcoming sections of the article:

$$\begin{aligned}
\delta &:= \delta(\overline{\mathscr{A}}) := \{i | \mathtt{K}_i(\overline{\mathscr{A}}) = 0, \mathtt{L}_i(\overline{\mathscr{A}}) > 0\},\\
\omega &:= \omega(\overline{\mathscr{A}}) := \{i | \mathtt{K}_i(\overline{\mathscr{A}}) = 0, \mathtt{L}_i(\overline{\mathscr{A}}) = 0\},\\
\gamma &:= \gamma(\overline{\mathscr{A}}) := \{i | \mathtt{K}_i(\overline{\mathscr{A}}) > 0, \mathtt{L}_i(\overline{\mathscr{A}}) = 0\}.
\end{aligned}$$

The set ω is known as the degenerate set. We say that the feasible point $\overline{\mathscr{A}}$ satisfies the strict complementarity condition if ω is empty. Throughout the article, we are considering the case where ω is nonempty. For convenience, we introduce the following notation which will be utilized in the upcoming sections of the article:

$$\begin{aligned}
\mathcal{F} &:= \bigcup_{i \in \mathbb{I}} \operatorname{co} \partial^* \Phi_i(\overline{\mathscr{A}}),\\
\mathcal{F}^i &:= \bigcup_{j \in \mathbb{I} \setminus \{i\}} \operatorname{co} \partial^* \Phi_j(\overline{\mathscr{A}}),\\
\mathcal{G} &:= \bigcup_{i \in \mathbb{I}_\Psi} \operatorname{co} \partial^* \Psi_i(\overline{\mathscr{A}}),\\
\mathcal{H} &:= \bigcup_{i \in \mathbb{J}} \operatorname{co} \partial^* \Theta_i(\overline{\mathscr{A}}) \cup \operatorname{co} \partial^* (-\Theta_i)(\overline{\mathscr{A}}),\\
\mathcal{G}_\delta &:= \bigcup_{i \in \delta} \operatorname{co} \partial^* \mathtt{K}_i(\overline{\mathscr{A}}) \cup \operatorname{co} \partial^* (-\mathtt{K}_i)(\overline{\mathscr{A}}),\\
\mathcal{H}_\gamma &:= \bigcup_{i \in \gamma} \operatorname{co} \partial^* \mathtt{L}_i(\overline{\mathscr{A}}) \cup \operatorname{co} \partial^* (-\mathtt{L}_i)(\overline{\mathscr{A}}),\\
\mathcal{G}_\omega &:= \bigcup_{i \in \omega} \operatorname{co} \partial^* \mathtt{K}_i(\overline{\mathscr{A}}),\\
\mathcal{H}_\omega &:= \bigcup_{i \in \omega} \operatorname{co} \partial^* \mathtt{L}_i(\overline{\mathscr{A}}),\\
(\mathcal{G}\mathcal{H})_\omega &:= \bigcup_{i \in \omega} \operatorname{co} \partial^* (-\mathtt{K}_i)(\overline{\mathscr{A}}) \cup \operatorname{co} \partial^* (-\mathtt{L}_i)(\overline{\mathscr{A}}),\\
\Gamma(\overline{\mathscr{A}}) &:= (\mathcal{F}^{i_0})^- \cap \mathcal{G}^- \cap \mathcal{H}^- \cap \mathcal{G}_\delta^- \cap \mathcal{H}_\gamma^- \cap \mathcal{G}_\omega^- \cap \mathcal{H}_\omega^- \cap \mathbb{S}_+^n,\\
\mathscr{T}(\overline{\mathscr{A}}) &:= (\mathcal{F}^{i_0})^- \cap \mathcal{G}^- \cap \mathcal{H}^- \cap \mathcal{G}_\delta^- \cap \mathcal{H}_\gamma^- \cap \mathcal{G}_\omega^- \cap \mathcal{H}_\omega^- \cap (\mathcal{G}\mathcal{H})_\omega^- \cap \mathbb{S}_+^n,\\
\Lambda(\overline{\mathscr{A}}) &:= (\mathcal{F}^{i_0})^s \cap \mathcal{G}^s \cap \mathcal{H}^s \cap \mathcal{G}_\delta^s \cap \mathcal{H}_\gamma^s \cap \mathcal{G}_\omega^s \cap \mathcal{H}_\omega^s \cap (\mathcal{G}\mathcal{H})_\omega^s,\\
\mathcal{S} &:= \{\mathscr{A} \in \mathbb{S}_+^n | \Psi(\mathscr{A}) \leqslant 0,\ \Theta(\mathscr{A}) := 0,\ \mathtt{K}(\mathscr{A}) \geqslant 0,\ \mathtt{L}(\mathscr{A}) \geqslant 0,\ \mathtt{K}_i(\mathscr{A})\mathtt{L}_i(\mathscr{A}) = 0,\ i \in \mathbb{M}\},\\
\mathcal{S}^i &:= \{\mathscr{A} \in \mathbb{S}_+^n | \Phi_j(\mathscr{A}) \leqslant \Phi_j(\overline{\mathscr{A}}),\ \forall j \in \mathbb{I} \setminus \{i\},\ \Psi(\mathscr{A}) \leqslant 0,\ \Theta(\mathscr{A}) = 0,\\
&\qquad \mathtt{K}(\mathscr{A}) \geqslant 0,\ \mathtt{L}(\mathscr{A}) \geqslant 0,\ \mathtt{K}_i(\mathscr{A})\mathtt{L}_i(\mathscr{A}) = 0,\ i \in \mathbb{M}\}.
\end{aligned}$$

We recall the following definition of NSMPEC-tailored ACQ, introduced by Upadhyay et al. [62].

Definition 3.4 (NSMPEC-tailored ACQ)**.** The NSMPEC-tailored ACQ is satisfied at $\overline{\mathscr{A}} \in \mathcal{S}$ if for every $i_0 \in \mathbb{I}$,

$$D^{i_0} := \text{cone co } \mathcal{F}^{i_0} + \text{cone co } \mathcal{G} + \text{cone co } \mathcal{H} + \text{cone co } \mathcal{G}_\delta + \text{cone co } \mathcal{H}_\gamma + \text{cone co } \mathcal{G}_\omega + \text{cone co } \mathcal{H}_\omega + \text{cone co } (\mathcal{G}\mathcal{H})_\omega - S_n^+$$

is closed and $\mathscr{T}(\overline{\mathscr{A}}) \subset T(\mathcal{S}^{i_0}, \overline{\mathscr{A}})$.

We recall the following definition of GS-stationary point of NSMPEC from [62].

Definition 3.5 (GS-stationary)**.** We say that $\overline{\mathscr{A}} \in \mathscr{F}$ is a generalized strong stationary (GS-stationary) point if there exist vectors $\nu = (\nu^\Phi, \nu^\Psi, \nu^\Theta, \nu^K, \nu^L) \in \mathbb{R}^{r+p+q+2m}$, $\tau = (\tau^\Theta, \tau^K, \tau^L) \in \mathbb{R}^{q+2m}$, and $\bar{U} \in S_+^n$, such that

$$\begin{aligned} 0 \in & \sum_{i\in\mathbb{I}} \nu_i^\Phi \operatorname{co} \partial^*\Phi(\overline{\mathscr{A}}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \operatorname{co} \partial^*\Psi_i(\overline{\mathscr{A}}) + \sum_{i\in\mathbb{J}} [\nu_i^\Theta \operatorname{co} \partial^*\Theta_i(\overline{\mathscr{A}}) + \tau_i^\Theta \operatorname{co} \partial^*(-\Theta_i)(\overline{\mathscr{A}})] \\ & + \sum_{i\in\mathbb{M}} [\nu_i^K \operatorname{co} \partial^*(-K_i)(\overline{\mathscr{A}}) + \nu_i^L \operatorname{co} \partial^*(-L_i)(\overline{\mathscr{A}})] + \sum_{i\in\mathbb{M}} [\tau_i^K \operatorname{co} \partial^* K_i(\overline{\mathscr{A}}) + \tau_i^L \operatorname{co} \partial^* L_i(\overline{\mathscr{A}})] - \bar{U}, \end{aligned}$$

$$\begin{aligned} &\langle \bar{U}, \overline{\mathscr{A}} \rangle = 0,\ \nu_i^\Phi > 0,\ i \in \mathbb{I},\ \nu_i^\Psi \geqslant 0,\ i \in \mathbb{I}_\Psi,\ \nu_i^\Theta,\ \tau_i^\Theta \geqslant 0,\ i \in \mathbb{J},\ \nu_i^K,\ \nu_i^L,\ \tau_i^K,\ \tau_i^L, \geqslant 0,\ i \in \mathbb{M}, \\ &\nu_\gamma^K = \nu_\delta^L = \tau_\gamma^K = \tau_\delta^L = 0,\ \forall i \in \omega,\ \ \tau_i^K = 0,\ \tau_i^L = 0. \end{aligned}$$

In the following theorem Upadhyay et al. [62] established that the GS-stationary condition for NSMPEC is the necessary first-order optimality condition for a locally Pareto efficient solution of NSMPEC.

Theorem 3.6. *Suppose that $\overline{\mathscr{A}}$ is a locally Pareto efficient solution of NSMPEC. Suppose that at $\overline{\mathscr{A}}$, Φ_i, $i \in \mathbb{I}$, Ψ_i, $i \in \mathbb{I}_\Psi$ $\pm\Theta_i$, $i \in \mathbb{J}$, $\pm K_i$, $i \in \mathbb{M}$, and $\pm L_i$, $i \in \mathbb{M}$ admit bounded USRC. Assume that NSMPEC-tailored ACQ holds at $\overline{\mathscr{A}}$. Then $\overline{\mathscr{A}}$ is a GS-stationary point.*

Under the assumptions of generalized convexity, Upadhyay et al. [62] established sufficient optimality conditions for weak Pareto efficient and Pareto efficient solutions of NSMPEC.

Theorem 3.7. *Let us assume that $\overline{\mathscr{A}} \in \mathscr{F}$ is a GS-stationary point of NSMPEC. Consider the index sets:*

$$\delta_\tau^+ := \{i \in \delta :\ \tau_i^K > 0\}, \quad \gamma_\tau^+ := \{i \in \gamma :\ \tau_i^L > 0\}.$$

Assume that Φ_i, $i \in \mathbb{I}$ are ∂^*-pseudoconvex and Ψ_i, $i \in \mathbb{I}_\Psi$, $\pm\Theta_i$, $i \in \mathbb{J}$, $(-K_i)$, $i \in (\delta \cup \omega)$, and $(-L_i)$, $i \in (\gamma \cup \omega)$, are ∂^*-quasiconvex at $\overline{\mathscr{A}}$. Then,

1. *if $\delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$, then $\overline{\mathscr{A}}$ is a weak Pareto efficient solution of NSMPEC;*
2. *if Φ_i, $i \in \mathbb{I}$, are strictly ∂^*-pseudoconvex at $\overline{\mathscr{A}}$, then $\overline{\mathscr{A}}$ is a Pareto efficient solution of NSMPEC.*

4. Wolfe duality

Within the section, we present the WMPEC dual model for the primal problem NSMPEC. Furthermore, we establish the duality theorems that relate WMPEC with NSMPEC. We introduce the index sets that are utilized in the sequel:

$$\omega_\tau^K := \{i \in \omega : \tau_i^K > 0,\ \tau_i^L = 0\}, \qquad \omega_\tau^L := \{i \in \omega : \tau_i^K = 0,\ \tau_i^L > 0\}.$$

Now, we formulate the WMPEC problem related to the NSMPEC as

$$\begin{aligned} \underset{(\mathscr{Z},\nu)}{\text{Maximize}} \Bigg\{ & \Phi(\mathscr{Z}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} [\nu_i^\Theta \Theta_i(\mathscr{Z}) + \tau_i^\Theta(-\Theta_i)(\mathscr{Z})] \\ & + \sum_{i\in\mathbb{M}} [\nu_i^K(-K_i)(\mathscr{Z}) + \nu_i^L(-L_i)(\mathscr{Z})] - \langle U, \mathscr{Z} \rangle \Bigg\}, \end{aligned}$$

subject to $\left(\mathscr{Z}, \nu^{\Phi}, \nu^{\Psi}, \nu^{\Theta}, \tau^{\Theta}, \nu^{K}, \nu^{L}, U\right) \in \mathscr{F}_W$, where $\Phi(\mathscr{Z}) = (\Phi_1(\mathscr{Z}), \dots, \Phi_r(\mathscr{Z}))$ and $\mathscr{F}_W$ denotes the set of all feasible solutions of WMPEC and is defined as:

$$\begin{aligned}\mathscr{F}_W := \Big\{ & \nu = (\nu^{\Phi}, \nu^{\Psi}, \nu^{\Theta}, \tau^{\Theta}, \nu^{K}, \nu^{L}, U) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^n_+, \\ & \mathscr{Z} \in \mathbb{S}^n_+ : \ 0 \in \sum_{i \in \mathbb{I}} \nu_i^{\Phi} \mathrm{co}\partial^* \Phi_i(\mathscr{Z}) + \sum_{i \in \mathbb{I}_\Psi} \nu_i^{\Psi} \mathrm{co}\partial^* \Psi_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} [\nu_i^{\Theta} \mathrm{co}\partial^* \Theta_i(\mathscr{Z}) \\ & \quad + \tau_i^{\Theta} \mathrm{co}\partial^*(-\Theta_i)(\mathscr{Z})] + \sum_{i \in \mathbb{M}} [\nu_i^{K} \mathrm{co}\partial^*(-K_i)(\mathscr{Z}) + \nu_i^{L} \mathrm{co}\partial^*(-L_i)(\mathscr{Z})] - U, \\ & \nu_i^{\Phi} > 0, i \in \mathbb{I},\ \nu_i^{\Psi} \geqslant 0,\ i \in \mathbb{I}_\Psi,\ \nu_i^{\Theta},\ \tau_i^{\Theta} \geqslant 0,\ i \in \mathbb{J},\ \nu_i^{K},\ \nu_i^{L},\ \tau_i^{K},\ \tau_i^{L} \geqslant 0,\ i \in \mathbb{M}, \\ & \nu_\gamma^{K} = \nu_\delta^{L} = \tau_\gamma^{K} = \tau_\delta^{L} = 0,\ \tau_i^{K} = \tau_i^{L} = 0, \forall i \in \omega \Big\}.\end{aligned}$$

In the subsequent theorems, we derive various duality theorems, such as weak, strong, and strict converse duality theorems that relate WMPEC and NSMPEC.

Theorem 4.1 (Weak duality). *Let $\mathscr{A} \in \mathscr{F}$ and let $(\mathscr{Z}, \nu) \in \mathscr{F}_W$. Suppose that Φ_i, $i \in \mathbb{I}$, Ψ_i, $i \in \mathbb{I}_\Psi$, $\pm\Theta_i$, $i \in \mathbb{J}$, $(-K_i)$, $i \in (\delta \cup \omega)$, $(-L_i)$, $i \in (\gamma \cup \omega)$ admit bounded USRC and are ∂^*-convex functions at $\mathscr{Z}$. Assume that $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$. Then, we have*

$$\begin{aligned}\Phi(\mathscr{A}) \geqslant \Phi(\mathscr{Z}) &+ \sum_{i \in \mathbb{I}_\Psi} \nu_i^{\Psi} \Psi_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} [\nu_i^{\Theta} \Theta_i(\mathscr{Z}) + \tau_i^{\Theta}(-\Theta_i)(\mathscr{Z})] \\ &+ \sum_{i \in \mathbb{M}} [\nu_i^{K}(-K_i)(\mathscr{Z}) + \nu_i^{L}(-L_i)(\mathscr{Z})] - \langle U, \mathscr{Z} \rangle.\end{aligned}$$

Proof. Let $\mathscr{A}$ be any feasible solution of NSMPEC. Since Φ_i, $i \in \mathbb{I}$ is ∂^*-convex at $\mathscr{Z}$, then we have

$$\Phi_i(\mathscr{A}) - \Phi_i(\mathscr{Z}) \geqslant \langle \xi_i^1, \mathscr{A} - \mathscr{Z} \rangle,\ \forall \xi_i^1 \in \partial^* \Phi_i(\mathscr{Z}),\ \forall i \in \mathbb{I}. \tag{4.1}$$

Similarly, by the ∂^*-convexity of $\Psi_i, i \in \mathbb{I}_\Psi$, $\pm\Theta_i, i \in \mathbb{J}$, $(-K_i), i \in (\delta \cup \omega)$, $(-L_i), i \in (\gamma \cup \omega)$ at $\mathscr{Z}$, we have

$$\Psi_i(\mathscr{A}) - \Psi_i(\mathscr{Z}) \geqslant \langle \xi_i^2, \mathscr{A} - \mathscr{Z} \rangle,\ \forall \xi_i^2 \in \partial^* \Psi_i(\mathscr{Z}),\ \forall i \in \mathbb{I}_\Psi, \tag{4.2}$$
$$\Theta_i(\mathscr{A}) - \Theta_i(\mathscr{Z}) \geqslant \langle \xi_i^3, \mathscr{A} - \mathscr{Z} \rangle,\ \forall\ \xi_i^3 \in \partial^* \Theta_i(\mathscr{Z}),\ \forall i \in \mathbb{J}, \tag{4.3}$$
$$(-\Theta_i)(\mathscr{A}) - (-\Theta_i)(\mathscr{Z}) \geqslant \langle \xi_i^4, \mathscr{A} - \mathscr{Z} \rangle,\ \forall \xi_i^4 \in \partial^*(-\Theta_i)(\mathscr{Z}),\ \forall i \in \mathbb{J}, \tag{4.4}$$
$$(-K_i)(\mathscr{A}) - (-K_i)(\mathscr{Z}) \geqslant \langle \xi_i^5, \mathscr{A} - \mathscr{Z} \rangle,\ \forall \xi_i^5 \in \partial^*(-K_i)(\mathscr{Z}),\ \forall i \in (\delta \cup \omega), \tag{4.5}$$
$$(-L_i)(\mathscr{A}) - (-L_i)(\mathscr{Z}) \geqslant \langle \xi_i^6, \mathscr{A} - \mathscr{Z} \rangle,\ \forall \xi_i^5 \in \partial^*(-L_i)(\mathscr{Z}),\ \forall i \in (\gamma \cup \omega). \tag{4.6}$$

If $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$, then multiplying (4.1)-(4.6) by $\nu_i^{\Phi} > 0,\ i \in \mathbb{I}$, $\nu_i^{\Psi} \geqslant 0,\ i \in \mathbb{I}_\Psi$, $\nu_i^{\Theta} \geqslant 0,\ i \in \mathbb{J}$, $\tau_i^{\Theta} \geqslant 0,\ i \in \mathbb{J}$, $\nu_i^{K} > 0,\ i \in (\delta \cup \omega)$, $\nu_i^{L} > 0,\ i \in (\gamma \cup \omega)$, respectively and then adding them subsequently, we get

$$\begin{aligned}&\sum_{i \in \mathbb{I}} \nu_i^{\Phi} \Phi_i(\mathscr{A}) - \sum_{i \in \mathbb{I}} \nu_i^{\Phi} \Phi_i(\mathscr{Z}) + \sum_{i \in \mathbb{I}_\Psi} \nu_i^{\Psi} \Psi_i(\mathscr{A}) - \sum_{i \in \mathbb{I}_\Psi} \nu_i^{\Psi} \Psi_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} \nu_i^{\Theta} \Theta_i(\mathscr{A}) \\ &\quad - \sum_{i \in \mathbb{J}} \nu_i^{\Theta} \Theta_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} \tau_i^{\Theta}(-\Theta_i)(\mathscr{A}) - \sum_{i \in \mathbb{J}} \tau_i^{\Theta}(-\Theta_i)(\mathscr{Z}) + \sum_{i \in \mathbb{M}} \nu_i^{K}(-K_i)(\mathscr{A}) \\ &\quad - \sum_{i \in \mathbb{M}} \nu_i^{K}(-K_i)(\mathscr{Z}) + \sum_{i \in \mathbb{M}} \nu_i^{L}(-L_i)(\mathscr{A}) - \sum_{i \in \mathbb{M}} \nu_i^{L}(-L_i)(\mathscr{Z}) \\ &\geqslant \left\langle \sum_{i \in \mathbb{I}} \nu_i^{\Phi} \xi_i^1 + \sum_{i \in \mathbb{I}_\Psi} \nu_i^{\Psi} \xi_i^2 + \sum_{i \in \mathbb{J}} [\nu_i^{\Theta} \xi_i^3 + \tau_i^{\Theta} \xi^4] + \sum_{i \in \mathbb{M}} [\nu_i^{K} \xi_i^5 + \nu_i^{L} \xi_i^6] - U,\ \mathscr{A} - \mathscr{Z} \right\rangle + \langle U,\ \mathscr{A} - \mathscr{Z} \rangle.\end{aligned}$$

Since $\mathscr{Z} \in \mathscr{F}_W$, there exist $\overline{\xi}_i^1 \in co\partial^*\Phi_i(\mathscr{Z}), i \in \mathbb{I}$, $\overline{\xi}_i^2 \in co\partial^*\Psi_i(\mathscr{Z}), i \in \mathbb{I}_\Psi$, $\overline{\xi}_i^3 \in co\partial^*\Theta_i(\mathscr{Z}), i \in \mathbb{J}$, $\overline{\xi}_i^4 \in co\partial^*(-\Theta_i)(\mathscr{Z}), i \in \mathbb{J}$, $\overline{\xi}_i^5 \in co\partial^*(-K_i)(\mathscr{Z}), i \in \mathbb{M}$, $\overline{\xi}_i^6 \in co\partial^*(-L_i)(\mathscr{Z}), i \in \mathbb{M}$, and $\overline{U} \in \mathbb{S}_+^n$, such that

$$\sum_{i\in\mathbb{I}} \nu_i^\Phi \overline{\xi}_i^1 + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \overline{\xi}_i^2 + \sum_{i\in\mathbb{J}} [\nu_i^\Theta \overline{\xi}_i^3 + \tau_i^\Theta \overline{\xi}_i^4] + \sum_{i\in\mathbb{M}} [\nu_i^K \overline{\xi}_i^5 + \nu_i^L \overline{\xi}_i^6] - \overline{U} = 0.$$

Moreover, $\langle U, \mathscr{A}\rangle \geqslant 0$, $\forall U, \mathscr{A} \in \mathbb{S}_+^n$. Therefore,

$$\begin{aligned}\sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{A}) - \sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{Z}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{A}) - \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \nu_i^\Theta \Theta_i(\mathscr{A})\\ - \sum_{i\in\mathbb{J}} \nu_i^\Theta \Theta_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \tau_i^\Theta(-\Theta_i)(\mathscr{A}) - \sum_{i\in\mathbb{J}} \tau_i^\Theta(-\Theta_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^K(-K_i)(\mathscr{A})\\ - \sum_{i\in\mathbb{M}} \nu_i^K(-K_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^L(-L_i)(\mathscr{A}) - \sum_{i\in\mathbb{M}} \nu_i^L(-L_i)(\mathscr{Z}) \geqslant -\langle U, \mathscr{Z}\rangle.\end{aligned}$$

Since $\mathscr{A} \in \mathscr{F}$, we have $\Psi_i(\mathscr{A}) \leqslant 0$, $i \in \mathbb{I}_\Psi$, $\Theta_i(\mathscr{A}) = 0$, $i \in \mathbb{J}$, $K_i(\mathscr{A}) \geqslant 0$, $i \in \mathbb{M}$, $L_i(\mathscr{A}) \geqslant 0$, $i \in \mathbb{M}$. Therefore,

$$\begin{aligned}\sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{A}) \geqslant \sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{Z}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} [\nu_i^\Theta \Theta_i(\mathscr{Z}) + \tau_i^\Theta(-\Theta_i)(\mathscr{Z})]\\ + \sum_{i\in\mathbb{M}} \nu_i^K(-K_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^L(-L_i)(\mathscr{Z}) - \langle U, \mathscr{Z}\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\Phi(\mathscr{A}) \geqslant \Phi(\mathscr{Z}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} [\nu_i^\Theta \Theta_i(\mathscr{Z}) + \tau_i^\Theta(-\Theta_i)(\mathscr{Z})]\\ + \sum_{i\in\mathbb{M}} [\nu_i^K(-K_i)(\mathscr{Z}) + \nu_i^L(-L_i)(\mathscr{Z})] - \langle U, \mathscr{Z}\rangle.\end{aligned}$$

□

Now we establish the strong duality theorem for WMPEC assuming the objective and the constraint functions are ∂^*-convex functions that admit bounded USRC and NSMPEC-tailored ACQ holds at an optimal solution of NSMPEC.

Theorem 4.2 (Strong duality). *Let $\overline{\mathscr{A}} \in \mathscr{F}$ be an optimal solution of NSMPEC and Φ_i, $i \in \mathbb{I}$, Ψ_i, $i \in \mathbb{I}_\Psi$ $\pm\Theta_i$, $i \in \mathbb{J}$, $-K_i$, $i \in (\delta \cup \omega)$, and $-L_i$, $i \in (\gamma \cup \omega)$ admit bounded USRC and are ∂^*-convex functions at $\overline{\mathscr{A}}$. Suppose that at $\overline{\mathscr{A}}$, NSMPEC-tailored ACQ holds. Then there exists*

$$\overline{\nu} = (\overline{\nu}^\Phi, \overline{\nu}^\Psi, \overline{\nu}^\Theta, \overline{\tau}^\Theta, \overline{\nu}^K, \overline{\nu}^L, \overline{U}) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}_+^n,$$

such that $(\overline{\mathscr{A}}, \overline{\nu})$ becomes an optimal solution of the WMPEC. Moreover, the corresponding objective values are equal.

Proof. Since $\overline{\mathscr{A}}$ is an optimal solution of NSMPEC and NSMPEC-tailored ACQ holds at $\overline{\mathscr{A}}$, then from Theorem 3.6 there exists

$$\overline{\nu} = (\overline{\nu}^\Phi, \overline{\nu}^\Psi, \overline{\nu}^\Theta, \overline{\tau}^\Theta, \overline{\nu}^K, \overline{\nu}^L, \overline{U}) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}_+^n,$$

such that $\overline{\mathscr{A}}$ is a GS-stationary point of NSMPEC. Thus, there exists $\overline{\xi}_i^1 \in co\partial^*\Phi_i(\overline{\mathscr{A}}), i \in \mathbb{I}$, $\overline{\xi}_i^2 \in co\partial^*\Psi_i(\overline{\mathscr{A}}), i \in \mathbb{I}_\psi$, $\overline{\xi}_i^3 \in co\partial^*\Theta_i(\overline{\mathscr{A}}), i \in \mathbb{J}$, $\overline{\xi}_i^4 \in co\partial^*(-\Theta_i)(\overline{\mathscr{A}}), i \in \mathbb{J}$, $\overline{\xi}_i^5 \in co\partial^*(-K_i)(\overline{\mathscr{A}}), i \in \mathbb{M}$, $\overline{\xi}_i^6 \in co\partial^*(-L_i)(\overline{\mathscr{A}}), i \in \mathbb{M}$, and $\overline{U} \in \mathbb{S}_+^n$, such that

$$\sum_{i\in\mathbb{I}} \overline{\nu}_i^\Phi \overline{\xi}_i^1 + \sum_{i\in\mathbb{I}_\Psi} \overline{\nu}_i^\Psi \overline{\xi}_i^2 + \sum_{i\in\mathbb{J}} [\overline{\nu}_i^\Theta \overline{\xi}_i^3 + \overline{\tau}_i^\Theta \overline{\xi}_i^4] + \sum_{i\in\mathbb{M}} [\overline{\nu}_i^K \overline{\xi}_i^5 + \overline{\nu}_i^L \overline{\xi}_i^6] - \overline{U} = 0,$$

$\langle \bar{U}, \overline{\mathscr{A}} \rangle = 0$, $\bar{\nu}_i^{\Phi} > 0, i \in \mathbb{I}$, $\bar{\nu}_i^{\Psi} \geqslant 0$, $i \in \mathbb{I}_{\Psi}$, $\bar{\nu}_i^{\Theta} \geqslant 0$, $\bar{\tau}_i^{\Theta} \geqslant 0$, $i \in \mathbb{J}$, $\bar{\nu}_i^{K} \geqslant 0$, $\bar{\nu}_i^{L} \geqslant 0$, $i \in \mathbb{M}$. Therefore, $(\overline{\mathscr{A}}, \bar{\nu}) \in \mathscr{F}_W$. By Theorem 4.1, for any $(\mathscr{Z}, \nu) \in \mathscr{F}_W$ we have

$$\begin{aligned} \Phi(\overline{\mathscr{A}}) \geqslant \Phi(\mathscr{Z}) &+ \sum_{i \in \mathbb{I}_{\Psi}} \nu_i^{\Psi} \Psi_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} [\nu_i^{\Theta} \Theta_i(\mathscr{Z}) + \tau_i^{\Theta}(-\Theta_i)(\mathscr{Z})] \\ &+ \sum_{i \in \mathbb{M}} [\nu_i^{K}(-K_i)(\mathscr{Z}) + \nu_i^{L}(-L_i)(\mathscr{Z})] - \langle U, \mathscr{Z} \rangle. \end{aligned} \tag{4.7}$$

Now, from the feasibility conditions of NSMPEC and of the dual WMPEC we have

$$\Psi_i(\overline{\mathscr{A}}) = 0,\ i \in \mathbb{I}_{\Psi},\ \Theta_i(\overline{\mathscr{A}}) = 0,\ i \in \mathbb{J}, \qquad K_i(\overline{\mathscr{A}}) = 0,\ \forall i \in (\delta \cup \omega),\ L_i(\overline{\mathscr{A}}) = 0,\ \forall i \in (\gamma \cup \omega).$$

Therefore, we have

$$\begin{aligned} \Phi(\overline{\mathscr{A}}) = \Phi(\overline{\mathscr{A}}) &+ \sum_{i \in \mathbb{I}_{\Psi}} \bar{\nu}_i^{\Psi} \Psi_i(\overline{\mathscr{A}}) + \sum_{i \in \mathbb{J}} [\bar{\nu}_i^{\Theta} \Theta_i(\overline{\mathscr{A}}) + \bar{\tau}_i^{\Theta}(-\Theta_i)(\overline{\mathscr{A}})] \\ &+ \sum_{i \in \mathbb{M}} [\bar{\nu}_i^{K}(-K_i)(\overline{\mathscr{A}}) + \bar{\nu}_i^{L}(-L_i)(\overline{\mathscr{A}})] - \langle \bar{U}, \overline{\mathscr{A}} \rangle. \end{aligned} \tag{4.8}$$

From equations (4.7) and (4.8) we have

$$\begin{aligned} &\Phi(\overline{\mathscr{A}}) + \sum_{i \in \mathbb{I}_{\Psi}} \bar{\nu}_i^{\Psi} \Psi_i(\overline{\mathscr{A}}) + \sum_{i \in \mathbb{J}} [\bar{\nu}_i^{\Theta} \Theta_i(\overline{\mathscr{A}}) + \bar{\tau}_i^{\Theta}(-\Theta_i)(\overline{\mathscr{A}})] + \sum_{i \in \mathbb{M}} [\bar{\nu}_i^{K}(-K_i)(\overline{\mathscr{A}}) + \bar{\nu}_i^{L}(-L_i)(\overline{\mathscr{A}})] - \langle \bar{U}, \overline{\mathscr{A}} \rangle \\ &\geqslant \Phi(\mathscr{Z}) + \sum_{i \in \mathbb{I}_{\Psi}} \nu_i^{\Psi} \Psi_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} [\nu_i^{\Theta} \Theta_i(\mathscr{Z}) + \tau_i^{\Theta}(-\Theta_i)(\mathscr{Z})] + \sum_{i \in \mathbb{M}} [\nu_i^{K}(-K_i)(\mathscr{Z}) + \nu_i^{L}(-L_i)(\mathscr{Z})] - \langle U, \mathscr{Z} \rangle. \end{aligned}$$

Hence, $(\overline{\mathscr{A}}, \bar{\nu})$ is an optimal solution of WMPEC. Moreover, the corresponding objective values are equal. □

Remark 4.3. It is evident that Theorems 4.1 and 4.2 generalize and extend Theorems 3.1 and 3.2, respectively, established in Pandey and Mishra [42] from Euclidean space $\mathbb{R}^n$ to S_+^n.

Now we derive the strict converse duality theorem for the WMPEC, where we assume the objective function to be ∂^*-convex and the hypothesis of the strong duality theorem holds.

Theorem 4.4 (Strict converse duality)**.** *Suppose that $\overline{\mathscr{A}}$ is a local weak Pareto efficient solution of NSMPEC. Let $(\hat{\mathscr{Z}}, \hat{\nu})$ be the global weak Pareto efficient solution of WMPEC. Suppose that the assumptions of strong duality theorem hold and Φ is strictly ∂^*-convex at $\hat{\mathscr{Z}}$. Assume that $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$. Then, we have $\overline{\mathscr{A}} = \hat{\mathscr{Z}}$.*

Proof. Let us assume that $\overline{\mathscr{A}} \neq \hat{\mathscr{Z}}$. By the Theorem 4.2 we have

$$\begin{aligned} \Phi(\overline{\mathscr{A}}) = \Phi(\overline{\mathscr{A}}) &+ \sum_{i \in \mathbb{I}_{\Psi}} \bar{\nu}_i^{\Psi} \Psi_i(\overline{\mathscr{A}}) + \sum_{i \in \mathbb{J}} [\bar{\nu}_i^{\Theta} \Theta_i(\overline{\mathscr{A}}) + \bar{\tau}_i^{\Theta}(-\Theta_i)(\overline{\mathscr{A}})] \\ &+ \sum_{i \in \mathbb{M}} [\bar{\nu}_i^{K}(-K_i)(\overline{\mathscr{A}}) + \bar{\nu}_i^{L}(-L_i)(\overline{\mathscr{A}})] - \langle \bar{U}, \overline{\mathscr{A}} \rangle. \\ = \Phi(\hat{\mathscr{Z}}) &+ \sum_{i \in \mathbb{I}_{\Psi}} \hat{\nu}_i^{\Psi} \Psi_i(\hat{\mathscr{Z}}) + \sum_{i \in \mathbb{J}} [\hat{\nu}_i^{\Theta} \Theta_i(\hat{\mathscr{Z}}) + \hat{\tau}_i^{\Theta}(-\Theta_i)(\hat{\mathscr{Z}})] \\ &+ \sum_{i \in \mathbb{M}} [\hat{\nu}_i^{K}(-K_i)(\hat{\mathscr{Z}}) + \hat{\nu}_i^{L}(-L_i)(\hat{\mathscr{Z}})] - \langle \hat{U}, \hat{\mathscr{Z}} \rangle. \end{aligned} \tag{4.9}$$

Since Φ is strictly ∂^*-convex at $\hat{\mathscr{Z}}$, therefore

$$\Phi_i(\overline{\mathscr{A}}) - \Phi_i(\hat{\mathscr{Z}}) > \langle \xi_i^1, \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle,\ \forall \xi_i^1 \in \partial^* \Phi(\hat{\mathscr{Z}}). \tag{4.10}$$

Similarly, by the ∂^*-convexity of $\Psi_i, i \in \mathbb{I}_\Psi$, $\pm\Theta_i, i \in \mathbb{J}$, $(-K_i), i \in (\delta\cup\omega)$, $(-L_i), i \in (\gamma\cup\omega)$ at $\hat{\mathscr{Z}}$ we have

$$\Psi_i(\overline{\mathscr{A}}) - \Psi_i(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^2, \overline{\mathscr{A}} - \hat{\mathscr{Z}}\rangle,\ \forall \xi_i^2 \in \partial^*\Psi_i(\hat{\mathscr{Z}}),\ \forall i \in \mathbb{I}_\Psi, \tag{4.11}$$

$$\Theta_i(\overline{\mathscr{A}}) - \Theta_i(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^3, \overline{\mathscr{A}} - \hat{\mathscr{Z}}\rangle,\ \forall \xi_i^3 \in \partial^*\Theta_i(\hat{\mathscr{Z}}),\ \forall i \in \mathbb{J}, \tag{4.12}$$

$$(-\Theta_i)(\overline{\mathscr{A}}) - (-\Theta_i)(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^4, \overline{\mathscr{A}} - \hat{\mathscr{Z}}\rangle,\ \forall \xi_i^4 \in \partial^*(-\Theta_i)(\hat{\mathscr{Z}}),\ \forall i \in \mathbb{J}, \tag{4.13}$$

$$(-K_i)(\overline{\mathscr{A}}) - (-K_i)(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^5, \overline{\mathscr{A}} - \hat{\mathscr{Z}}\rangle,\ \forall \xi_i^5 \in \partial^*(-K_i)(\hat{\mathscr{Z}}),\ \forall i \in (\delta\cup\omega), \tag{4.14}$$

$$(-L_i)(\overline{\mathscr{A}}) - (-L_i)(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^6, \overline{\mathscr{A}} - \hat{\mathscr{Z}}\rangle,\ \forall \xi_i^6 \in \partial^*(-L_i)(\hat{\mathscr{Z}}),\ \forall i \in (\gamma\cup\omega). \tag{4.15}$$

If $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$, then multiplying (4.10)-(4.15) by $\hat{v}_i^\Phi > 0, i \in \mathbb{I}$, $\hat{v}_i^\Psi \geqslant 0, i \in \mathbb{I}_\Psi$, $\hat{v}_i^\Theta \geqslant 0, i \in \mathbb{J}$, $\hat{\tau}_i^\Theta \geqslant 0, i \in \mathbb{J}$, $\hat{v}_i^K > 0, i \in (\delta\cup\omega)$, $\hat{v}_i^L > 0, i \in (\gamma\cup\omega)$, respectively and then adding them subsequently we get

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\overline{\mathscr{A}})\\
&\quad - \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta(-\Theta_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta(-\Theta_i)(\hat{\mathscr{Z}})\\
&\quad + \sum_{i\in\mathbb{M}} \hat{v}_i^K(-K_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{M}} \hat{v}_i^K(-K_i)(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{M}} \hat{v}_i^L(-L_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{M}} \hat{v}_i^L(-L_i)(\hat{\mathscr{Z}})\\
&> \left\langle \sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \xi_i^1 + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \xi_i^2 + \sum_{i\in\mathbb{J}} [\hat{v}_i^\Theta \xi_i^3 + \hat{\tau}_i^\Theta \xi^4] + \sum_{i\in\mathbb{M}} [\hat{v}_i^K \xi_i^5 + \hat{v}_i^L \xi_i^6] - \hat{U},\ \overline{\mathscr{A}} - \hat{\mathscr{Z}} \right\rangle + \langle \hat{U},\ \overline{\mathscr{A}} - \hat{\mathscr{Z}}\rangle.
\end{aligned}$$

Since $\hat{\mathscr{Z}} \in \mathscr{F}_W$, there exist $\hat{\xi}_i^1 \in \operatorname{co}\partial^*\Phi_i(\hat{\mathscr{Z}})$, $i \in \mathbb{I}$, $\hat{\xi}_i^2 \in \operatorname{co}\partial^*\Psi_i(\hat{\mathscr{Z}})$, $i \in \mathbb{I}_\Psi$, $\hat{\xi}_i^3 \in \operatorname{co}\partial^*\Theta_i(\hat{\mathscr{Z}})$, $i \in \mathbb{J}$, $\hat{\xi}_i^4 \in \operatorname{co}\partial^*(-\Theta_i)(\hat{\mathscr{Z}})$, $i \in \mathbb{J}$, $\hat{\xi}_i^5 \in \operatorname{co}\partial^*(-K_i)(\hat{\mathscr{Z}})$, $i \in \mathbb{M}$, $\hat{\xi}_i^6 \in \operatorname{co}\partial^*(-L_i)(\hat{\mathscr{Z}})$, $i \in \mathbb{M}$, and $U^* \in \mathbb{S}_+^n$ such that

$$\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \hat{\xi}_i^1 + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \hat{\xi}_i^2 + \sum_{i\in\mathbb{J}} [\hat{v}_i^\Theta \hat{\xi}_i^3 + \hat{\tau}_i^\Theta \hat{\xi}_i^4] + \sum_{i\in\mathbb{M}} [\hat{v}_i^K \hat{\xi}_i^5 + \hat{v}_i^L \hat{\xi}_i^6] - U^* = 0.$$

Moreover, $\langle \hat{U},\ \overline{\mathscr{A}}\rangle \geqslant 0$, $\forall \hat{U}, \overline{\mathscr{A}} \in \mathbb{S}_+^n$. Therefore,

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\overline{\mathscr{A}})\\
&\quad - \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta(-\Theta_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta(-\Theta_i)(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{M}} \hat{v}_i^K(-K_i)(\overline{\mathscr{A}})\\
&\quad - \sum_{i\in\mathbb{M}} \hat{v}_i^K(-K_i)(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{M}} \hat{v}_i^L(-L_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{M}} \hat{v}_i^L(-L_i)(\hat{\mathscr{Z}}) > -\langle \hat{U},\ \hat{\mathscr{Z}}\rangle.
\end{aligned}$$

Since $\overline{\mathscr{A}} \in \mathscr{F}$ we have

$$\Psi_i(\overline{\mathscr{A}}) \leqslant 0, i \in \mathbb{I}_\Psi,\ \Theta_i(\overline{\mathscr{A}}) = 0, i \in \mathbb{J},\ K_i(\overline{\mathscr{A}}) \geqslant 0, i \in \mathbb{M},\ L_i(\overline{\mathscr{A}}) \geqslant 0, i \in \mathbb{M}.$$

Therefore,

$$\begin{aligned}
\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\overline{\mathscr{A}}) > &\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} [\hat{v}_i^\Theta \Theta_i(\hat{\mathscr{Z}}) + \hat{\tau}_i^\Theta(-\Theta_i)(\hat{\mathscr{Z}})]\\
&+ \sum_{i\in\mathbb{M}} \hat{v}_i^K(-K_i)(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{M}} \hat{v}_i^L(-L_i)(\hat{\mathscr{Z}}) - \langle \hat{U},\ \hat{\mathscr{Z}}\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Phi(\overline{\mathscr{A}}) > \Phi(\hat{\mathscr{Z}}) &+ \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} [\hat{v}_i^\Theta \Theta_i(\hat{\mathscr{Z}}) + \hat{\tau}_i^\Theta(-\Theta_i)(\hat{\mathscr{Z}})]\\
&+ \sum_{i\in\mathbb{M}} [\hat{v}_i^K(-K_i)(\hat{\mathscr{Z}}) + \hat{v}_i^L(-L_i)(\hat{\mathscr{Z}})] - \langle \hat{U}, \hat{\mathscr{Z}}\rangle,
\end{aligned}$$

which contradicts (4.9). Therefore, $\overline{\mathscr{A}} = \hat{\mathscr{Z}}$. This completes the proof. □

Remark 4.5. Theorem 4.4 generalizes Theorem 4.5 established in Mishra et al. [37] for a more general programming problem NSMPEC. For $\mathbb{J} = \emptyset$ and $\mathbb{M} = \emptyset$, Theorem 4.4 reduces to Theorem 4.5 established in [37].

The subsequent example illustrates the significance of the weak duality theorem of WMPEC for the problem NSMPEC.

Example 4.6. Consider the following multiobjective programming problem with equilibrium constraints

$$\begin{aligned} &\text{Minimize } \ \Phi(\mathscr{A}) = (\Phi_1(\mathscr{A}), \Phi_2(\mathscr{A})) := (a_2, a_1), \\ &\text{subject to } \ \Psi(\mathscr{A}) := -a_3 \leqslant 0, \ \mathtt{K}(\mathscr{A}) := 2a_2 \geqslant 0, \ \mathtt{L}(\mathscr{A}) := a_1 \geqslant 0, \end{aligned} \tag{P$_1$}$$

where $\Phi_i : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$ $(i = 1, 2)$, $\mathtt{K} : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, $\mathtt{L} : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, and $\mathscr{A} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \in \mathbb{S}^2_+$. The set of all feasible solutions of (P$_1$) is defined as

$$\mathscr{F}_{(P_1)} := \left\{ \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \in \mathbb{S}^n_+ : a_1 \geqslant 0 \ a_2 \geqslant 0, \ a_3 \geqslant 0, \ a_1 a_3 - a_2^2 \geqslant 0 \right\}.$$

Let $\mathscr{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then for $\nu_1^\Phi = 2$, $\nu_2^\Phi = 2$, $\nu^\Psi = 0$, $\nu^K = 1$, and $\nu^L = 1$, $\mathscr{Z}$ is a feasible solution of the Wolfe dual (D_1) of the problem (P$_1$). Moreover, $\Phi_i, i \in \{1, 2\}$, Ψ, $-\mathtt{K}$, and $-\mathtt{L}$ are ∂^*-convex at $\mathscr{Z}$ and $\omega^K_\tau \cup \omega^L_\tau \cup \delta^+_\tau \cup \gamma^+_\tau = \emptyset$, such that

$$\begin{aligned} \Phi(\mathscr{A}) \geqslant \Phi(\mathscr{Z}) &+ \sum_{i \in \mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} [\nu_i^\Theta \Theta_i(\mathscr{Z}) + \tau_i^\Theta(-\Theta_i)(\mathscr{Z})] \\ &+ \sum_{i \in \mathbb{M}} [\nu_i^K(-\mathtt{K}_i)(\mathscr{Z}) + \nu_i^L(-\mathtt{L}_i)(\mathscr{Z})] - \langle \mathtt{U}, \mathscr{Z} \rangle. \end{aligned}$$

Hence, Theorem 4.1 holds at the feasible point $\mathscr{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ of (D_1).

5. Mond-Weir duality

Now, corresponding to the primal problem NSMPEC, we formulate the Mond-Weir-type MWMPEC dual model as

$$\begin{aligned} &\underset{(\mathscr{Z}, \nu)}{\text{Maximize}} \ \Phi(\mathscr{Z}) = (\Phi_1(\mathscr{Z}), \dots, \Phi_r(\mathscr{Z})), \\ &\text{subject to } (\mathscr{Z}, \nu) \in \mathscr{F}_{MW}, \end{aligned}$$

where $\mathscr{F}_{MW}$ denotes the set of all feasible solutions of MWMPEC and is defined as

$$\begin{aligned} \mathscr{F}_{MW} = \Big\{ &\nu = (\nu^\Phi, \nu^\Psi, \nu^\Theta, \tau^\Theta, \nu^K, \nu^L, \mathtt{U}) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^n_+, \\ &\mathscr{Z} \in \mathbb{S}^n_+ : \ 0 \in \sum_{i \in \mathbb{I}} \nu_i^\Phi \operatorname{co}\partial^* \Phi_i(\mathscr{Z}) + \sum_{i \in \mathbb{I}_\Psi} \nu_i^\Psi \operatorname{co}\partial^* \Psi_i(\mathscr{Z}) + \sum_{i \in \mathbb{J}} [\nu_i^\Theta \operatorname{co}\partial^* \Theta_i(\mathscr{Z}) \\ &\quad + \tau_i^\Theta \operatorname{co}\partial^*(-\Theta_i)(\mathscr{Z})] + \sum_{i \in \mathbb{M}} [\nu_i^K \operatorname{co}\partial^*(-\mathtt{K}_i)(\mathscr{Z}) + \nu_i^L \operatorname{co}\partial^*(-\mathtt{L}_i)(\mathscr{Z})] - \mathtt{U}, \\ &\langle \mathtt{U}, \mathscr{Z} \rangle = 0, \ \Psi_i(\mathscr{Z}) \geqslant 0, \ i \in \mathbb{I}_\Psi, \ \Theta_i(\mathscr{Z}) = 0, \ i \in \mathbb{J}, \ \mathtt{K}_i(\mathscr{Z}) \leqslant 0, \ i \in (\delta \cup \omega), \ \mathtt{L}_i(\mathscr{Z}) \leqslant 0, \\ &i \in (\gamma \cup \omega), \ \nu_i^\Phi > 0, i \in \mathbb{I}, \ \nu_i^\Psi \geqslant 0, \ i \in \mathbb{I}_\Psi, \ \nu_i^\Theta, \ \tau_i^\Theta \geqslant 0, \ i \in \mathbb{J}, \ \nu_i^K, \ \nu_i^L, \ \tau_i^K, \ \tau_i^L \geqslant 0, \ i \in \mathbb{M}, \\ &\nu_\gamma^K = \nu_\delta^L = \tau_\gamma^K = \tau_\delta^L = 0; \ \tau_i^K = \tau_i^L = 0, \forall i \in \omega \ \Big\}. \end{aligned}$$

In the subsequent theorems, we present weak, strong, and strict converse duality theorems that relate MWMPEC and NSMPEC.

Theorem 5.1 (Weak duality)**.** *Let $\mathscr{A}$ be a feasible solution of NSMPEC. Let $(\mathscr{Z},\nu)$ be a feasible solution of the dual MWMPEC of NSMPEC. Suppose that Φ_i, $i \in \mathbb{I}$, Ψ_i, $i \in \mathbb{I}_\Psi$, $\pm\Theta_i$ $i \in \mathbb{J}$, $-K_i$, $i \in (\delta \cup \omega)$, $-L_i$, $i \in (\gamma \cup \omega)$ admit bounded USRC and are ∂^*-convex at $\mathscr{Z}$. Assume that $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$. Then $\Phi(\mathscr{A}) \geqslant \Phi(\mathscr{Z})$.*

Proof. Since Φ is ∂^*-convex at $\mathscr{Z}$, then

$$\Phi_i(\mathscr{A}) - \Phi_i(\mathscr{Z}) \geqslant \langle \xi_i^1, \mathscr{A} - \mathscr{Z}\rangle,\ \forall \xi_i^1 \in \partial^*\Phi_i(\mathscr{Z}). \tag{5.1}$$

Similarly, by the ∂^*-convexity of Ψ_i, $i \in \mathbb{I}_\Psi$, $\pm\Theta_i, i \in \mathbb{J}$, $(-K_i)$, $i \in (\delta \cup \omega)$, $(-L_i)$, $i \in (\gamma \cup \omega)$ at $\mathscr{Z}$ we have

$$\Psi_i(\mathscr{A}) - \Psi_i(\mathscr{Z}) \geqslant \langle \xi_i^2, \mathscr{A} - \mathscr{Z}\rangle,\ \forall \xi_i^2 \in \partial^*\Psi_i(\mathscr{Z}),\ \forall i \in \mathbb{I}_\Psi, \tag{5.2}$$

$$\Theta_i(\mathscr{A}) - \Theta_i(\mathscr{Z}) \geqslant \langle \xi_i^3, \mathscr{A} - \mathscr{Z}\rangle,\ \forall \xi_i^3 \in \partial^*\Theta_i(\mathscr{Z}),\ \forall i \in \mathbb{J}, \tag{5.3}$$

$$(-\Theta_i)(\mathscr{A}) - (-\Theta_i)(\mathscr{Z}) \geqslant \langle \xi_i^4, \mathscr{A} - \mathscr{Z}\rangle,\ \forall \xi_i^4 \in \partial^*(-\Theta_i)(\mathscr{Z}),\ \forall i \in \mathbb{J}, \tag{5.4}$$

$$(-K_i)(\mathscr{A}) - (-K_i)(\mathscr{Z}) \geqslant \langle \xi_i^5, \mathscr{A} - \mathscr{Z}\rangle,\ \forall \xi_i^5 \in \partial^*(-K_i)(\mathscr{Z}),\ \forall i \in (\delta \cup \omega), \tag{5.5}$$

$$(-L_i)(\mathscr{A}) - (-L_i)(\mathscr{Z}) \geqslant \langle \xi_i^6, \mathscr{A} - \mathscr{Z}\rangle,\ \forall \xi_i^6 \in \partial^*(-L_i)(\mathscr{Z}),\ \forall i \in (\gamma \cup \omega). \tag{5.6}$$

If $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$, then multiplying (5.1)-(5.6) by $\nu_i^\Phi > 0, i \in \mathbb{I}$, $\nu_i^\Psi \geqslant 0, i \in \mathbb{I}_\Psi$, $\nu_i^\Theta \geqslant 0, i \in \mathbb{J}$, $\tau_i^\Theta \geqslant 0, i \in \mathbb{J}$, $\nu_i^K > 0, i \in (\delta \cup \omega)$, $\nu_i^L > 0, i \in (\gamma \cup \omega)$, respectively and then adding them subsequently we get

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{A}) - \sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{Z}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{A}) - \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \nu_i^\Theta \Theta_i(\mathscr{A}) \\
&\quad - \sum_{i\in\mathbb{J}} \nu_i^\Theta \Theta_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \tau_i^\Theta (-\Theta_i)(\mathscr{A}) - \sum_{i\in\mathbb{J}} \tau_i^\Theta (-\Theta_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^K (-K_i)(\mathscr{A}) \\
&\quad - \sum_{i\in\mathbb{M}} \nu_i^K (-K_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^L (-L_i)(\mathscr{A}) - \sum_{i\in\mathbb{M}} \nu_i^L (-L_i)(\mathscr{Z}) \\
&\geqslant \left\langle \sum_{i\in\mathbb{I}} \nu_i^\Phi \xi_i^1 + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \xi_i^2 + \sum_{i\in\mathbb{J}} [\nu_i^\Theta \xi_i^3 + \tau_i^\Theta \xi^4] + \sum_{i\in\mathbb{M}} [\nu_i^K \xi_i^5 + \nu_i^L \xi_i^6] - U,\ \mathscr{A} - \mathscr{Z} \right\rangle + \langle U,\ \mathscr{A} - \mathscr{Z}\rangle.
\end{aligned}$$

Since $\mathscr{Z}$ is feasible solution of the dual problem MWMPEC, then there exist $\overline{\xi}_i^1 \in co\partial^*\Phi_i(\mathscr{Z})$, $i \in \mathbb{I}$, $\overline{\xi}_i^2 \in co\partial^*\Psi_i(\mathscr{Z})$, $i \in \mathbb{I}_\Psi$, $\overline{\xi}_i^3 \in co\partial^*\Theta_i(\mathscr{Z})$, $i \in \mathbb{J}$, $\overline{\xi}_i^4 \in co\partial^*(-\Theta_i)(\mathscr{Z})$, $i \in \mathbb{J}$, $\overline{\xi}_i^5 \in co\partial^*(-K_i)(\mathscr{Z})$, $i \in \mathbb{M}$, $\overline{\xi}_i^6 \in co\partial^*(-L_i)(\mathscr{Z})$, $i \in \mathbb{M}$, and $\overline{U} \in \mathbb{S}_+^n$ such that

$$\sum_{i\in\mathbb{I}} \nu_i^\Phi \overline{\xi}_i^1 + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \overline{\xi}_i^2 + \sum_{i\in\mathbb{J}} [\nu_i^\Theta \overline{\xi}_i^3 + \tau_i^\Theta \overline{\xi}_i^4] + \sum_{i\in\mathbb{M}} [\nu_i^K \overline{\xi}_i^5 + \nu_i^L \overline{\xi}_i^6] - \overline{U} = 0.$$

Moreover, $\langle U,\ \mathscr{A}\rangle \geqslant 0$, $\forall U, \mathscr{A} \in \mathbb{S}_+^n$, and $\langle U,\ \mathscr{Z}\rangle = 0$, $\forall U \in \mathbb{S}_+^n$, $\forall \mathscr{Z} \in \mathscr{F}_{MW}$. Therefore,

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{A}) - \sum_{i\in\mathbb{I}} \nu_i^\Phi \Phi_i(\mathscr{Z}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{A}) - \sum_{i\in\mathbb{I}_\Psi} \nu_i^\Psi \Psi_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \nu_i^\Theta \Theta_i(\mathscr{A}) \\
&\quad - \sum_{i\in\mathbb{J}} \nu_i^\Theta \Theta_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \tau_i^\Theta (-\Theta_i)(\mathscr{A}) - \sum_{i\in\mathbb{J}} \tau_i^\Theta (-\Theta_i)(\mathscr{Z}) \\
&\quad + \sum_{i\in\mathbb{M}} \nu_i^K (-K_i)(\mathscr{A}) - \sum_{i\in\mathbb{M}} \nu_i^K (-K_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^L (-L_i)(\mathscr{A}) - \sum_{i\in\mathbb{M}} \nu_i^L (-L_i)(\mathscr{Z}) \geqslant 0.
\end{aligned}$$

Since $\mathscr{A} \in \mathscr{F}$ we have

$$\Psi_i(\mathscr{A}) \leqslant 0,\ i \in \mathbb{I}_\Psi,\ \Theta_i(\mathscr{A}) = 0,\ i \in \mathbb{J},\ K_i(\mathscr{A}) \geqslant 0,\ i \in \mathbb{M},\ L_i(\mathscr{A}) \geqslant 0,\ i \in \mathbb{M}.$$

Therefore,

$$\sum_{i\in\mathbb{I}} \nu_i^{\Phi}\Phi_i(\mathscr{A}) \geqslant \sum_{i\in\mathbb{I}} \nu_i^{\Phi}\Phi_i(\mathscr{Z}) + \sum_{i\in\mathbb{I}_\Psi} \nu_i^{\Psi}\Psi_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \nu_i^{\Theta}\Theta_i(\mathscr{Z}) + \sum_{i\in\mathbb{J}} \tau_i^{\Theta}(-\Theta_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^{K}(-K_i)(\mathscr{Z}) + \sum_{i\in\mathbb{M}} \nu_i^{L}(-L_i)(\mathscr{Z}).$$

Since $\mathscr{Z} \in \mathscr{F}_{MW}$ we have

$$\Psi_i(\mathscr{Z}) \geqslant 0,\ i \in \mathbb{I}_\Psi,\ \Theta_i(\mathscr{Z}) = 0,\ i \in \mathbb{J},\ K_i(\mathscr{Z}) \leqslant 0,\ i \in \mathbb{M},\ L_i(\mathscr{Z}) \leqslant 0,\ i \in \mathbb{M}.$$

Hence, $\Phi(\mathscr{A}) \geqslant \Phi(\mathscr{Z})$. □

Now we establish the strong duality theorem for MWMPEC assuming the objective and the constraint functions are ∂^*-convex functions that admit bounded USRC and NSMPEC-tailored ACQ holds at a locally Pareto efficient solution of NSMPEC.

Theorem 5.2 (Strong duality)**.** *Let $\overline{\mathscr{A}}$ be a locally Pareto efficient solution NSMPEC. Let at $\overline{\mathscr{A}}$, Φ_i, $i \in \mathbb{I}$, Ψ_i, $i \in \mathbb{I}_\Psi$ $\pm\Theta_i$, $i \in \mathbb{J}$, $-K_i$, $i \in (\delta \cup \omega)$, and $-L_i$, $i \in (\gamma \cup \omega)$ admit bounded USRC and are ∂^*-convex functions at $\overline{\mathscr{A}}$. Suppose that at $\overline{\mathscr{A}}$, NSMPEC-tailored ACQ holds. Then there exists*

$$\overline{\nu} = (\overline{\nu}^{\Phi}, \overline{\nu}^{\Psi}, \overline{\nu}^{\Theta}, \overline{\tau}^{\Theta}, \overline{\nu}^{K}, \overline{\nu}^{L}, \overline{U}) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^n_+,$$

such that $(\overline{\mathscr{A}}, \overline{\nu})$ is an optimal solution of MWMPEC. Moreover, the corresponding objective values are equal.

Proof. From the assumptions of the theorem, $\overline{\mathscr{A}}$ is a locally Pareto efficient solution of NSMPEC and NSMPEC-tailored ACQ holds at $\overline{\mathscr{A}}$. Then from Theorem 3.6 there exists

$$\overline{\nu} = (\overline{\nu}^{\Phi}, \overline{\nu}^{\Psi}, \overline{\nu}^{\Theta}, \overline{\tau}^{\Theta}, \overline{\nu}^{K}, \overline{\nu}^{L}, \overline{U}) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^n_+,$$

such that $\overline{\mathscr{A}}$ is a GS-stationary point of NSMPEC. Thus, there exists $\overline{\xi}_i^1 \in \mathrm{co}\partial^*\Phi_i(\overline{\mathscr{A}})$, $\overline{\xi}_i^2 \in \mathrm{co}\partial^*\Psi_i(\overline{\mathscr{A}})$, $\overline{\xi}_i^3 \in \mathrm{co}\partial^*\Theta_i(\overline{\mathscr{A}})$, $\overline{\xi}_i^4 \in \mathrm{co}\partial^*(-\Theta_i)(\overline{\mathscr{A}})$, $\overline{\xi}_i^5 \in \mathrm{co}\partial^*(-K_i)(\overline{\mathscr{A}})$, $\overline{\xi}_i^6 \in \mathrm{co}\partial^*(-L_i)(\overline{\mathscr{A}})$, and $\overline{U} \in \mathbb{S}^n_+$ such that

$$\sum_{i\in\mathbb{I}} \overline{\nu}_i^{\Phi}\overline{\xi}_i^1 + \sum_{i\in\mathbb{I}_\Psi} \overline{\nu}_i^{\Psi}\overline{\xi}_i^2 + \sum_{i\in\mathbb{J}} [\overline{\nu}_i^{\Theta}\overline{\xi}_i^3 + \overline{\tau}_i^{\Theta}\overline{\xi}_i^4] + \sum_{i\in\mathbb{M}} [\overline{\nu}_i^{K}\overline{\xi}_i^5 + \overline{\nu}_i^{L}\overline{\xi}_i^6] - \overline{U} = 0,$$

$\langle \overline{U}, \overline{\mathscr{A}} \rangle = 0$, $\overline{\nu}_i^{\Phi} > 0$, $i \in \mathbb{I}$, $\overline{\nu}_i^{\Psi} \geqslant 0$, $i \in \mathbb{I}_\Psi$, $\overline{\nu}_i^{\Theta}$, $\overline{\tau}_i^{\Theta} \geqslant 0$, $i \in \mathbb{J}$, $\overline{\nu}_i^{K}$, $\overline{\nu}_i^{L}$, $i \in \mathbb{M}$. Therefore, $(\overline{\mathscr{A}}, \overline{\nu}) \in \mathscr{F}_{MW}$. Using Theorem 5.1, for any feasible solution $(\mathscr{Z}, \nu)$ of MWMPEC we have $\Phi(\overline{\mathscr{A}}) \geqslant \Phi(\mathscr{Z})$. Hence, $(\overline{\mathscr{A}}, \overline{\nu})$ is an optimal solution of MWMPEC. Moreover, the respective objective values are equal. This completes the proof. □

Remark 5.3. It is evident that Theorems 5.1 and 5.2 generalize and extend Theorems 3.3 and 3.4, respectively, established in Pandey and Mishra [42] from Euclidean space $\mathbb{R}^n$ to $\mathbb{S}^n_+$.

Now we establish the strict converse duality theorem for the MWMPEC, where we assume the objective function to be ∂^*-convex and the hypothesis of the strong duality theorem holds.

Theorem 5.4 (Strict converse duality)**.** *Suppose that $\overline{\mathscr{A}}$ is a local weak Pareto efficient solution of NSMPEC and $(\hat{\mathscr{Z}}, \hat{\nu})$ is a global weak Pareto efficient solution of MWMPEC. If the assumptions of the strong duality theorem hold and Φ is strictly ∂^*-convex at $\hat{\mathscr{Z}}$, then we have $\overline{\mathscr{A}} = \hat{\mathscr{Z}}$.*

Proof. Let us assume that $\overline{\mathscr{A}} \neq \hat{\mathscr{Z}}$. By Theorem 5.2 there exists

$$\overline{\nu} = (\overline{\nu}^{\Phi}, \overline{\nu}^{\Psi}, \overline{\nu}^{\Theta}, \overline{\tau}^{\Theta}, \overline{\nu}^{K}, \overline{\nu}^{L}, \overline{U}) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^n_+,$$

such that $(\overline{\mathscr{A}}, \overline{\nu}) \in \mathscr{F}_{MW}$ and

$$\Phi(\overline{\mathscr{A}}) = \Phi(\hat{\mathscr{Z}}). \tag{5.7}$$

Since Φ is strictly ∂^*-convex at $\hat{\mathscr{Z}}$, therefore

$$\Phi_i(\overline{\mathscr{A}}) - \Phi_i(\hat{\mathscr{Z}}) > \langle \xi_i^1, \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle, \ \forall \xi_i^1 \in \partial^*\Phi(\hat{\mathscr{Z}}). \tag{5.8}$$

Similarly, by the ∂^*-convexity of $\Psi_i, i \in \mathbb{I}_\Psi, \pm\Theta_i, i \in \mathbb{J}, (-K_i), i \in (\delta \cup \omega), (-L_i), i \in (\gamma \cup \omega)$ at $\hat{\mathscr{Z}}$, we have

$$\Psi_i(\overline{\mathscr{A}}) - \Psi_i(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^2, \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle, \ \forall \xi_i^2 \in \partial^*\Psi_i(\hat{\mathscr{Z}}), \ \forall i \in \mathbb{I}_\Psi, \tag{5.9}$$

$$\Theta_i(\overline{\mathscr{A}}) - \Theta_i(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^3, \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle, \ \forall \xi_i^3 \in \partial^*\Theta_i(\hat{\mathscr{Z}}), \ \forall i \in \mathbb{J}, \tag{5.10}$$

$$(-\Theta_i)(\overline{\mathscr{A}}) - (-\Theta_i)(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^4, \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle, \ \forall \xi_i^4 \in \partial^*(-\Theta_i)(\hat{\mathscr{Z}}), \ \forall i \in \mathbb{J}, \tag{5.11}$$

$$(-K_i)(\overline{\mathscr{A}}) - (-K_i)(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^5, \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle, \ \forall \xi_i^5 \in \partial^*(-K_i)(\hat{\mathscr{Z}}), \ \forall i \in (\delta \cup \omega), \tag{5.12}$$

$$(-L_i)(\overline{\mathscr{A}}) - (-L_i)(\hat{\mathscr{Z}}) \geqslant \langle \xi_i^6, \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle, \ \forall \xi_i^6 \in \partial^*(-L_i)(\hat{\mathscr{Z}}), \ \forall i \in (\gamma \cup \omega). \tag{5.13}$$

If $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$, then multiplying (5.8)-(5.13) by $\hat{v}_i^\Phi > 0, i \in \mathbb{I}$, $\hat{v}_i^\Psi \geqslant 0, i \in \mathbb{I}_\Psi$, $\hat{v}_i^\Theta \geqslant 0, i \in \mathbb{J}$, $\hat{\tau}_i^\Theta \geqslant 0, i \in \mathbb{J}$, $\hat{v}_i^K > 0, i \in (\delta \cup \omega)$, $\hat{v}_i^L > 0, i \in (\gamma \cup \omega)$, respectively and then adding them subsequently we get

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\overline{\mathscr{A}}) \\
&\quad - \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta (-\Theta_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta (-\Theta_i)(\hat{\mathscr{Z}}) \\
&\quad + \sum_{i\in\mathbb{M}} \hat{v}_i^K (-K_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{M}} \hat{v}_i^K (-K_i)(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{M}} \hat{v}_i^L (-L_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{M}} \hat{v}_i^L (-L_i)(\hat{\mathscr{Z}}) \\
&> \left\langle \sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \xi_i^1 + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \xi_i^2 + \sum_{i\in\mathbb{J}} [\hat{v}_i^\Theta \xi_i^3 + \hat{\tau}_i^\Theta \xi^4] + \sum_{i\in\mathbb{M}} [\hat{v}_i^K \xi_i^5 + \hat{v}_i^L \xi_i^6] - \hat{U}, \ \overline{\mathscr{A}} - \hat{\mathscr{Z}} \right\rangle + \langle \hat{U}, \ \overline{\mathscr{A}} - \hat{\mathscr{Z}} \rangle.
\end{aligned}$$

Since $\hat{\mathscr{Z}} \in \mathscr{F}_{MW}$, there exist $\hat{\xi}_i^1 \in co\partial^*\Phi_i(\mathscr{Z})$, $\hat{\xi}_i^2 \in co\partial^*\Psi_i(\mathscr{Z})$, $\hat{\xi}_i^3 \in co\partial^*\Theta_i(\mathscr{Z})$, $\hat{\xi}_i^4 \in co\partial^*(-\Theta_i)(\mathscr{Z})$, $\hat{\xi}_i^5 \in co\partial^*(-K_i)(\mathscr{Z})$, $\hat{\xi}_i^6 \in co\partial^*(-L_i)(\mathscr{Z})$, and $U^* \in \mathbb{S}_+^n$ such that

$$\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \hat{\xi}_i^1 + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \hat{\xi}_i^2 + \sum_{i\in\mathbb{J}} [\hat{v}_i^\Theta \hat{\xi}_i^3 + \hat{\tau}_i^\Theta \hat{\xi}_i^4] + \sum_{i\in\mathbb{M}} [\hat{v}_i^K \hat{\xi}_i^5 + \hat{v}_i^L \hat{\xi}_i^6] - U^* = 0, \qquad \langle U^*, \ \hat{\mathscr{Z}} \rangle = 0.$$

Moreover, $\langle \hat{U}, \ \overline{\mathscr{A}} \rangle \geqslant 0, \ \forall \overline{\mathscr{A}}, \hat{U} \in \mathbb{S}_+^n$. Therefore,

$$\begin{aligned}
&\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{I}_\Psi} \hat{v}_i^\Psi \Psi_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\overline{\mathscr{A}}) \\
&\quad - \sum_{i\in\mathbb{J}} \hat{v}_i^\Theta \Theta_i(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta (-\Theta_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{J}} \hat{\tau}_i^\Theta (-\Theta_i)(\hat{\mathscr{Z}}) \\
&\quad + \sum_{i\in\mathbb{M}} \hat{v}_i^K (-K_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{M}} \hat{v}_i^K (-K_i)(\hat{\mathscr{Z}}) + \sum_{i\in\mathbb{M}} \hat{v}_i^L (-L_i)(\overline{\mathscr{A}}) - \sum_{i\in\mathbb{M}} \hat{v}_i^L (-L_i)(\hat{\mathscr{Z}}) > 0.
\end{aligned}$$

Since $\overline{\mathscr{A}} \in \mathscr{F}$, we have

$$\Psi_i(\overline{\mathscr{A}}) \leqslant 0, i \in \mathbb{I}_\Psi, \ \Theta_i(\overline{\mathscr{A}}) = 0, i \in \mathbb{J}, \ K_i(\overline{\mathscr{A}}) \geqslant 0, i \in \mathbb{M}, \ L_i(\overline{\mathscr{A}}) \geqslant 0, i \in \mathbb{M},$$

and $\hat{\mathscr{Z}}$ is a feasible solution MWMPEC, then we have

$$\Psi_i(\hat{\mathscr{Z}}) \geqslant 0, i \in \mathbb{I}_\Psi, \ \Theta_i(\hat{\mathscr{Z}}) = 0, i \in \mathbb{J}, \ K_i(\hat{\mathscr{Z}}) \leqslant 0, i \in \mathbb{M}, \ L_i(\hat{\mathscr{Z}}) \leqslant 0, i \in \mathbb{M}.$$

Therefore,

$$\sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\overline{\mathscr{A}}) > \sum_{i\in\mathbb{I}} \hat{v}_i^\Phi \Phi_i(\hat{\mathscr{Z}}).$$

Hence,

$$\Phi(\overline{\mathscr{A}}) > \Phi(\hat{\mathscr{Z}}),$$

which contradicts (5.7). Therefore, $\overline{\mathscr{A}} = \hat{\mathscr{Z}}$. This completes the proof. □

Remark 5.5. Theorem 5.4 extends the scope of Theorem 4.10 established in Mishra et al. [37] for a more general programming problem NSMPEC. For $\mathbb{J} = \emptyset$ and $\mathbb{M} = \emptyset$, Theorem 5.4 reduces to Theorem 4.10 derived in [37].

The subsequent example illustrates the importance of the weak duality theorem of MWMPEC for the considered problem NSMPEC.

Example 5.6. Consider the following multiobjective programming problem with equilibrium constraints

$$\begin{aligned} &\text{Minimize } \ \Phi(\mathscr{A}) = (\Phi_1(\mathscr{A}), \Phi_2(\mathscr{A})) := (a_2, a_1), \\ &\text{subject to } \ \Psi(\mathscr{A}) := -a_3 \leqslant 0, \ \mathtt{K}(\mathscr{A}) := 2a_2 \geqslant 0, \ \mathtt{L}(\mathscr{A}) := a_1 \geqslant 0, \end{aligned} \tag{P$_2$}$$

where $\Phi_i : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$ $(i = 1, 2)$, $\mathtt{K} : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, $\mathtt{L} : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, and $\mathscr{A} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \in \mathbb{S}^2_+$. The set of all feasible solutions of (P$_2$) is defined as

$$\mathscr{F}_{(P_2)} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \in \mathbb{S}^n_+ : a_1 \geqslant 0 \ a_2 \geqslant 0, \ a_3 \geqslant 0, \ a_1 a_3 - a_2^2 \geqslant 0 \right\}.$$

The Mond-Weir type dual of the problem (P$_2$) is given by

$$\begin{aligned} &\text{Maximize } \ \Phi(\mathscr{Z}) = (\Phi_1(\mathscr{Z}), \Phi_2(\mathscr{Z})) := (z_2, z_1), \\ &\text{subject to } \ \Psi(\mathscr{Z}) = -z_3 \geqslant 0, \ \mathtt{K}(\mathscr{Z}) = 2z_2 \leqslant 0, \ \mathtt{L}(\mathscr{Z}) = z_1 \leqslant 0, \end{aligned} \tag{D$_2$}$$

where $\Phi_i : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$ $(i = 1, 2)$, $\mathtt{K} : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, $\mathtt{L} : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, and $\mathscr{Z} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2_+$. The set of all feasible solutions of (D$_2$) is defined as

$$\mathscr{F}_{(D_2)} = \left\{ \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2_+ : z_1 \leqslant 0, \ z_2 \leqslant 0, \ z_3 \leqslant 0, \ z_1 z_3 - z_2^2 \geqslant 0 \right\}.$$

There exist $\nu_1^\Phi = 2$, $\nu_2^\Phi = 2$, $\nu^\Psi = 0$, $\nu^\mathtt{K} = 1$, $\nu^\mathtt{L} = 1$, $\mathtt{U} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ such that $\mathscr{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathscr{F}_{(D_2)}$. Moreover, $\Phi_i, i \in \{1, 2\}$, Ψ, $-\mathtt{K}$, $-\mathtt{L}$, are ∂^*-convex at $\mathscr{Z}$ and $\omega^\mathtt{K}_\tau \cup \omega^\mathtt{L}_\tau \cup \delta^+_\tau \cup \gamma^+_\tau = \emptyset$. Hence, by Theorem 5.1 we have $\Phi(\mathscr{A}) \geqslant \Phi(\mathscr{Z})$.

In the following section, we shall discuss an application of semidefinite multiobjective programming problems with equilibrium constraints in approximating K-Means-type clustering problems. Moreover, we have considered a particular case of three data points to show the significance of weak and strong duality theorems in approximating K-Means-type clustering problems.

6. Applicaton in approximating K-means-type clustering problems

One of the fundamental clustering problems is to assign n points into K clusters based on minimal sum-of-squared distances (in short, MSSC) (see, for instance, [43]). Let S be a set of n points in a d-dimensional Euclidean space, denoted by $S = \{s_i = (s_{i1}, \ldots, s_{id})^T \in \mathbb{R}^d, \ i = 1, \ldots, n\}$. The task of MSSC is to find an assignment of the n points into K disjoint clusters $S = (S_1, \ldots, S_K)$ centred at cluster centres $c_j (j = 1, \ldots, K)$ such that the total sum-of-squared Euclidean distances from each point s_i to its assigned cluster centroid c_j and

$$f(S, S) = \sum_{j=1}^{K} \sum_{i=1}^{|S_j|} \| \ s_i^{(j)} - c_j \ \|^2$$

is minimized. Here $|S_j|$ is the number of points in S_j and $s_i^{(j)}$ is the i^{th} point in S_j. Note that if the cluster centres are known, then the function $f(S, S)$ achieves its minimum when each point is assigned to its closest cluster centre. Therefore, MSSC can be described by the following bilevel programming problem

$$\underset{\mathbf{c}_1,\dots,\mathbf{c}_k}{\text{Minimize}} \sum_{i=1}^{n} \min\left\{\|\mathbf{s}_i - \mathbf{c}_1\|^2, \dots, \|\mathbf{s}_i - \mathbf{c}_k\|^2\right\}. \tag{6.1}$$

On the other hand, if the points in cluster S_j are fixed, then the function

$$f\left(S_j, S_j\right) = \sum_{i=1}^{|S_j|} \left\|\mathbf{s}_i^{(j)} - \mathbf{c}_j\right\|^2$$

is minimized when

$$\mathbf{c}_j = \frac{1}{|S_j|} \sum_{i=1}^{|S_j|} \mathbf{s}_i^{(j)}.$$

Let $X = [x_{ij}] \in \mathbb{R}^{n \times K}$ be the assignment matrix defined by

$$x_{ij} = \begin{cases} 1, & \text{if } s_i \text{ is assigned to } S_j, \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, the cluster center of the cluster S_j, as the mean of all the points in the cluster, is defined by

$$\mathbf{c}_j = \frac{\sum_{l=1}^{n} x_{lj}\mathbf{s}_l}{\sum_{l=1}^{n} x_{lj}}.$$

Using this fact, we can represent (6.1) as

$$\begin{aligned} &\underset{x_{ij}}{\text{Minimize}} \sum_{j=1}^{k}\sum_{i=1}^{n} x_{ij} \left\| s_i - \frac{\sum_{l=1}^{n} x_{lj}\, s_l}{\sum_{l=1}^{n} x_{lj}} \right\|^2, \\ &\text{subject to } \sum_{j=1}^{k} x_{ij} = 1,\ \forall i \in \{1, \dots, n\}, && (6.2) \\ &\qquad\qquad \sum_{i=1}^{n} x_{ij} \geqslant 1,\ \forall j \in \{1, \dots, k\},\ x_{ij} \in \{0, 1\},\ \forall i, j \in \{1, \dots, n\}\}. && (6.3) \end{aligned}$$

The constraint (6.2) ensures that each point $\mathbf{s}_i$ is assigned to one and only one cluster, and constraint (6.3) ensures that there are exactly K clusters. Let $Z := [z_{ij}] = X(X^TX)^{-1}X^T$. Following Peng and Wei [43] the above problem can be modelled as a semidefinite programming problem as

$$\begin{aligned} &\text{Minimize} \quad \text{trace}(W_sW_s^T(I - Z)), \\ &\text{subject to} \quad Ze = e,\ \text{trace}(Z) = K,\ Z \geqslant 0,\ Z = Z^2,\ Z \in \mathbb{S}_+^n, \end{aligned}$$

where $W_s \in \mathbb{R}^{n\times d}$ denotes the matrix whose ith row is s_i^T, I is the $n \times n$ identity matrix, e is $n \times 1$ vector with all entries equal to 1, and $Z \geqslant 0$ means the componentwise inequality.

The constraint $Z = Z^2$ implies that $z_{ij} = z_{ij}^2$, which can be written as $z_{ij}(1 - z_{ij}) = 0$. Since $z_{ij} \geqslant 0$ and $1 - z_{ij} \geqslant 0$, the above problem can be reformulated as

$$\text{Minimize} \quad \Phi(Z) := \text{trace}(W_sW_s^T(I - Z)),$$

$$\begin{aligned} \text{subject to } \ & \Theta^{(1)}(Z) := ((z_{11}+\cdots+z_{1n}-1),\ldots,(z_{n1}+\cdots+z_{nn}-1)) = 0,\\ & \Theta^{(2)}(Z) := z_{11}+z_{22}+\cdots+z_{nn}-K = 0,\\ & \mathtt{K}_{ij}(Z) := z_{ij} \geqslant 0, \ \forall\, i,j \in \{1,\ldots,n\},\\ & \mathtt{L}_{ij}(Z) = (1-z_{ij}) \geqslant 0, \ \forall\, i,j \in \{1,\ldots,n\},\\ & \mathtt{K}_{ij}(Z)\mathtt{L}_{ij}(Z) = 0, \ \forall\, i,j \in \{1,\ldots,n\}, \end{aligned}$$

which is in the form of semidefinite mathematical programming problems with equilibrium constraints. In particular, we take $S = \{(0,0)^T, (1,0)^T, (2,0) \in \mathbb{R}^2\}$, and $K = 2$ in the following example to demonstrate the significance of the established results.

Example 6.1. Let us consider the following set of points given by $S = \{(0,0)^T, (1,0)^T, (2,0)^T \in \mathbb{R}^2\}$ and let $K = 2$ be the number of clusters. Then we have

$$W_s = \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4 \end{bmatrix} \ \text{ and } \ \Phi(Z) := \operatorname{trace}(W_s W_s^T(I-Z)) = 5 - z_{22} - 4z_{33}.$$

Thus the K-Means-type clustering problem can be formulated as a semidefinite programming problem with equilibrium constraints as

$$\begin{aligned} \text{Minimize } \ & \Phi(Z) := 5 - z_{22} - 4z_{33},\\ \text{subject to } \ & \Theta_1(Z) := z_{11}+z_{12}+z_{13}-1 = 0,\\ & \Theta_2(Z) := z_{21}+z_{22}+z_{23}-1 = 0,\\ & \Theta_3(Z) := z_{31}+z_{32}+z_{33}-1 = 0,\\ & \Theta_4(Z) := z_{11}+z_{22}+z_{33}-2 = 0,\\ & \mathtt{K}_{ij}(Z) := z_{ij} \geqslant 0, \ \forall\, i,j \in \{1,2,3\},\\ & \mathtt{L}_{ij}(Z) = (1-z_{ij}) \geqslant 0, \ \forall\, i,j \in \{1,2,3\},\\ & \mathtt{K}_{ij}(Z)\mathtt{L}_{ij}(Z) = 0, \ \forall\, i,j \in \{1,2,3\}, \end{aligned} \tag{P}$$

where $\Phi : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, $\Theta_t : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, $t \in \{1,2,3,4\}$, $\mathtt{K}_{ij}, \mathtt{L}_{ij} : \mathbb{S}^n_+ \to \overline{\mathbb{R}}$, $\forall i,j \in \{1,2,3\}$, and $Z = \begin{bmatrix} z_{11} & z_{12} & z_{13}\\ z_{21} & z_{21} & z_{23}\\ z_{31} & z_{32} & z_{33}\end{bmatrix} \in \mathbb{S}^3_+$. It is evident that $\overline{Z} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ satisfies all the constraints and hence is a feasible solution to the considered problem. Moreover, we calculate the USRC of each function of the problem at $\overline{Z}$, given by

$$\partial^*\Phi(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -4 \end{bmatrix} \right\}, \qquad \partial^*\Theta_1(\overline{Z}) = \left\{ \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*\Theta_2(\overline{Z}) = \left\{ \begin{bmatrix} 0 & \frac{1}{2} & 0\\ \frac{1}{2} & 1 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0 \end{bmatrix} \right\}, \qquad \partial^*\Theta_3(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & \frac{1}{2}\\ 0 & 0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \right\},$$

$$\partial^*\Theta_4(\overline{Z}) = \left\{ \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} \right\}, \qquad \partial^*(-\Theta_1)(\overline{Z}) = \left\{ \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & 0 & 0\\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*(-\Theta_2)(\overline{Z}) = \left\{ \begin{bmatrix} 0 & -\frac{1}{2} & 0\\ -\frac{1}{2} & -1 & -\frac{1}{2}\\ 0 & -\frac{1}{2} & 0 \end{bmatrix} \right\}, \qquad \partial^*(-\Theta_3)(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & -\frac{1}{2}\\ 0 & 0 & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix} \right\},$$

$$\partial^*(-\Theta_4)(\overline{Z}) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}, \qquad \partial^*K_{11}(\overline{Z}) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*K_{12}(\overline{Z}) = \partial^*K_{21}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*K_{13}(\overline{Z}) = \partial^*K_{31}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*K_{22}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*K_{33}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

$$\partial^*K_{23}(\overline{Z}) = \partial^*K_{32}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*L_{11}(\overline{Z}) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*L_{12}(\overline{Z}) = \partial^*L_{21}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*L_{13}(\overline{Z}) = \partial^*L_{31}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*L_{22}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*L_{33}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\},$$

$$\partial^*L_{23}(\overline{Z}) = \partial^*L_{32}(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*(-K_{11})(\overline{Z}) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*(-K_{12})(\overline{Z}) = \partial^*(-K_{21})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*(-K_{13})(\overline{Z}) = \partial^*(-K_{31})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*(-K_{22})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*(-K_{33})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\},$$

$$\partial^*(-K_{23})(\overline{Z}) = \partial^*(-K_{32})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*(-L_{11})(\overline{Z}) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*(-L_{12})(\overline{Z}) = \partial^*(-L_{21})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*(-L_{13})(\overline{Z}) = \partial^*(-L_{31})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*(-L_{22})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*(-L_{33})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

$$\partial^*(-L_{23})(\overline{Z}) = \partial^*(-L_{32})(\overline{Z}) = \left\{ \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \right\}.$$

It can be verified that there exist $\xi^{\Phi} \in \operatorname{co}\partial^*\Phi(\overline{Z})$, $\xi^{\Theta}_t \in \operatorname{co}\partial^*\Theta_t(\overline{Z})$, $t \in \{1,2,3,4\}$, $\eta^{\Theta}_t \in \operatorname{co}\partial^*(-\Theta_t)(\overline{Z})$, $\forall t \in \{1,2,3,4\}$, $\xi^K_{ij} \in \operatorname{co}\partial^*K_{ij}(\overline{Z})$, $\eta^K_{ij} \in \operatorname{co}\partial^*(-K_{ij})(\overline{Z})$, $\xi^L_{ij} \in \operatorname{co}\partial^*L_{ij}(\overline{Z})$, $\eta^L_{ij} \in \operatorname{co}\partial^*(-L_{ij})(\overline{Z})$, $\forall i,j \in \{1,2,3\}$, and $U \in \mathbb{S}^3_+$, such that $\overline{Z}$ is GS-stationary point. Moreover, related to the primal problem (P) the

Wolfe-type dual model (D_W) can be formulated as

$$\underset{(Y,\lambda)}{\text{Maximize}} \left\{ \Phi(Y) + \sum_{t\in\mathbb{J}} [\nu_t^{\Theta}\Theta_t(Y) + \tau_t^{\Theta}(-\Theta_t)(Y)] + \sum_{i,j\in\mathbb{T}} [\nu_{ij}^{K}(-K_{ij})(Y) + \nu_{ij}^{L}(-L_{ij})(Y)] - \langle U, Y\rangle \right\}, \quad \text{subject to } (Y,\lambda) \in \mathscr{F}_W, \tag{D_W}$$

where $\Phi(Y) = \text{trace}(W_s W_s^T(I-Y))$, $\mathbb{J} = \{1,2,3,4\}$, $\mathbb{T} = \{1,2,3\}$ and $\mathscr{F}_W$ denotes the set of all feasible solutions of the Wolfe-type dual problem (D_W) related to the primal problem (P) and is defined as

$$\begin{aligned}\mathscr{F}_W := \Big\{ & \lambda = (\nu^{\Phi}, \nu^{\Theta}, \tau^{\Theta}, \nu^{K}, \nu^{L}, U) \in \mathbb{R}\times\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^9\times\mathbb{R}^9\times\mathbb{S}^3_+, \\ & Y \in \mathbb{S}^3_+ :\ 0 \in \nu_i^{\Phi} co\partial^*\Phi_i(Y) + \sum_{t\in\mathbb{J}} [\nu_t^{\Theta} co\partial^*\Theta_t(Y) + \tau_t^{\Theta} co\partial^*(-\Theta_t)(Y)] \\ & \quad + \sum_{i,j\in\mathbb{T}} [\nu_{ij}^{K} co\partial^*(-K_{ij})(Y) + \nu_{ij}^{L} co\partial^*(-L_{ij})(Y)] - U, \\ & \nu_i^{\Phi} > 0, i \in \mathbb{I},\ \nu_t^{\Theta},\ \tau_t^{\Theta} \geqslant 0,\ t\in\mathbb{J},\ \nu_{ij}^{K},\ \nu_{ij}^{L},\ \tau_{ij}^{K},\ \tau_{ij}^{L} \geqslant 0,\ i,j\in\mathbb{T},\ \nu_{\gamma}^{K} = \nu_{\delta}^{L} = \tau_{\gamma}^{K} = \tau_{\delta}^{L} = 0 \Big\}.\end{aligned}$$

Furthermore, related to the primal problem (P) the Mond-Weir-type dual model (D_W) can be formulated as

$$\underset{(\mathscr{Z},\lambda)}{\text{Maximize}}\ \Phi(Y) = \text{trace}(W_s W_s^T(I-Y)), \quad \text{subject to } (Y,\lambda) \in \mathscr{F}_{MW}, \tag{D_M}$$

where $\mathbb{J} = \{1,2,3,4\}$, $\mathbb{T} = \{1,2,3\}$ and $\mathscr{F}_{MW}$ denotes the set of all feasible solutions of the Mond-Weir-type dual model (D_M) related to the primal problem (P) and is defined as

$$\begin{aligned}\mathscr{F}_{MW} = \Big\{ & \lambda = (\nu^{\Phi}, \nu^{\Theta}, \tau^{\Theta}, \nu^{K}, \nu^{L}, U) \in \mathbb{R}\times\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^9\times\mathbb{R}^9\times\mathbb{S}^3_+, \\ & Y \in \mathbb{S}^3_+ :\ 0 \in \nu_i^{\Phi} co\partial^*\Phi_i(Y) + \sum_{t\in\mathbb{J}} [\nu_t^{\Theta} co\partial^*\Theta_t(Y) + \tau_t^{\Theta} co\partial^*(-\Theta_t)(Y)] \\ & \quad + \sum_{i,j\in\mathbb{T}} [\nu_{ij}^{K} co\partial^*(-K_{ij})(Y) + \nu_{ij}^{L} co\partial^*(-L_{ij})(Y)] - U, \\ & \langle U, Y\rangle = 0,\ \Theta_t(Y) = 0,\ t\in\mathbb{J},\ K_{ij}(Y) \leqslant 0,\ ij \in (\delta\cup\omega),\ L_{ij}(Y) \leqslant 0,\ ij\in(\gamma\cup\omega), \\ & \nu_i^{\Phi} > 0, i \in \mathbb{I},\ \nu_i^{\Theta},\ \tau_i^{\Theta} \geqslant 0,\ i\in\mathbb{J},\ \nu_{ij}^{K},\ \nu_{ij}^{L},\ \tau_{ij}^{K},\ \tau_{ij}^{L} \geqslant 0,\ \forall i,j\in\mathbb{T},\ \nu_{\gamma}^{K} = \nu_{\delta}^{L} = \tau_{\gamma}^{K} = \tau_{\delta}^{L} = 0 \Big\}.\end{aligned}$$

Since $\bar{Z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a GS-stationary point and $\omega_\tau^K \cup \omega_\tau^L \cup \delta_\tau^+ \cup \gamma_\tau^+ = \emptyset$ of the primal problem (P). Moreover, Φ, $\pm\Theta_t$, $t\in\mathbb{J}$, $-K_{ij}$, $ij \in (\delta\cup\omega)$, and $-L_{ij}$, $ij\in(\gamma\cup\omega)$ admit bounded USRC and are ∂^*-convex functions at $\bar{Z}$. Since $Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a feasible solution to the Wolfe-type dual (D_W) such that

$$\Phi(Z) \geqslant \Phi(Y) + \sum_{t\in\mathbb{J}} [\nu_t^{\Theta}\Theta_t(Y) + \tau_t^{\Theta}(-\Theta_t)(Y)] + \sum_{i,j\in\mathbb{T}} [\nu_{ij}^{K}(-K_{ij})(Y) + \nu_{ij}^{L}(-L_{ij})(Y)] - \langle U, Y\rangle.$$

Hence weak duality theorem, i.e., Theorem 4.1, holds for the considered problem (P). Furthermore, NSMPEC-tailored ACQ is satisfied at $\bar{Z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and there exists $\bar{\lambda} = (\bar{\nu}^{\Phi}, \bar{\nu}^{\Theta}, \bar{\tau}^{\Theta}, \bar{\nu}^{K}, \bar{\nu}^{L}, \bar{U}) \in \mathbb{R}\times$

$\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{S}^3_+$ such that $(\bar{Z}, \bar{\lambda})$ is a feasible solution of the dual problem. Hence, the strong duality theorem, i.e., Theorem 4.2, holds for the considered problem (P). Similarly, for the Mond-Weir-type dual (D_M) related to the primal problem (P), $Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a feasible solution of the dual (D_M) such that $\Phi(Z) \geqslant \Phi(Y)$. Hence weak duality theorem, i.e., Theorem 5.1, holds for the the considered problem (P). Moreover, there exists $\tilde{\lambda} = (\tilde{\nu}^{\Phi}, \tilde{\nu}^{\Theta}, \tilde{\tau}^{\Theta}, \tilde{\nu}^{K}, \tilde{\nu}^{L}, \tilde{U}) \in \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{S}^3_+$ such that $(\tilde{Z}, \tilde{\lambda})$ is a feasible solution of the dual problem (D_M). Hence, the strong duality theorem, i.e., Theorem 5.2, holds for the considered problem (P).

7. Conclusions and future directions

In this article, we explored various duality results for a class of non-smooth semidefinite multiobjective programming problems with equilibrium constraints (NSMPEC). We have presented Wolfe-type (WMPEC) and Mond-Weir-type (MWMPEC) dual models related to the primal problem NSMPEC and established weak, strong and strict converse duality theorems for the corresponding dual models.

The various results established in this article extend many familiar results existing in the literature from Euclidean space to the space of symmetric positive semidefinite matrices. Especially, the duality theorems established in this article extend the duality theorems established in [27, 42] for Euclidean space $\mathbb{R}^n$ to $\mathbb{S}^n_+$. Moreover, as an application of the results established in this article, we have considered the problem of assigning n points into K clusters based on minimal sum-of-squared distances. In particular, for a certain set of points, given by $S = \{(0,0)^T, (1,0)^T, (2,0)^T \in \mathbb{R}^2\}$, we have considered the problem of assigning the points in S into two clusters. Moreover, we have formulated Wolfe-type and Mond-Weir-type dual models and demonstrated the significance of weak and strong duality theorems for the considered problem.

The various results introduced throughout the article leave various avenues for future research. For example, given the work presented by Lai et al. [29], studying duality results for non-smooth semidefinite multiobjective programming problems with vanishing constraints would be interesting.

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