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New types of convergence of double sequences in neutrosophic fuzzy G-metric spaces

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Abstract

In this study, we present statistical convergence, statistical limit points, and statistical cluster points of double sequences in neutrosophic fuzzy G-metric space with order q, extending the notion of neutrosophic fuzzy metric space. We support our assertions with relevant theorems and elucidate them through illustrative examples. Following the establishment of statistical convergence and the scrutiny of its properties within these spaces, we explore the concepts of lacunary statistical convergence and strongly lacunary convergence of double sequences, while also investigating the relationships among them.

Keywords: Neutrosophic normed spaces, g-metric space, statistical convergence, statistical Cauchy sequence, statistical limit points, statistical cluster points.

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1. Introduction and preliminaries

Statistical convergence was initially explored by Fast [12]. Mursaleen and Edely [31] later extended this concept to double sequences. Fridy and Orhan [13] investigated lacunary statistical convergence using lacunary sequences, which significantly impacted various scientific domains. Çakan and Altay [5] illustrated multidimensional parallels to the findings of Fridy and Orhan [13]. Additional studies on lacunary statistical convergence are available in works by Khan et al. [22], Patterson and Savaş [35], Tripathy and Baruah [39], Tripathy and Dutta [40], and others.

Zadeh [42] introduced the concept of fuzzy sets in 1965, which have found application across diverse fields like artificial intelligence, robotics, and control theory. In Zadeh's framework, a fuzzy set assigns a membership value between 0 and 1 to each element of a crisp universe set. Kramosil and Michálek [29] proposed the concept of fuzzy metric space as a generalization of the traditional metric space.

George and Veeramani [15] proposed adjustments to the fuzzy metric space introduced by Kramosil and Michálek [29], while Deng [7] suggested modifications inspired by Grabiec [16]. Concurrently,

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Atanassov [2] presented intuitionistic fuzzy sets, and Park [34] expanded this concept into intuitionistic fuzzy metric spaces, drawing on the groundwork laid by George and Veeramani [15].

In 1998, Samarandache [36] introduced the foundational principles of neutrosophic sets. These sets expand upon classical set theory by incorporating an intermediate membership function. Similar generalizations include fuzzy sets [42] and intuitionistic fuzzy sets [2]. Neutrosophic sets are defined by their independence regarding membership, non-membership, and hesitation functions. Samarandache and Kanasamy in [41] pioneered neutrosophic algebraic structures. Bera and Mahapatra [3] were the first to introduce neutrosophic soft linear spaces. Bera and Mahapatra [4] investigated neutrosophic soft norm linear spaces, convexity, metrics [42], and Cauchy sequences. Kirişci and Şimsek [27] defined the neutrosophic metric space using continuous triangular norms and conorms. Various topological and structural characteristics of neutrosophic metric spaces have been explored. Recently, statistical convergence has found notable applications, as exemplified in [9–11, 17, 18, 20, 21, 23–26, 28].

In various scientific fields, the distance function holds fundamental importance. Given the complexity of modern datasets, it becomes imperative to broaden the definition of distance functions. Initially, Gähler [14] introduced the concept of a 2-metric space as a nonlinear extension of the ordinary metric space, using the area of a triangle in $\mathbb{R}^2$ as an illustration. However, it was later discovered that there isn't a straightforward relationship between the 2-metric space and the ordinary metric space. Ha et al. [19] demonstrated that unlike an ordinary metric, a 2-metric may not be a continuous function of its variables, leading Dhage [8] to propose a new type of generalized metric space known as D-metric space. Subsequent verification in [32] revealed issues with the topological structure of D-metric spaces. To address this, Mustafa and Sims [33] introduced the G-metric space, which employs the perimeter of a triangle in $\mathbb{R}^2$ as a more suitable example. This novel approach significantly influenced the field of metric spaces. Building upon this advancement, authors in [30, 38] extended the concepts to generalized fuzzy metric spaces and intuitionistic generalized fuzzy metric spaces, respectively. Similarly, Zhou et al. [43] introduced the probabilistic version of G-metric space known as the Menger probabilistic G-metric space.

Alternatively, Choi et al. [6] extended the concept of distance between two points by considering $q+1$ points instead of two, introducing the notion of $\mathfrak{g}$-metric with order q. Building on this notion, Abazari [1] introduced the Menger probabilistic $\mathfrak{g}$-metric space as a generalization of the Menger probabilistic G-metric space, investigating statistical convergence with respect to the strong topology using the concept of q-dimensional asymptotic density of subsets of $\mathbb{N}$. These advancements prompt the exploration of extending the concept of $\mathfrak{g}$-metric to neutrosophic fuzzy settings. Additionally, within this generalized framework, there arises a question regarding the validity of statistical convergence incorporating the q-dimensional asymptotic density.

Inspired by the aforementioned concepts, this paper examines the properties the neutrosophic fuzzy version of the $\mathfrak{g}$-metric space, termed as the neutrosophic fuzzy G-metric space with order q. This space serves as a broader extension of the neutrosophic generalized fuzzy metric space. Leveraging the concept of q-dimensional asymptotic density [1], we delve into the statistical convergence and statistical Cauchy criteria of sequences within this defined space. Furthermore, we examine the statistical limit points and statistical cluster points of sequences in this context.

Now let us review a few definitions and notations we will use in this paper. Throughout this study, $\mathbb{R}^+$ stands for the set of non-negative real numbers.

Definition 1.1 ([33])**.** Consider an arbitrary non-empty set $\mathcal{U}$ and a mapping $G : \mathcal{U}^3 \longrightarrow \mathbb{R}^+$. We term the pair $(\mathcal{U}, G)$ a G-metric space if, for any $t, \alpha, \beta, \gamma \in \mathcal{U}$, the subsequent conditions are valid:

(G-1) $G(t, \alpha, \beta) = 0$ if $t = \alpha = \beta$;
(G-2) $G(t, t, \alpha) > 0$ if $t \neq \alpha$;
(G-3) $G(t, t, \alpha) \leqslant G(t, \alpha, \beta)$ if $\alpha \neq \beta$;
(G-4) $G(t, \alpha, \beta) = G(\alpha, \beta, t) = G(t, \beta, \alpha) = \cdots$ (symmetry in all three variables);
(G-5) $G(t, \alpha, \beta) \leqslant G(t, \gamma, \gamma) + G(\gamma, \alpha, \beta)$.

In this scenario, the function G is referred to as a G-metric on the set $\mathcal{U}$.

Example 1.2. Consider $(\mathcal{U}, d)$ as an ordinary metric space. Let $G : \mathcal{U}^3 \longrightarrow \mathbb{R}^+$ be defined as

$$G(t, \alpha, \beta) = \frac{1}{2}(d(t, \alpha) + d(\alpha, \beta) + d(t, \beta)).$$

Under this definition, $(\mathcal{U}, G)$ forms a G-metric space.

Remark 1.3. Let $(\mathcal{U}, G)$ represent a G-metric space. Now, consider the function $d_G : \mathcal{U}^2 \longrightarrow \mathbb{R}^+$, given by:

$$d_G(t, \alpha) = \frac{1}{3}(G(t, \alpha, \alpha) + G(t, t, \alpha)).$$

With this formulation, $(\mathcal{U}, d_G)$ constitutes an ordinary metric space.

Definition 1.4 ([37])**.** A function $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is termed a continuous t-norm under the following conditions:

(1) $*$ is both commutative and associative;
(2) $\varkappa = \varkappa * 1$ for any $0 \leqslant \varkappa \leqslant 1$;
(3) for all $0 \leqslant \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \leqslant 1$, if $\varkappa_1 \leqslant \varkappa_3$ and $\varkappa_2 \leqslant \varkappa_4$, then $\varkappa_1 * \varkappa_2 \leqslant \varkappa_3 * \varkappa_4$;
(4) $*$ exhibits continuity.

A binary function $\circledast : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is termed a continuous t-conorm if it satisfies the following conditions:

(1′) $\circledast$ is commutative and associative;
(2′) $\varkappa = \varkappa * 0$ for any $0 \leqslant \varkappa \leqslant 1$;
(3′) for each $0 \leqslant \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \leqslant 1$, if $\varkappa_1 \leqslant \varkappa_3$ and $\varkappa_2 \leqslant \varkappa_4$, then $\varkappa_1 \circledast \varkappa_2 \leqslant \varkappa_3 \circledast \varkappa_4$;
(4′) $\circledast$ is continuous.

Example 1.5. For $\varkappa_1, \varkappa_2 \in [0, 1]$, the following holds.

(1) $\varkappa_1 * \varkappa_2 = \min\{\varkappa_1, \varkappa_2\}$ and $\varkappa_1 * \varkappa_2 = \varkappa_1 \cdot \varkappa_2$ are continuous t-norms.
(2) $\varkappa_1 \circledast \varkappa_2 = \max\{\varkappa_1, \varkappa_2\}$ and $\varkappa_1 \circledast \varkappa_2 = \min\{\varkappa_1 + \varkappa_2, 1\}$ are continuous t-conorms on $[0, 1]$.

Remark 1.6. The concepts of t-norms and t-conorms serve as the foundational frameworks employed to define fuzzy intersections and unions, respectively.

Definition 1.7 ([27])**.** Suppose $\mathcal{U}$ is a non-empty set, $*$ and $\diamond$ represent continuous t-norm and continuous t-conorm, respectively, and F, G, H are fuzzy sets on $\mathcal{U}^2 \times (0, \infty)$. The six-tuple $(\mathcal{U}, F, G, H, *, \circledast)$ is denoted as a neutrosophic fuzzy metric space (NFMS) if for all $t, \alpha, \beta \in \mathcal{U}$ and $s, w > 0$, the following conditions are satisfied:

(NF-1) $0 \leqslant F(t, \alpha, s) \leqslant 1, 0 \leqslant G(t, \alpha, s) \leqslant 1, 0 \leqslant H(t, \alpha, s) \leqslant 1$;
(NF-2) $F(t, \alpha, s) + G(t, \alpha, s) + H(t, \alpha, s) \leqslant 3$;
(NF-3) $F(t, \alpha, s) = 1 \Longleftrightarrow t = \alpha$;
(NF-4) $F(t, \alpha, s) = F(\alpha, t, s)$;
(NF-5) $F(t, \beta, s) * F(\beta, \alpha, w) \leqslant F(t, \alpha, s + w)$;
(NF-6) $F(t, \alpha, .) : (0, \infty) \longrightarrow (0, 1]$ is continuous;
(NF-7) $\lim_{s\to\infty} F(t, \alpha, s) = 1$ $(\forall s > 0)$;
(NF-8) $G(t, \alpha, s) = 0 \Longleftrightarrow t = y$;
(NF-9) $G(t, \alpha, s) = G(y, \alpha, s)$;
(NF-10) $G(t, \beta, s) \circledast G(\beta, \alpha, w) \geqslant G(t, \alpha, s + w)$;
(NF-11) $G(t, \alpha, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous;
(NF-12) $\lim_{s\to\infty} G(t, \alpha, s) = 0$ $(\forall s > 0)$;

(NF-13) $H(t,\alpha,s)=0 \Longleftrightarrow t=\alpha$;
(NF-14) $H(t,\alpha,s)=H(\alpha,t,s)$;
(NF-15) $H(t,\beta,s)\circledast H(\beta,\alpha,w)\geqslant H(t,\alpha,s+w)$;
(NF-16) $H(t,\alpha,.):(0,\infty)\longrightarrow(0,1]$ is continuous;
(NF-17) $\lim_{s\to\infty}H(t,\alpha,s)=0\ (\forall s>0)$.

In this scenario, the triplet (F,G,H) is termed a neutrosophic fuzzy metric (NFM) on $\mathcal{U}$.

Definition 1.8 ([6]). Consider a non-empty set $\mathcal{U}$. A function $g:\mathcal{U}^{q+1}\longrightarrow\mathbb{R}^+$, where $\mathcal{U}^q=\prod_{j=1}^q\mathcal{U}^j$, characterizes a g-metric with order q on $\mathcal{U}$ under the following conditions:

(g_1) $g(t_0,t_1,\ldots,t_q)=0$ iff $t_0=t_1=\ldots=t_q$;
(g_2) $g(t_0,t_1,\ldots,t_q)=g\left(t_{\pi(0)},t_{\pi(1)},\ldots,t_{\pi(q)}\right)$ for any permutation π on $\{0,1,\ldots,q\}$;
(g_3) $g(t_0,t_1,\ldots,t_q)\leqslant g(\alpha_0,\alpha_1,\ldots,\alpha_q)$ for any $(t_0,t_1,\ldots,t_q),(\alpha_0,\alpha_1,\ldots,\alpha_q)\in\mathcal{U}^{q+1}$ with $\{t_0,t_1,\ldots,t_q\}\subsetneq\{\alpha_0,\alpha_1,\ldots,\alpha_q\}$;
(g_4) for each $t_0,t_1,\ldots,t_l,\alpha_0,\alpha_1,\ldots,\alpha_m,\beta\in\mathcal{U}$ with $l+m+1=q$,

$$g(t_0,t_1,\ldots,t_l,\beta,\ldots,\beta)+g(\alpha_0,\alpha_1,\ldots,\alpha_m,\beta,\ldots,\beta)\geqslant g(t_0,t_1,\ldots,t_l,\alpha_0,\alpha_1,\ldots,\alpha_m).$$

The pair $(\mathcal{U},g)$ is referred to as a g-metric space. When $q=1$ and $q=2$, a g-metric reduces to the ordinary metric and G-metric, respectively.

Example 1.9 ([6]). Consider $(\mathcal{U},d)$ as an ordinary metric space. Given $g:\mathcal{U}^{q+1}\longrightarrow\mathbb{R}^+$ defined by

$$g(t_0,t_1,\ldots,t_q)=\max_{0\leqslant m,n\leqslant q}\{|t_m-t_n|\}$$

for every $t_0,t_1,\ldots,t_q\in\mathcal{U}$, $(\mathcal{U},g)$ forms a g-metric space.

Definition 1.10 ([6]). A g-metric on $\mathcal{U}$ exhibits multiplicity independence if

$$g(t_0,t_1,\ldots,t_q)=g(\alpha_0,\alpha_1,\ldots,\alpha_q)$$

for each $(t_0,t_1,\ldots,t_q),(\alpha_0,\alpha_1,\ldots,\alpha_q)\in\mathcal{U}^{q+1}$ with $\{t_0,t_1,\ldots,t_q\}=\{\alpha_0,\alpha_1,\ldots,\alpha_q\}$.

Definition 1.11 ([6]). In a g-metric space $(\mathcal{U},g)$, the g-ball with center at $t\in\mathcal{U}$ and radius $r>0$ is defined as:

$$\mathbf{B}_g(t,r)=\{\alpha\in\mathcal{U}:g(t,\alpha,\ldots,\alpha)<r\}.$$

Definition 1.12 ([6]). Suppose that $(\mathcal{U},g)$ be a g-metric space and (t_k) be a sequence in $\mathcal{U}$. Then

(a) (t_k) is called to be g-convergent to some $t\in\mathcal{U}$ if, $\forall\xi>0$, $\exists K\in\mathbb{N}$ such that

$$g\left(t_{r_1},t_{r_2},\ldots,t_{r_q},x\right)<\xi,\ \forall r_1,r_2,\ldots,r_q\geqslant K.$$

(b) (t_k) is considered g-Cauchy if, $\forall\xi>0$, $\exists M\in\mathbb{N}$ such that

$$g\left(t_{r_0},t_{r_1},\ldots,t_{r_q}\right)<\xi,\ \forall r_0,r_1,\ldots,r_q\geqslant M.$$

In a g-metric space $(\mathcal{U},g)$ completeness is defined as the property where every g-Cauchy sequence converges to a limit in $\mathcal{U}$.

Definition 1.13. Assume $\mathcal{U}$ is a non-empty set, $*$ and $\diamond$ represent continuous t-norm and continuous t-conorm, respectively, and F,G,H are fuzzy sets on $\mathcal{U}^3\times(0,\infty)$. The six-tuple $(\mathcal{U},F,G,H,*,\circledast)$ is termed a neutrosophic generalized fuzzy metric space (NGFM-space) if, for all $t,\alpha,\beta,\gamma\in\mathcal{U}$ and $s,w>0$, subsequent conditions are valid:

(a) $F(t,\alpha,\beta,s)+G(t,\alpha,\beta,s)+H(t,\alpha,\beta,s)\leqslant 3$;
(b) $F(t,t,\alpha,s)>0$ for $t\neq\alpha$;
(c) $F(t,t,\alpha,s)\geqslant F(t,\alpha,\beta,s)$ for $\alpha\neq\beta$;
(d) $F(t,\alpha,\beta,s)=1 \Longleftrightarrow t=\alpha=\beta$;
(e) $F(t,\alpha,\beta,s)=F(\pi(t,\alpha,\beta),s)$, where π represents the permutation function;
(f) $F(t,\gamma,\gamma,s)*F(\gamma,\alpha,\beta,w)\leqslant F(t,\alpha,\beta,s+w)$;
(g) $F(t,\alpha,\beta,.):(0,\infty)\longrightarrow[0,1]$ is continuous;
(h) F is non-decreasing on $(0,\infty)$, $\lim_{s\to\infty}F(t,\alpha,\beta,s)=1$ and $\lim_{s\to 0}F(t,\alpha,\beta,s)=0$;
(i) $G(t,t,\alpha,s)<1$ for $t\neq\alpha$;
(j) $G(t,t,\alpha,s)\leqslant G(t,\alpha,\beta,s)$ for $\alpha\neq\beta$;
(k) $G(t,\alpha,\beta,s)=0 \Longleftrightarrow t=\alpha=\beta$;
(l) $G(t,\alpha,\beta,s)=G(\pi(t,\alpha,\beta),s)$, where π represents the permutation function;
(m) $G(t,\gamma,\gamma,s)\circledast G(\gamma,\alpha,\beta,w)\geqslant G(t,\alpha,\beta,s+w)$;
(n) $G(t,\alpha,\beta,.):(0,\infty)\longrightarrow[0,1]$ a continuous;
(o) G is non-increasing on $(0,\infty)$, $\lim_{s\to\infty}G(t,\alpha,\beta,s)=0$ and $\lim_{s\to 0}G(t,\alpha,\beta,s)=1$;
(p) $H(t,t,\alpha,s)<1$ for $t\neq\alpha$;
(q) $H(t,t,\alpha,s)\leqslant H(t,\alpha,\beta,s)$ for $\alpha\neq\beta$;
(r) $H(t,\alpha,\beta,s)=0 \Longleftrightarrow t=\alpha=\beta$;
(s) $H(t,\alpha,\beta,s)=H(\pi(t,\alpha,\beta),s)$, where π represents the permutation function;
(t) $H(t,\gamma,\gamma,s)\circledast H(\gamma,\alpha,\beta,w)\geqslant H(t,\alpha,\beta,s+w)$;
(u) $H(t,\alpha,\beta,.):(0,\infty)\longrightarrow[0,1]$ a continuous;
(v) H is non-increasing on $(0,\infty)$, $\lim_{s\to\infty}H(t,\alpha,\beta,s)=0$, and $\lim_{s\to 0}H(t,\alpha,\beta,s)=1$.

The triplet (F,G,H) is referred to as a neutrosophic generalized fuzzy metric (NGFM) on $\mathcal{U}$.

Now, we give the concept of Neutrosophic Fuzzy g-Metric space (NFGM), integrating continuous t-norms, continuous t-conorms, and neutrosophic fuzzy sets.

Definition 1.14. Consider an arbitrary non-empty set $\mathcal{U}$, with continuous t-norm $(*)$, continuous t-conorm $(\circledast)$, and fuzzy sets Θ,Ω,Ξ defined on on $\mathcal{U}^{q+1}\times(0,\infty)$. We denote the the six-tuple $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ as a neutrosophic fuzzy generalized metric space (abbreviated as NFGMS) with order q if the subsequent conditions are valid for all $s,w\in(0,\infty)$:

(NFG-1) $\Theta(t_0,t_1,\dots,t_q,s)+\Omega(t_0,t_1,\dots,t_q,s)+\Xi(t_0,t_1,\dots,t_q,s)\leqslant 3$ for every $t_0,t_1,\dots,t_q\in\mathcal{U}$;
(NFG-2) $\Theta(t_0,t_0,\dots,t_0,t_1,s)>0$ for $t_0\neq t_1$, $\forall t_0,t_1\in\mathcal{U}$;
(NFG-3) $\Theta(t_0,t_1,\dots,t_q,s)\geqslant\Theta(\alpha_0,\alpha_1,\dots,\alpha_q,s)$, $\forall(t_0,t_1,\dots,t_q),(\alpha_0,\alpha_1,\dots,\alpha_q)\in\mathcal{U}^{q+1}$ with $\{t_0,t_1,\dots,t_q\}\subsetneq\{\alpha_0,\alpha_1,\dots,\alpha_q\}$;
(NFG-4) $\Theta(t_0,t_1,\dots,t_q,s)=1\Longleftrightarrow t_0=t_1=\dots=t_q$;
(NFG-5) $\Theta(t_0,t_1,\dots,t_q,s)=\Theta\left(t_{\pi(0)},t_{\pi(1)},\dots,t_{\pi(q)},s\right)$ for any permutation π on $\{0,1,\dots,q\}$;
(NFG-6) for each $t_0,t_1,\dots,t_g$, $\alpha_0,\alpha_1,\dots,\alpha_h,\beta\in\mathcal{U}$ with $g+h+1=q$,

$$\Theta(t_0,t_1,\dots,t_g,\beta,\dots,\beta,s)*\Theta(\alpha_0,\alpha_1,\dots,\alpha_h,\beta,\dots,\beta,w)\leqslant\Theta(t_0,t_1,\dots,t_g,\alpha_0,\alpha_1,\dots,\alpha_h,s+w);$$

(NFG-7) $\Theta(t_0,t_1,\dots,t_q,.):(0,\infty)\longrightarrow[0,1]$ is a continuous function;
(NFG-8) $\lim_{s\to\infty}\Theta(t_0,t_1,\dots,t_q,s)=1$ for all $t_0,t_1,\dots,t_q\in\mathcal{U}$;
(NFG-9) $\Omega(t_0,t_0,\dots,t_0,t_1,s)<1$ for $t_0\neq t_1$, $\forall t_0,t_1\in\mathcal{U}$;
(NFG-10) $\Omega(t_0,t_1,\dots,t_q,s)\leqslant\Omega(\alpha_0,\alpha_1,\dots,\alpha_q,s)$, $\forall(t_0,t_1,\dots,t_q),(\alpha_0,\alpha_1,\dots,\alpha_q)\in\mathcal{U}^{q+1}$ with $\{t_0,t_1,\dots,t_q\}\subsetneq\{\alpha_0,\alpha_1,\dots,\alpha_q\}$;
(NFG-11) $\Omega(t_0,t_1,\dots,t_q,s)=0\Longleftrightarrow t_0=t_1=\cdots=t_q$;
(NFG-12) $\Omega(t_0,t_1,\dots,t_q,s)=\Omega\left(t_{\pi(0)},t_{\pi(1)},\dots,t_{\pi(q)},s\right)$ for any permutation π on $\{0,1,\dots,q\}$;

(NFG-13) for each $t_0, t_1, \ldots, t_g$, $\alpha_0, \alpha_1, \ldots, \alpha_h, \beta \in \mathcal{U}$ with $g+h+1=q$,

$$\Omega\left(t_0, t_1, \ldots, t_g, \beta, \ldots, \beta, s\right) \circledast \Omega\left(\alpha_0, \alpha_1, \ldots, \alpha_h, \beta, \ldots, \beta, w\right) \geqslant \Omega\left(t_0, t_1, \ldots, t_g, \alpha_0, \alpha_1, \ldots, \alpha_h, s+w\right);$$

(NFG-14) $\Omega\left(t_0, t_1, \ldots, t_q, .\right) : (0, \infty) \longrightarrow [0,1]$ is a continuous function;
(NFG-15) $\lim_{s \to \infty} \Omega\left(t_0, t_1, \ldots, t_q, s\right) = 0$ for all $t_0, t_1, \ldots, t_q \in \mathcal{U}$;
(NFG-16) $\Xi\left(t_0, t_0, \ldots, t_0, t_1, s\right) < 1$ for $t_0 \neq t_1$, $\forall t_0, t_1 \in \mathcal{U}$;
(NFG-17) $\Xi\left(t_0, t_1, \ldots, t_q, s\right) \leqslant \Xi\left(\alpha_0, \alpha_1, \ldots, \alpha_q, s\right)$, $\forall \left(t_0, t_1, \ldots, t_q\right), \left(\alpha_0, \alpha_1, \ldots, \alpha_q\right) \in \mathcal{U}^{q+1}$ with $\{t_0, t_1, \ldots, t_q\} \subsetneq \{\alpha_0, \alpha_1, \ldots, \alpha_q\}$;
(NFG-18) $\Xi\left(t_0, t_1, \ldots, t_q, s\right) = 0 \Longleftrightarrow t_0 = t_1 = \cdots = t_q$;
(NFG-19) $\Xi\left(t_0, t_1, \ldots, t_q, s\right) = \Xi\left(t_{\pi(0)}, t_{\pi(1)}, \ldots, t_{\pi(q)}, s\right)$ for any permutation π on $\{0, 1, \ldots, q\}$;
(NFG-20) for each $t_0, t_1, \ldots, t_g$, $\alpha_0, \alpha_1, \ldots, \alpha_h, l \in \mathcal{U}$ with $g+h+1=q$,

$$\Xi\left(t_0, t_1, \ldots, t_g, l, \ldots, l, s\right) \circledast \Xi\left(\alpha_0, \alpha_1, \ldots, \alpha_h, l, \ldots, l, w\right) \geqslant \Xi\left(t_0, t_1, \ldots, t_g, \alpha_0, \alpha_1, \ldots, \alpha_h, s+w\right);$$

(NFG-21) $\Xi\left(t_0, t_1, \ldots, t_q, .\right) : (0, \infty) \longrightarrow [0,1]$ is a continuous function;
(NFG-22) $\lim_{s \to \infty} \Xi\left(t_0, t_1, \ldots, t_q, s\right) = 0$ for all $t_0, t_1, \ldots, t_q \in \mathcal{U}$.

Moreover, we designate the triple (Θ, Ω, Ξ) as the neutrosophic fuzzy generalized metric NFGM on $\mathcal{U}$. To prevent ambiguity, we denote the six-tuple $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ as the NFGMS, rather than simply NFGMS with order q.

Example 1.15. Suppose that $(\mathcal{U}, g)$ be a g-metric space with order q. For $s > 0$, identify

$$\Theta\left(t_0, t_1, \ldots, t_q, s\right) = \frac{s}{s + g\left(t_0, t_1, \ldots, t_q\right)}, \qquad \Omega\left(t_0, t_1, \ldots, t_q, s\right) = \frac{g\left(t_0, t_1, \ldots, t_q\right)}{s + g\left(t_0, t_1, \ldots, t_q\right)},$$

and

$$\Xi\left(t_0, t_1, \ldots, t_q, s\right) = \frac{s}{g\left(t_0, t_1, \ldots, t_q\right)},$$

where $\varkappa_1 * \varkappa_2 = \varkappa_1 \cdot \varkappa_2$ and $\varkappa_1 \circledast \varkappa_2 = \min\{\varkappa_1 + \varkappa_2, 1\}$, $\forall \varkappa_1, \varkappa_2 \in [0,1]$. Then, $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ forms a NFGMS.

We can note that the same example holds true for $\varkappa_1 * \varkappa_2 = \min\{\varkappa_1, \varkappa_2\}$ and $\varkappa_1 \circledast \varkappa_2 = \max\{\varkappa_1, \varkappa_2\}$, $\forall \varkappa_1, \varkappa_2 \in [0,1]$. This metric space $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ is derived from the g-metric, recognized as the standard NFGMS.

Definition 1.16. The triple (Θ, Ω, Ξ) on NFGMS $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ is termed multiplicity independent if, for all $s > 0$, the following conditions are met:

$$\Theta\left(t_0, t_1, \ldots, t_q, s\right) = \Theta\left(\alpha_0, \alpha_1, \ldots, \alpha_q, s\right), \qquad \Omega\left(t_0, t_1, \ldots, t_q, s\right) = \Omega\left(\alpha_0, \alpha_1, \ldots, \alpha_q, s\right),$$

and

$$\Xi\left(t_0, t_1, \ldots, t_q, s\right) = \Xi\left(\alpha_0, \alpha_1, \ldots, \alpha_q, s\right),$$

where $\left(t_0, t_1, \ldots, t_q\right), \left(\alpha_0, \alpha_1, \ldots, \alpha_q\right) \in \mathcal{U}^{q+1}$ with $\{t_0, t_1, \ldots, t_q\} = \{\alpha_0, \alpha_1, \ldots, \alpha_q\}$.

Remark 1.17.

(a) When $q = 1$, the NFGMS simplifies to NFMS, and when $q = 2$, it simplifies to NGFMS. At these values, the multiplicity independence aligns with the symmetries present in the respective metrics.
(b) In Definition 1.14, by permitting conditions (NFG-3), (NFG-10) and (NFG-17) for $\{t_0, t_1, \ldots, t_q\} \subsetneq \{\alpha_0, \alpha_1, \ldots, \alpha_q\}$, the triple (Θ, Ω, Ξ) on NFGMS $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ achieves multiplicity independence.

Lemma 1.18. *Let* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ *denote an NFGMS. Then, for all* $t_0,t_1,\dots,t_q\in\mathcal{U}$, $\Theta(t_0,t_1,\dots,t_q,s)$, $\Omega(t_0,t_1,\dots,t_q,s)$, *and* $\Xi(t_0,t_1,\dots,t_q,s)$ *are non-decreasing, non-increasing functions, and non-increasing functions w.r.t.* s*, respectively.*

In the subsequent remark, we demonstrate that from any provided NFMS, it is possible to construct an NFGMS under specific limitations.

Remark 1.19. Let $(\mathcal{U},F,G,H*,\circledast)$ to be an NFMS satisfying $\lim_{s\to\infty}F(t,\rho,s)=1$, $\lim_{s\to\infty}G(t,\rho,s)=0$, and $\lim_{s\to\infty}H(t,\rho,s)=0$, $\forall t,\rho\in\mathcal{U}$. Then $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ is an NFGMS, where

$$\begin{aligned}\Theta(t_0,t_1,\dots,t_q,s)&=\min_{0\leqslant m,n\leqslant q}\{F(t_m,t_n,s)\},\\ \Omega(t_0,t_1,\dots,t_q,s)&=\max_{0\leqslant m,n\leqslant q}\{G(t_m,t_n,s)\},\\ \Xi(t_0,t_1,\dots,t_q,s)&=\max_{0\leqslant m,n\leqslant q}\{H(t_m,t_n,s)\}.\end{aligned}$$

We can see that (Θ,Ω,Ξ) is multiplicity independent in this case.

Remark 1.20. Consider $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ to be an NFGMS. Then $(\mathcal{U},F,G,H,*,\circledast)$ is an NFMS, where (F,G,H) is defined by

$$\begin{aligned}F(t,y,s)&=\min\left\{\Theta(\rho_0,\rho_1,\dots,\rho_q,s):\rho_j\in\{t,y\},\ j=0,1,\dots,q\right\},\\ G(t,y,s)&=\max\left\{\Omega(\rho_0,\rho_1,\dots,\rho_q,s):\rho_j\in\{t,y\},\ j=0,1,\dots,q\right\},\\ H(t,y,s)&=\max\left\{\Xi(\rho_0,\rho_1,\dots,\rho_q,s):\rho_j\in\{t,y\},\ j=0,1,\dots,q\right\}.\end{aligned}$$

Proposition 1.21. *Under the NFGMS* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$*, where* $\varkappa*\varkappa=\varkappa$ *and* $\varkappa\circledast\varkappa=\varkappa$ *for all* $\varkappa\in[0,1]$*, the following statements hold:*

(a)

$$\begin{aligned}\Theta(\underbrace{t,t,\dots,t}_{u\text{-times}},y,\dots,y,s)&\geqslant\Theta\left(t,y,\dots,y,\frac{s}{2^{u-1}}\right),\\ \Omega(\underbrace{t,t,\dots,t}_{u\text{-times}},y,\dots,y,s)&\leqslant\Omega\left(t,y,\dots,y,\frac{s}{2^{u-1}}\right),\\ \Xi(\underbrace{t,t,\dots,t}_{u\text{-times}},y,\dots,y,s)&\leqslant\Xi\left(t,y,\dots,y,\frac{s}{2^{u-1}}\right);\end{aligned}$$

(b)

$$\begin{aligned}\Theta(\underbrace{t,t,\dots,t}_{u\text{-times}},y,\dots,y,s)&\geqslant\Theta\left(y,t,\dots,t,\frac{s}{2^{q-1}}\right),\\ \Omega(\underbrace{t,t,\dots,t}_{u\text{-times}},y,\dots,y,s)&\leqslant\Omega\left(y,t,\dots,t,\frac{s}{2^{q-1}}\right),\\ \Xi(\underbrace{t,t,\dots,t}_{u\text{-times}},y,\dots,y,s)&\leqslant\Xi\left(y,t,\dots,t,\frac{s}{2^{q-1}}\right).\end{aligned}$$

The following outlines the implementation of a topology induced by (Θ,Ω,Ξ) in the NFGMS.

Definition 1.22. Consider $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ as an NFGMS and $t_0\in\mathcal{U}$. The open ball with center t_0 and radius $r\in(0,1)$ w.r.t. $s>0$, denoted as $\mathcal{B}_{t_0}^{(\Theta,\Omega,\Xi)}(s,r)$ is defined as

$$\mathcal{B}_{t_0}^{(\Theta,\Omega,\Xi)}(s,r)=\{\alpha\in\mathcal{U}:\Theta(t_0,\alpha,\alpha,\dots,\alpha,s)>1-r,\ \Omega(t_0,\alpha,\alpha,\dots,\alpha,s)<r,\ \text{and}\ \Xi(t_0,\alpha,\alpha,\dots,\alpha,s)<r\}.$$

Remark 1.23. Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ be an NFGMS. Define

$$\mathcal{T}^{(\Theta,\Omega,\Xi)} = \left\{\mathbf{K} \subset \mathcal{U} : \text{ for each } t_0 \in \mathbf{K}, \exists r \in (0,1) \text{ and } s > 0 \text{ such that } \mathcal{B}_{t_0}^{(\Theta,\Omega,\Xi)}(s,r) \subset \mathbf{K}\right\}.$$

Then $\mathcal{T}^{(\Theta,\Omega,\Xi)}$ forms a topology on $\mathcal{U}$ induced by (Θ,Ω,Ξ). It's evident that the set $\left\{\mathcal{B}_{t_0}^{(\Theta,\Omega,\Xi)}\left(\frac{1}{m},\frac{1}{m}\right)\right\}$ is a local base at $t_0 \in \mathcal{U}$, thus $\mathcal{T}^{(\Theta,\Omega,\Xi)}$ is first countable. Additionally, every open ball is an open set in the topology $\mathcal{T}^{(\Theta,\Omega,\Xi)}$.

Proposition 1.24. *Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ be a NFGMS. Suppose $\Theta(t_0, t_1, \ldots, t_q, s) > 1 - \varkappa$, $\Omega(t_0, t_1, \ldots, t_q, s) < \varkappa$ and $\Xi(t_0, t_1, \ldots, t_q, s) < \varkappa$ for some $\varkappa \in (0,1)$ and $s > 0$. Then*

(a) *if $q(\{t_0, t_1, \ldots, t_q\}) \geqslant 3$, then $t_j \in \mathcal{B}_{t_0}^{(\Theta,\Omega,\Xi)}(s,r)$ for each $j \in \{0,1,\ldots,q\}$;*

(b) *if (Θ,Ω,Ξ) is multiplicity independent and $q(\{t_0, t_1, \ldots, t_q\}) \geqslant 2$, then $t_j \in \mathcal{B}_{t_0}^{(\Theta,\Omega,\Xi)}(s,r)$ for each $j \in \{0,1,\ldots,q\}$.*

Theorem 1.25. *Consider $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ be an NFGMS, where $\varkappa * \varkappa = \varkappa$ and $\varkappa \circledast \varkappa = \varkappa$ for all $\varkappa \in [0,1]$. Then $\left(\mathcal{U}, \mathcal{T}^{(\Theta,\Omega,\Xi)}\right)$ is Hausdorff.*

A double sequence $\theta_2 = \theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_r) and (l_s) such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \ \text{ and } \ l_0 = 0,\ h_s = l_s - l_{s-1} \to \infty, \quad r,s \to \infty.$$

We will utilize the following notation $k_{rs} := k_r l_s$, $h_{rs} := h_r h_s$ and θ_{rs} is identified by

$$I_{rs} := \{(k,l) : k_{r-1} < k \leqslant k_r \text{ and } l_{s-1} < l \leqslant l_s\}, \quad q_r := \frac{k_r}{k_{r-1}}, \quad q_s := \frac{l_s}{l_{s-1}}, \quad \text{and} \quad q_{rs} := q_r q_s.$$

Throughout the paper, by $\theta_2 = \theta_{r,s} = \{(k_r, l_s)\}$ we will denote a double lacunary sequence of positive real numbers, unless otherwise stated.

2. Statistical convergence of double sequences in NFGMS

In this section, our goal is to delve into the idea of statistical convergence of double sequences within the NFGMS $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$. To do so, let's revisit some key concepts.

Given $K \subseteq \mathbb{N}$, the asymptotic density of set A, denoted as $d(K)$, is determined by

$$d(K) = \lim_{u\to\infty} \frac{1}{u} |\{m \leqslant u : m \in K\}|$$

if the limit exists. A real sequence (w_u) is considered statistically convergent to $w_0 \in \mathbb{R}$ if

$$d(\{u \in \mathbb{N} : |w_u - w_0| > \kappa\}) = 0$$

supplies for each $\kappa > 0$ and the limit is demonstrated by $st - w_u = w_0$ (see [12]).

Definition 2.1 ([1]). Let's define the q-product of $\mathbb{N}$ as $\mathbb{N}^q = \prod_{r=1}^{q} \mathbb{N}^r$. Consider $Y \subseteq \mathbb{N}^q$ and define $Y(u)$ as

$$Y(u) = \{(r_1, r_2, \ldots, r_q) \in Y : r_1, r_2, \ldots, r_q \leqslant u\}.$$

Then, the q-dimensional asymptotic density of the set Y is given by

$$d_q(Y) = \lim_{u\to\infty} \frac{q!}{u^q} |Y(u)|.$$

For a subset $K \subseteq \mathbb{N}$, the q-dimensional asymptotic density of the set K is given as

$$d_q(K) = \lim_{u\to\infty} \frac{q!}{u^q} |K(u)|,$$

where

$$K(u) = \{(u_1, u_2, \ldots, u_q) \in K^q : u_1, u_2, \ldots, u_q \leqslant u\}.$$

Definition 2.2 ([1]). Suppose $K = \{u_t : t \in \mathbb{N}\}$ is a subset of $\mathbb{N}$. Then, K is termed statistically dense in $\mathbb{N}$ if

$$d_q(K) = \lim_{u\to\infty} \frac{q!}{u^q}|K(u)| = 1.$$

Now, we can present our main result as follows.

Definition 2.3. Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ denote an NFGMS. A sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ is considered convergent to some $t \in \mathcal{U}$ w.r.t. the (Θ, Ω, Ξ) if, for each $\varkappa \in (0, 1)$ and $s > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$\Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) > 1 - \varkappa,\ \Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \varkappa$$

and

$$\Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \varkappa$$

for all $r_1, r_2, \dots, r_q \geqslant n_0$ and $u_1, u_2, \dots, u_q \geqslant n_0$.

Theorem 2.4. *Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ be an NFGMS and $\mathcal{T}^{(\Theta,\Omega,\Xi)}$ be the topology on $\mathcal{U}$. Then a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ converges to t iff $\Theta\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 1$, $\Omega\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 0$, and $\Xi\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 0$ as $\alpha, \beta \to \infty$ for every $s > 0$.*

Proof. Assuming that $(t_{\alpha\beta})$ converges to t. Then, for every $s > 0$ and $\varkappa \in (0, 1)$, $\exists n_0 \in \mathbb{N}$ such that $t_{\alpha\beta} \in \mathcal{B}_t^{(\Theta,\Omega,\Xi)}(s, \varkappa)$, $\forall \alpha, \beta \geqslant n_0$. As a result, for all $\alpha, \beta \geqslant n_0$, we have

$$\Theta\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) > 1 - \varkappa,\ \ \Omega\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) < \varkappa,\ \text{ and }\ \Xi\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) < \varkappa.$$

This yields that $1 - \Theta\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) < \varkappa$, $\Omega\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) < \varkappa$, and $\Xi(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s) < \varkappa$. Consequently, we get

$$\Theta\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 1,\ \ \Omega\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 0$$

and $\Xi\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 0$ as $\alpha, \beta \to \infty$. Conversely, assume that

$$\Theta\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 1,\ \ \Omega\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 0$$

and $\Xi\left(t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, t, s\right) \to 0$ as $\alpha, \beta \to \infty$ for all $s > 0$. Then, for given $\varkappa \in (0, 1)$, $\exists n_0 \in \mathbb{N}$ so that

$$1 - \Theta\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) < \varkappa,\ \ \Omega\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) < \varkappa$$

and $\Xi\left(t, t_{\alpha\beta}, t_{\alpha\beta}, \dots, t_{\alpha\beta}, s\right) < \varkappa$, $\forall \alpha, \beta \geqslant n_0$. Hence $t_{\alpha\beta} \in \mathcal{B}_t^{(\Theta,\Omega,\Xi)}(s, \varkappa)$, $\forall \alpha, \beta \geqslant n_0$. Thus $(t_{\alpha\beta})$ is convergent to t. □

Definition 2.5. In an NFGMS $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$, a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ is considered Cauchy w.r.t. the (Θ, Ω, Ξ) if, for every $\varkappa \in (0, 1)$ and $s > 0$, $\exists n_0 \in \mathbb{N}$ such that $\Theta\left(t_{r_0u_0}, t_{r_1u_1}, \dots, t_{r_qu_q}, s\right) > 1 - \varkappa$, $\Omega\left(t_{r_0u_0}, t_{r_1u_1}, \dots, t_{r_qu_q}, s\right) < \varkappa$ and $\Xi\left(t_{r_0u_0}, t_{r_1u_1}, \dots, t_{r_qu_q}, s\right) < \varkappa$ for all $r_0, r_1, \dots, r_q \geqslant n_0$ and $u_0, u_1, \dots, u_q \geqslant n_0$.

An NFGMS $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ is regarded as complete if every Cauchy sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ is convergent.

Theorem 2.6. *Consider $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ to be an NFGMS. Then, every convergent sequence $(t_{\alpha\beta})$ is Cauchy in $\mathcal{U}$.*

Definition 2.7. Consider $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ as an NFGMS, with $t \in \mathcal{U}$. For each $\varkappa \in (0, 1)$ and $s > 0$, we define the $(s, \varkappa)$-vicinity of $t \in \mathcal{U}$ w.r.t. (Θ, Ω, Ξ) as

$$\mathcal{V}_t^{(\Theta,\Omega,\Xi)}(s, \varkappa) = \{(t_1, t_2, \dots, t_q) \in \mathcal{U}^q : \Theta\left(t, t_1, t_2, \dots, t_q, s\right) > 1 - \varkappa \text{ and } \Omega\left(t, t_1, t_2, \dots, t_q, s\right) < \varkappa, \Xi\left(t, t_1, t_2, \dots, t_q, s\right) < \varkappa\}.$$

Definition 2.8. Let $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ represents an NFGMS. A sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ is statistically convergent to a $t\in\mathcal{U}$ w.r.t. (Θ,Ω,Ξ) if, for each $\varkappa\in(0,1)$ and $s>0$,

$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\leqslant 1-\varkappa \right.$$
$$\left.\text{or } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa,\ \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa\}\right)=0,$$

or equivalently

$$\lim_{\alpha,\beta\to\infty}\frac{q!}{(\alpha\beta)^q}\big|\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q: r_1,r_2,\dots,r_q\leqslant\alpha,$$
$$u_1,u_2,\dots,u_q\leqslant\beta,\ \Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\leqslant 1-\varkappa \text{ or}$$
$$\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa,\ \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa\}\big|=0.$$

In this scenario, we denote the convergence as $t_{\alpha\beta}\overset{st_2-(\Theta,\Omega,\Xi)}{\rightarrow}t$ or $st_2-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$. The collection of all statistically convergent sequences in NFGMS is symbolized as $st_2^{(\Theta,\Omega,\Xi)}$.

Lemma 2.9. *In an NFGMS $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ with a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$, under a fixed $s>0$ and $\varkappa\in(0,1)$, the following assertions hold true interchangeably.*

(a) $st_2-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$.

(b)

$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\varkappa\right.$$
$$\left.\textit{and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa,\ \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa\}\right)=1.$$

(c)

$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\leqslant 1-\varkappa\}\right)=0$$

and

$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa\}\right)=0,$$
$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa\}\right)=0.$$

(d)

$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\varkappa\}\right)=1$$

and

$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa\}\right)=1,$$
$$d_q\left(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa\}\right)=1.$$

Theorem 2.10. *Given an NFGMS $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ with a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$, and for a fixed $s>0$ and $\varkappa\in(0,1)$, the following statements are equivalent:*

(a) $st_2-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$;

(b) $d_q\left(\left\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q}\right)\notin\mathcal{V}_t^{(\Theta,\Omega,\Xi)}(s,\varkappa)\right\}\right)=0.$

Proof. The proof follows directly from the definition of statistical convergence. □

Theorem 2.11. *Let* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ *represent an NFGMS. Assume* $(t_{\alpha\beta})$ *is a sequence in* $\mathcal{U}$ *such that* $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta,\Omega,\Xi)$*. For any* $\varkappa\in(0,1)$ *and* $s>0$*, it holds that:*

$$d_q\left(\left\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q : t_{r_f u_f}\notin\mathcal{B}_t^{(\Theta,\Omega,\Xi)}(s,\varkappa)\right\}\right)=0,$$

for all $f\in\{1,2,\dots,q\}$.

Proof. Given $\varkappa\in(0,1)$ and $s>0$, define

$$\begin{aligned}\mathbf{K}(s,\varkappa) = (\{\,((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q : \Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\varkappa\\ \text{and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa,\ \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa\}),\end{aligned}$$

and

$$\mathbf{L}(s,\varkappa)=\left\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q : t_{r_f u_f}\in\mathcal{B}_t^{(\Theta,\Omega,\Xi)}(s,\varkappa)\right\},$$

for all $f\in\{1,2,\dots,q\}$. Since $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$, so $d_q(\mathbf{K}(s,\varkappa)))=1$. Suppose $(r_1,r_2,\dots,r_q)$, $(u_1,u_2,\dots,u_q)\in\mathbf{K}(s,\varkappa)$. Then

$$\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\varkappa \text{ and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa,$$

$\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa$. According to Proposition 1.24, we deduce that $t_{r_f u_f}\in\mathcal{B}_t^{(\Theta,\Omega,\Xi)}(s,\varkappa)$ for each $f\in\{1,2,\dots,q\}$. Thus $(r_1,r_2,\dots,r_q)$, $(u_1,u_2,\dots,u_q)\in\mathbf{L}(s,\varkappa)$, and consequently $\mathbf{K}(s,\varkappa)\subseteq\mathbf{L}(s,\varkappa)$. This implies $d_q(\mathbf{L}(s,\varkappa))=1$, leading to the desired conclusion. □

Theorem 2.12. *Let* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ *represent an NFGMS. If a sequence* $(t_{\alpha\beta})$ *in* $\mathcal{U}$ *is convergent to* $t\in\mathcal{U}$*, then* $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$.

Proof. Assume that $(t_{\alpha\beta})$ converges to $t\in\mathcal{U}$. For each $\varkappa\in(0,1)$ and $s>0$, $\exists w_0\in\mathbb{N}$ such that $\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\varkappa$ and $\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa$, $\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa$, for all $r_0,r_1,r_2,\dots,r_q\geqslant w_0$ and $u_1,u_2,\dots,u_q\geqslant w_0$. Let's define

$$\begin{aligned}\mathbf{K}(w,\rho)=\{\,&((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q : r_1,r_2,\dots,r_q\leqslant w,\\ &u_1,u_2,\dots,u_q\leqslant\rho,\ \Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\varkappa,\\ &\text{and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa,\ \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\varkappa\}.\end{aligned}$$

Obviously,

$$|\mathbf{K}(w,\rho)|\geqslant\begin{pmatrix} w\rho-w_0^2\\ q\end{pmatrix}. \Longrightarrow \lim_{w,\rho\to\infty}\frac{q!}{(w\rho)^q}|\mathbf{K}(w,\rho)|\geqslant\lim_{w,\rho\to\infty}\frac{q!}{(w\rho)^q}\begin{pmatrix} w\rho-w_0^2\\ q\end{pmatrix}=1.$$

Hence, $\lim_{w,\rho\to\infty}\frac{q!}{(w\rho)^q}\left|(\mathbf{K}(w,\rho))^c\right|=0$. This gives that $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$. □

We offer the following example to illustrate that the converse of Theorem 2.12 does not hold.

Example 2.13. Consider the NFGMS $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ as defined in Example 1.15, where $\mathcal{U}=\mathbb{R}$ and $g:\mathbb{R}^{q+1}\longrightarrow\mathbb{R}^+$ so that $g\,(t_0,t_1,\dots,t_q)=\max_{0\leqslant m,h\leqslant q}\{|t_m-t_h|\}$. Now, let's define the sequence $(t_{\alpha\beta})$ in $\mathbb{R}$ as

$$t_{\alpha\beta}=\begin{cases}\alpha\beta, & \text{if } k=m^3,\ \beta=n^3,\\ 1, & \text{if not},\end{cases}$$

where $m,n\in\mathbb{N}$. Then, $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=1\,(\Theta,\Omega,\Xi)$, but not convergent.

Theorem 2.14. *In an NFGMS* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$*, if* $(t_{\alpha\beta})$ *is statistically convergent sequence in* $\mathcal{U}$*, then the statistical limit* $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}$ *is unique.*

Proof. Suppose that $st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta,\Omega,\Xi)$ and $st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = p\,(\Theta,\Omega,\Xi)$. Our objective is to demonstrate that $t = p$. Given $\varkappa \in (0,1)$, let's select $\varkappa_1 \in (0,1)$ such that $(1-\varkappa_1) * (1-\varkappa_1) > 1-\varkappa$ and $\varkappa_1 \circledast \varkappa_1 < \varkappa$. Now, for a given $s > 0$, we examine the subsequent sets:

$$\begin{aligned}
\mathbf{A}\,(s,\varkappa_1) := \Big\{ &((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \leqslant 1-\varkappa_1 \\
&\text{or } \Omega\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \geqslant \varkappa_1, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \geqslant \varkappa_1 \Big\},\\
\mathbf{B}\,(s,\varkappa_1) := \Big\{ &((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \leqslant 1-\varkappa_1 \\
&\text{or } \Omega\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \geqslant \varkappa_1, \Xi\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \geqslant \varkappa_1 \Big\},\\
\mathbf{A}^c\,(s,\varkappa_1) := \Big\{ &((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) > 1-\varkappa_1 \\
&\text{and } \Omega\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) < \varkappa_1, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) < \varkappa_1 \Big\},\\
\mathbf{B}^c\,(s,\varkappa_1) := \Big\{ &((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) > 1-\varkappa_1 \\
&\text{and } \Omega\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) < \varkappa_1, \Xi\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) < \varkappa_1 \Big\}.
\end{aligned}$$

Since $st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta,\Omega,\Xi)$ and $st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = p\,(\Theta,\Omega,\Xi)$, we have $d_q\,(\mathbf{A}\,(s,\varkappa_1)) = 0$ and $d_q\,(\mathbf{B}\,(s,\varkappa_1)) = 0$. Also, by Lemma 2.9, we have$d_q\,(\mathbf{A}^c\,(s,\varkappa_1)) = d_q\,(\mathbf{B}^c\,(s,\varkappa_1)) = 1$. Thus

$$d_q\,(\mathbf{A}\,(s,\varkappa_1) \cup \mathbf{B}\,(s,\varkappa_1)) = 0,$$

implies

$$d_q\,((\mathbf{A}\,(s,\varkappa_1) \cup \mathbf{B}\,(s,\varkappa_1))^c) = d_q\,(\mathbf{A}\,(s,\varkappa_1)^c \cap \mathbf{B}\,(s,\varkappa_1)^c) = 1.$$

Let $(r_1,r_2,\dots,r_q)$, $(u_1,u_2,\dots,u_q) \in \mathbf{A}\,(s,\varkappa_1)^c \cap \mathbf{B}\,(s,\varkappa_1)^c$. Utilizing (NFG-3), (NFG-6), and part (3) of Definition 1.4, we obtain

$$\begin{aligned}
\Theta(t,p,\dots,p,s) &\geqslant \Theta\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) * \Theta\left(t_{r_qu_q}, p, p,\dots,p,\frac{s}{2}\right)\\
&\geqslant \Theta\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) * \Theta\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right)\\
&\geqslant (1-\varkappa_1) * (1-\varkappa_1) > 1-\varkappa.
\end{aligned}$$

Utilizing (NFG-10), (NFG-13), and part (3′) of Definition 1.4, we get

$$\begin{aligned}
\Omega(t,p,\dots,p,s) &\leqslant \Omega\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \circledast \Omega\left(t_{r_qu_q}, p, p,\dots,p,\frac{s}{2}\right)\\
&\leqslant \Omega\left(t, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \circledast \Omega\left(p, t_{r_1u_1}, t_{r_2u_2},\dots,t_{r_qu_q},\frac{s}{2}\right) \leqslant \varkappa_1 \circledast \varkappa_1 < \varkappa.
\end{aligned}$$

Also, utilizing (NFG-17), (NFG-20), and part (3′) of Definition 1.4, we get

$$\Xi(t,p,\dots,p,s) < \varkappa.$$

Since $\varkappa \in (0,1)$ is arbitrary, we conclude that

$$\Theta(t,p,\dots,p,s) = 1 \text{ and } \Omega(t,p,\dots,p,s) = 0, \Xi(t,p,\dots,p,s) < \varkappa, \quad \forall s > 0.$$

Hence $t = p$. □

Definition 2.15. Consider $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ as an NFGMS and $(t_{\alpha_w\beta_\kappa})$ as a subsequence of $(t_{\alpha\beta})$ in $\mathcal{U}$. Then $(t_{\alpha_w\beta_\kappa})$ is termed a statistically dense subsequence of $(t_{\alpha\beta})$ if the index set $\{(\alpha_w,\beta_\kappa) : w,\kappa \in \mathbb{N}\}$ is statistically dense in $\mathbb{N}^2$, denoted by $d_q\,(\{(\alpha_w,\beta_\kappa) : w,\kappa \in \mathbb{N}\}) = 1$.

Theorem 2.16. *Consider* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ *as an NFGMS and let* $(t_{\alpha\beta})$ *denote a sequence in* $\mathcal{U}$. *In light of this, the following assertions are interchangeable.*

(1) $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$, *for some* $t\in\mathcal{U}$.

(2) *There is a convergent sequence* $(y_{\alpha\beta})$ *in* $\mathcal{U}$ *with* $t_{\alpha\beta}=y_{\alpha\beta}$ *for almost all* α,β.

(3) *There is a subsequence* $(t_{\alpha_w\beta_\kappa})$ *of* $(t_{\alpha\beta})$ *such that* $(t_{\alpha_w\beta_\kappa})$ *is statistically dense and* $(t_{\alpha_w\beta_\kappa})$ *is convergent.*

Proof.

(1)$\Longrightarrow$(2) Let $t\in\mathcal{U}$ and $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$. According to Lemma 2.9, for any $\varkappa\in(0,1)$ and $s>0$, we have

$$d_q\big(\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\varkappa \\ \text{and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\eta, \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\eta\}\big)=1.$$

For $\tau\in\mathbb{N}$, establish the set

$$K(s,\tau)=\Big\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\frac{1}{\tau} \\ \text{and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\frac{1}{\tau}, \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\frac{1}{\tau}\Big\}.$$

It is evident that, $K(s,\tau+1)$ is a subset of $K(s,\tau)$ for each $\tau\in\mathbb{N}$. Since $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$, we conclude that

$$d_q(K(s,\tau))=1\ (\tau\in\mathbb{N}). \tag{2.1}$$

Select an arbitrary $(r_1^1,r_2^1,\dots,r_q^1)$, $(u_1^1,u_2^1,\dots,u_q^1)\in K(s,1)$. Let

$$m_1=\max\left\{r_1^1,r_2^1,\dots,r_q^1\right\},\qquad n_1=\max\left\{u_1^1,u_2^1,\dots,u_q^1\right\}.$$

Since (2.1) supplies, there exist $(r_1^2,r_2^2,\dots,r_q^2),(u_1^2,u_2^2,\dots,u_q^2)\in K(s,2)$, and

$$m_2=\max\left\{r_1^2,r_2^2,\dots,r_q^2\right\},\qquad n_2=\max\left\{u_1^2,u_2^2,\dots,u_q^2\right\},$$

such that $m_2>m_1$, and $n_2>n_1$ for all $\alpha\geqslant m_2$, $\beta\geqslant n_2$ we have

$$\lim_{\alpha,\beta\to\infty}\frac{q!}{(\alpha\beta)^q}\Big|\Big\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:r_1,r_2,\dots,r_q\leqslant\alpha, \\ u_1,u_2,\dots,u_q\leqslant\beta, \Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\frac{1}{2} \\ \text{and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\frac{1}{2}, \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\frac{1}{2}\Big\}\Big|>\frac{1}{2}.$$

Continuing this process by (2.1), there exist $(r_1^3,r_2^3,\dots,r_q^3),(u_1^3,u_2^3,\dots,u_q^3)\in K(s,3)$ with

$$m_3=\max\left\{r_1^3,r_2^3,\dots,r_q^3\right\}>m_2\quad\text{and}\quad n_3=\max\left\{u_1^3,u_2^3,\dots,u_q^3\right\}>n_2,$$

such that for all $\alpha\geqslant m_3$, $\beta\geqslant n_3$ we obtain

$$\lim_{\alpha,\beta\to\infty}\frac{q!}{(\alpha\beta)^q}\Big|\Big\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:r_1,r_2,\dots,r_q\leqslant\alpha, \\ u_1,u_2,\dots,u_q\leqslant\beta, \Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)>1-\frac{1}{3} \\ \text{and } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\frac{1}{3}, \Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)<\frac{1}{3}\Big\}\Big|>\frac{2}{3}.$$

By continuing this process, we can build an increasing sequence (m_τ) and (n_τ) of natural numbers, $m_\tau = \max\left\{r_1^\tau, r_2^\tau, \dots, r_q^\tau\right\}$ and $n_\tau = \max\left\{u_1^\tau, u_2^\tau, \dots, u_q^\tau\right\}$ such that $\left(r_1^\tau, r_2^\tau, \dots, r_q^\tau\right), \left(u_1^\tau, u_2^\tau, \dots, u_q^\tau\right) \in K(s,\tau)$ and for all $\alpha \geqslant m_\tau$, $\beta \geqslant n_\tau$ we have

$$\lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q}\Big|\Big\{ ((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \dots, r_q \leqslant \alpha,$$
$$u_1, u_2, \dots, u_q \leqslant \beta, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) > 1 - \frac{1}{\tau}$$
$$\text{and } \Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \frac{1}{\tau}, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \frac{1}{\tau}\Big\}\Big| > \frac{\tau-1}{\tau}.$$

Now, let's define the sets:

$$\mathcal{T}_1 = \left\{(c,d) \in \mathbb{N}^2 : 1 < c < m_1, 1 < d < n_1\right\},$$
$$\mathcal{T}_2 = \bigcup_{\tau\in\mathbb{N}} \big\{c = \max\{\alpha_1, \alpha_2, \dots, \alpha_q\}, d = \max\{\beta_1, \beta_2, \dots, \beta_q\} : (\alpha_1, \alpha_2, \dots, \alpha_q),$$
$$(\beta_1, \beta_2, \dots, \beta_q) \in K(s,\tau),\ m_\tau \leqslant c < m_{\tau+1},\ n_\tau \leqslant d < n_{\tau+1}\big\}.$$

Let's define $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and the sequence

$$y_{cd} = \begin{cases} t_{cd}, & \text{if } (c,d) \in \mathcal{T}, \\ t, & \text{if not.} \end{cases}$$

For $\varkappa \in (0,1)$, select $\tau \in \mathbb{N}$ so that $\frac{1}{\tau} < \varkappa$ and so $1 - \frac{1}{\tau} > 1 - \varkappa$. This implies that the sequence (y_{cd}) converges to t w.r.t. (Θ, Ω, Ξ). For fixed $\alpha, \beta \in \mathbb{N}$ and $m_\tau \leqslant \alpha < m_{\tau+1}$, $n_\tau \leqslant \beta < n_{\tau+1}$, we obtain

$$\big\{ ((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q :$$
$$r_1, r_2, \dots, r_q \leqslant \alpha, u_1, u_2, \dots, u_q \leqslant \beta, t_{r_wu_\kappa} \neq y_{r_wu_\kappa}, w, \kappa \in \{1, 2, \dots, q\}\big\}$$
$$\subseteq \{((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \dots, r_q \leqslant \alpha, u_1, u_2, \dots, u_q \leqslant \beta\}$$
$$\setminus \big\{ ((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \dots, r_q \leqslant \alpha,$$
$$u_1, u_2, \dots, u_q \leqslant \beta, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) > 1 - \frac{1}{\tau} \text{ and}$$
$$\Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \frac{1}{\tau}, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \frac{1}{\tau}\big\}.$$

This implies that

$$\lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q} |\{((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q :$$
$$r_1, r_2, \dots, r_q \leqslant \alpha, u_1, u_2, \dots, u_q \leqslant \beta, t_{r_wu_\kappa} \neq y_{r_wu_\kappa}, w, \kappa \in \{1, 2, \dots, q\}\}|$$
$$\leqslant \lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q} |\{((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q :$$
$$r_1, r_2, \dots, r_q \leqslant \alpha, u_1, u_2, \dots, u_q \leqslant \beta\}|$$
$$- \lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q} |\{((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \dots, r_q \leqslant \alpha,$$
$$u_1, u_2, \dots, u_q \leqslant \beta, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) > 1 - \frac{1}{\tau} \text{ and}$$
$$\Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \frac{1}{\tau}, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \dots, t_{r_qu_q}, s\right) < \frac{1}{\tau}\Big\}\Big|$$
$$\leqslant 1 - \lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q}\Big|\Big\{ ((r_1, r_2, \dots, r_q), (u_1, u_2, \dots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q :$$

$$r_1, r_2, \ldots, r_q \leqslant \alpha,\, u_1, u_2, \ldots, u_q \leqslant \beta,\, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) > 1 - \frac{1}{\tau} \text{ and}$$
$$\left.\left.\Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \frac{1}{\tau},\, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \frac{1}{\tau}\right\}\right| \leqslant \frac{1}{\tau} < \varkappa.$$

As $\varkappa$ is chosen arbitrarily, we deduce that

$$d_q\left(\{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \right.$$
$$\left. r_1, r_2, \ldots, r_q \leqslant \alpha,\, u_1, u_2, \ldots, u_q \leqslant \beta,\, t_{r_w u_\kappa} \neq y_{r_w u_\kappa},\ w, \kappa \in \{1, 2, \ldots, q\}\}\right) = 0.$$

Consequently, $t_{\alpha\beta} = y_{\alpha\beta}$ for almost all α, β.

(2)$\Longrightarrow$(3) Consider a convergent sequence $\left(y_{\alpha\beta}\right)$ in $\mathcal{U}$ such that $t_{\alpha\beta} = y_{\alpha\beta}$ for almost all α, β. Then, the set $A = \left\{(\alpha, \beta) \in \mathbb{N}^2 : t_{\alpha\beta} = y_{\alpha\beta}\right\}$ has $d_q(A) = 1$. Consequently, $\left(y_{\alpha\beta}\right)_{(\alpha,\beta)\in A}$ forms a statistically dense subsequence of $\left(t_{\alpha\beta}\right)$, which is convergent.

(3)$\Longrightarrow$(1) Let $\left(t_{\alpha_w\beta_\kappa}\right)$ be a subsequence of (t_k) such that $\left(t_{\alpha_w\beta_\kappa}\right)$ is statistically dense and $\left(t_{\alpha_w\beta_\kappa}\right)$ is statistically convergent to $t \in \mathcal{U}$. We define the index set $\mathbb{K} = \{(\alpha_w, \beta_\kappa) : w, \kappa \in \mathbb{N}\}$. It follows that $d_q(\mathbb{K}) = 1$. Now, for any $\varkappa \in (0, 1)$ and $s > 0$, we get

$$\begin{aligned}
&\{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \\
&\quad r_1, r_2, \ldots, r_q \leqslant \alpha,\, u_1, u_2, \ldots, u_q \leqslant \beta,\, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) > 1 - \varkappa \\
&\quad \text{and } \Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa,\, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa\} \\
&\supseteq \{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{K}^q \times \mathbb{K}^q : \\
&\quad r_1, r_2, \ldots, r_q \leqslant \alpha,\, u_1, u_2, \ldots, u_q \leqslant \beta,\, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) > 1 - \varkappa \\
&\quad \text{and } \Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa,\, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa\}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q} \left|\{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \ldots, r_q \leqslant \alpha,\right. \\
&\qquad u_1, u_2, \ldots, u_q \leqslant \beta,\, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) > 1 - \varkappa \text{ and} \\
&\qquad \left.\Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa,\, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa\}\right| \\
&\geqslant \lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q} \left|\{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{K}^q \times \mathbb{K}^q : r_1, r_2, \ldots, r_q \leqslant \alpha,\right. \\
&\qquad u_1, u_2, \ldots, u_q \leqslant \beta,\, \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) > 1 - \varkappa \text{ and} \\
&\qquad \left.\Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa,\, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa\}\right| = 1.
\end{aligned}$$

Thus, $st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta, \Omega, \Xi)$. □

The above Theorem 2.16 directly implies the following corollary.

Corollary 2.17. *Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ be a NFGMS and let $\left(t_{\alpha\beta}\right)$ be a sequence in $\mathcal{U}$ such that $\left(t_{\alpha\beta}\right)$ is statistically convergent. Then, $\left(t_{\alpha\beta}\right)$ possesses a convergent subsequence.*

The opposite of the statement above is not universally valid; in other words, there can be a non-statistically convergent sequence that contains a convergent subsequence. The following example illustrates our point.

Example 2.18. Let $(\mathbb{R}, g)$ denote a g-metric space with order q, where

$$g\left(t_0, t_1, \ldots, t_q\right) = \max_{0\leqslant m,h\leqslant q} \{|t_m - t_h|\},\ \forall t_0, t_1, \ldots, t_q \in \mathbb{R}.$$

Consider the triple (Θ,Ω,Ξ) as defined in Example 1.15. Let $\varkappa_1 * \varkappa_2 = \min\{\varkappa_1,\varkappa_2\}$ and $\varkappa_1 \circledast \varkappa_2 = \max\{\varkappa_1,\varkappa_2\}$, $\forall \varkappa_1,\varkappa_2 \in [0,1]$. Then $(\mathbb{R},\Theta,\Omega,\Xi,*,\circledast)$ forms an NFGMS. Consider $(t_{\alpha\beta})$ as

$$t_{\alpha\beta} = \begin{cases} \frac{1}{\alpha\beta}, & \text{if } \alpha = m^2, \beta = n^2, \\ (\alpha\beta)^2, & \text{if not,} \end{cases}$$

where $m,n \in \mathbb{N}$. Then $(t_{m^2n^2})$ constitutes a subsequence of $(t_{\alpha\beta})$ and it is convergent to 0. It is noteworthy that $(t_{\alpha\beta})$ is not statistically convergent.

3. Statistically Cauchy double sequences in NFGMS

In this section, we present the notion of statistically Cauchy double sequences within the framework of NFGMS $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ and explore several associated properties.

Definition 3.1. A sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ is statistically Cauchy w.r.t. (Θ,Ω,Ξ) if, for each $\varkappa \in (0,1)$ and $s > 0$, $\exists U = U(\varkappa) \in \mathbb{N}$, $V = V(\varkappa) \in \mathbb{N}$ such that

$$\begin{aligned} d_q\big(((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)) \in \mathbb{N}^q \times \mathbb{N}^q :& \\ \Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},s\right) \leqslant 1-\varkappa \text{ or } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},s\right) \geqslant \varkappa& \\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},s\right) \geqslant \varkappa\big) = 0.& \end{aligned}$$

Next, we explore the connection between statistically convergent and statistically Cauchy sequences in an NFGMS as follows.

Theorem 3.2. *Consider $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ as an NFGMS and $(t_{\alpha\beta})$ as a sequence in $\mathcal{U}$ such that $(t_{\alpha\beta})$ is statistically convergent. Thus, $(t_{\alpha\beta})$ is also statistically Cauchy.*

Proof. Let $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta,\Omega,\Xi)$. For given $\varkappa \in (0,1)$, select $\varkappa_1 \in (0,1)$ so that $(1-\varkappa_1)*(1-\varkappa_1) > 1-\varkappa$ and $\varkappa_1 \circledast \varkappa_1 < \varkappa$. For $s>0$, let's consult the following sets:

$$\begin{aligned} P(\varkappa_1) = \Big\{ ((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)) \in \mathbb{N}^q\times\mathbb{N}^q : \Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,\frac{s}{2}\right) \leqslant 1-\varkappa_1 \\ \text{or } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,\frac{s}{2}\right) \geqslant \varkappa_1, \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,\frac{s}{2}\right) \geqslant \varkappa_1\Big\} \end{aligned}$$

and

$$\begin{aligned} P(\varkappa_1)^c = \Big\{ (r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q) \in \mathbb{N}^q\times\mathbb{N}^q : \Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,\frac{s}{2}\right) > 1-\varkappa_1 \\ \text{or } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,\frac{s}{2}\right) < \varkappa_1, \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,\frac{s}{2}\right) < \varkappa_1\Big\}. \end{aligned}$$

Since $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta,\Omega,\Xi)$, so $d_q\left(P(\varkappa_1)\right) = 0$ and $d_q\left(P(\varkappa_1)^c\right) = 1$. Let

$$(m_1,m_2,\dots,m_q),\ (n_1,n_2,\dots,n_q) \in P(\varkappa_1)^c.$$

Then, we have

$$\Theta\left(t_{m_1n_1},t_{m_2n_2},\dots,t_{m_qn_q},t,\frac{s}{2}\right) > 1-\varkappa_1$$

and

$$\Omega\left(t_{m_1n_1},t_{m_2n_2},\dots,t_{m_qn_q},t,\frac{s}{2}\right) < \varkappa_1, \quad \Xi\left(t_{m_1n_1},t_{m_2n_2},\dots,t_{m_qn_q},t,\frac{s}{2}\right) < \varkappa_1.$$

Fix $m_\alpha, n_\beta \in \mathbb{N}$, for some $\alpha,\beta \in \{1,2,\dots,q\}$. Then

$$\Theta\left(t_{m_\alpha n_\beta},t,\dots,t,\frac{s}{2}\right) \geqslant \Theta\left(t_{m_1n_1},t_{m_2n_2},\dots,t_{m_qn_q},t,\frac{s}{2}\right) > 1-\varkappa_1$$

and

$$\Omega\left(t_{m_\alpha n_\beta}, t, \ldots, t, \frac{s}{2}\right) \leqslant \Omega\left(t_{m_1 n_1}, t_{m_2 n_2}, \ldots, t_{m_q n_q}, t, \frac{s}{2}\right) < \varkappa_1,$$
$$\Xi\left(t_{m_\alpha n_\beta}, t, \ldots, t, \frac{s}{2}\right) \leqslant \Xi\left(t_{m_1 n_1}, t_{m_2 n_2}, \ldots, t_{m_q n_q}, t, \frac{s}{2}\right) < \varkappa_1.$$

For $(r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q) \in P(\varkappa_1)^c$, we have

$$\Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) \geqslant \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, \frac{s}{2}\right) * \Theta\left(t_{m_\alpha n_\beta}, t, \ldots, t, \frac{s}{2}\right)$$
$$> (1-\varkappa_1) * (1-\varkappa_1) > 1-\varkappa$$

and

$$\Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) \leqslant \Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, \frac{s}{2}\right) \circledast \Omega\left(t_{m_\alpha n_\beta}, t, \ldots, t, \frac{s}{2}\right) < \varkappa_1 \circledast \varkappa_1 < \varkappa,$$
$$\Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) \leqslant \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, \frac{s}{2}\right) \circledast \Xi\left(t_{m_\alpha n_\beta}, t, \ldots, t, \frac{s}{2}\right) < \varkappa_1 \circledast \varkappa_1 < \varkappa.$$

Therefore, this suggests that

$$P(\varkappa_1)^c \subseteq \{(r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) > 1-\varkappa \text{ and}$$
$$\Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) < \varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) < \varkappa\}.$$

Consequently,

$$d_q\left(P(\varkappa_1)^c\right) \leqslant d_q\big(((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q :$$
$$\Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) > 1-\varkappa \text{ and}$$
$$\Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) < \varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) < \varkappa\big).$$

Therefore,

$$d_q\big(((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) > 1-\varkappa \text{ and}$$
$$\Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) < \varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) < \varkappa\big) = 1,$$

and thus

$$d_q\big(((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) \leqslant 1-\varkappa \text{ or}$$
$$\Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) \geqslant \varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{m_\alpha n_\beta}, s\right) \geqslant \varkappa\big) = 0.$$

The theorem's proof is now concluded. □

The converse of Theorem 3.2 does not hold. To illustrate this point, consider the following example.

Example 3.3. Take $\mathcal{U} = (0, 1]$. Let's examine $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$, the NFGMS as outlined in Example 2.13. Now, look at the sequence $(t_{\alpha\beta})$ defined by

$$t_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = k^3, \beta = l^3, \\ \frac{1}{\alpha\beta}, & \text{if not,} \end{cases}$$

where $k, l \in \mathbb{N}$. Then $(t_{\alpha\beta})$ is statistically Cauchy sequence, but not statistically convergent.

Definition 3.4. An NFGMS $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ is termed statistically complete iff every statistically Cauchy sequence in $\mathcal{U}$ is statistically convergent.

Theorem 3.5. *If an NFGMS $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ is statistically complete, then it is complete.*

Proof. Suppose $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ is statistically complete. Let $(t_{\alpha\beta})$ be a Cauchy sequence in $\mathcal{U}$. Then, it is statistically Cauchy in $\mathcal{U}$. Since $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ is statistically complete, $(t_{\alpha\beta})$ is statistically convergent. By Corollary 2.17, we have a convergent subsequence $(t_{\alpha_m\beta_n})$ of $(t_{\alpha\beta})$. Let $(t_{\alpha_m\beta_n})$ converges to t. For given $\varkappa \in (0,1)$, there are $\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4 \in (0,1)$ so that $(1-\varkappa_2)*(1-\varkappa_2) > 1-\varkappa_1$, $\varkappa_3 \circledast \varkappa_3 < \varkappa_1$, $(1-\varkappa_1)*(1-\varkappa_4) > 1-\varkappa$ and $\varkappa_1 \circledast \varkappa_4 < \varkappa$. Take $\varkappa_5 = \min\{\varkappa_2,\varkappa_3\}$. Since $(t_{\alpha\beta})$ is Cauchy, for given $s>0$, $\exists \alpha_0,\beta_0 \in \mathbb{N}$ such that for all $r_0,r_1,r_2,\ldots,r_q \geqslant \alpha_0$ and $u_1,u_2,\ldots,u_q \geqslant \beta_0$, so we have

$$\Theta\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{4}\right) > 1-\varkappa_5,$$

and

$$\Omega\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{4}\right) < \varkappa_5,\ \Xi\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{4}\right) < \varkappa_5.$$

Also, $(t_{\alpha_m\beta_n})$ is convergent to t. So, there exist $\alpha_1,\beta_1 \in \mathbb{N}$ such that

$$\Theta\left(t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},t,\frac{s}{4}\right) > 1-\varkappa_4$$

and

$$\Omega\left(t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},t,\frac{s}{4}\right) < \varkappa_4, \quad \Xi\left(t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},t,\frac{s}{4}\right) < \varkappa_4,$$

for all $r_{\alpha_1},r_{\alpha_2},\ldots,r_{\alpha_q} \geqslant \alpha_1$ and $u_{\alpha_1},u_{\alpha_2},\ldots,u_{\alpha_q} \geqslant \beta_1$. Let $U=\max\{\alpha_0,\alpha_1\}$, $V=\max\{\beta_0,\beta_1\}$. For $r_0,r_1,\ldots,r_q,r_{\alpha_1},r_{\alpha_2},\ldots,r_{\alpha_q} \geqslant U$, $u_0,u_1,\ldots,u_q,u_{\beta_1},u_{\beta_2},\ldots,u_{\beta_q} \geqslant V$, using (NFG-3) and (NFG-6) we have

$$\begin{aligned}
&\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\\
&\geqslant \Theta\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{2}\right) * \Theta\left(t_{r_0u_0},t_{r_0u_0},\ldots,t_{r_0u_0},t,\frac{s}{2}\right)\\
&\geqslant \Theta\left(t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},t,\frac{s}{4}\right) * \Theta\left(t_{r_0u_0},t_{r_0u_0},\ldots,t_{r_0u_0},t_{r_{\alpha_q}u_{\beta_q}},\frac{s}{4}\right)\\
&\quad * \Theta\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{2}\right)\\
&\geqslant \Theta\left(t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},t,\frac{s}{4}\right) * \Theta\left(t_{r_0u_0},t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},\frac{s}{4}\right)\\
&\quad * \Theta\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{4}\right)\\
&> (1-\varkappa_4)*(1-\varkappa_5)*(1-\varkappa_5) > (1-\varkappa_4)*(1-\varkappa_2)*(1-\varkappa_2) > (1-\varkappa_4)*(1-\varkappa_1) > (1-\varkappa).
\end{aligned}$$

Again, using (NFG-10) and (NFG-13), we have

$$\begin{aligned}
&\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\\
&\leqslant \Omega\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{2}\right) \circledast \Omega\left(t_{r_0u_0},t_{r_0u_0},\ldots,t_{r_0u_0},t,\frac{s}{2}\right)\\
&\leqslant \Omega\left(t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},t,\frac{s}{4}\right) \circledast \Omega\left(t_{r_0u_0},t_{r_0u_0},\ldots,t_{r_0u_0},t_{r_{\alpha_q}u_{\beta_q}},\frac{s}{4}\right)\\
&\quad \circledast \Omega\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{2}\right)\\
&\leqslant \Omega\left(t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},t,\frac{s}{4}\right) \circledast \Omega\left(t_{r_0u_0},t_{r_{\alpha_1}u_{\beta_1}},t_{r_{\alpha_2}u_{\beta_2}},\ldots,t_{r_{\alpha_q}u_{\beta_q}},\frac{s}{4}\right)\\
&\quad \circledast \Omega\left(t_{r_0u_0},t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},\frac{s}{4}\right) < \varkappa_4 \circledast \varkappa_5 \circledast \varkappa_5 < \varkappa_4 \circledast \varkappa_3 \circledast \varkappa_3 < \varkappa_4 \circledast \varkappa_1 < \varkappa.
\end{aligned}$$

Likewise, we derive $\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right) < \varkappa$. This suggests that $(t_{\alpha\beta})$ converges to t, thus affirming the completeness of $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$. □

Definition 3.6. Consider $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ to be an NFGMS. A sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ is termed statistically bounded w.r.t. (Θ,Ω,Ξ) if, for any $t_0\in\mathcal{U}$, there exist $\varkappa_0\in(0,1)$ and $s_0>0$ such that

$$d_q\big(((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s_0\right)\leqslant 1-\varkappa_0 \text{ and } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s_0\right)\geqslant\varkappa_0,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s_0\right)\geqslant\varkappa_0\big)=0.$$

Theorem 3.7. *If a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ is statistically Cauchy, then it is statistically bounded.*

Proof. Let the sequence $(t_{\alpha\beta})$ is statistically Cauchy in $\mathcal{U}$. Then, for each $\varkappa\in(0,1)$ and $s>0$, $\exists U=U(\varkappa)\in\mathbb{N}$, $V=V(\varkappa)\in\mathbb{N}$ such that the set

$$\mathbf{K}(s,\varkappa)=\big\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},s\right)>1-\varkappa \text{ or } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},s\right)<\varkappa,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},s\right)<\varkappa\big\}$$

has asymptotic density 1, namely $d_q(\mathbf{K}(s,\varkappa))=1$. Fix $t_0\in\mathcal{U}$ and let $\Theta\left(t_{UV},t_{UV},\dots,t_{UV},t_0,\frac{s}{2}\right)=\alpha$ and $\Omega\left(t_{UV},t_{UV},\dots,t_{UV},t_0,\frac{s}{2}\right)=\gamma$, $\Xi\left(t_{UV},t_{UV},\dots,t_{UV},t_0,\frac{s}{2}\right)=\psi$. Since $\alpha,\gamma,\psi\in(0,1)$, there exist $\beta,\delta,\tau\in(0,1)$ such that $(1-\varkappa)*\alpha>1-\beta$, $\varkappa\circledast\gamma<\delta$ and $\varkappa\circledast\psi<\tau$. Let $(r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)\in\mathbf{K}(s,\varkappa)$. Then

$$\begin{aligned}\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)&\geqslant\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},\frac{s}{2}\right)*\Theta\left(t_{UV},t_{UV},\dots,t_{UV},t_0,\frac{s}{2}\right)\\&>(1-\varkappa)*\alpha>1-\beta,\end{aligned}$$

and

$$\begin{aligned}\Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)&\leqslant\Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_{UV},\frac{s}{2}\right)\circledast\Omega\left(t_{UV},t_{UV},\dots,t_{UV},t_0,\frac{s}{2}\right)\\&<\varkappa\circledast\gamma<\delta,\end{aligned}$$

Similarly, we get $\Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)<\tau$. Take $\varkappa_0=\max\{\beta,\delta,\tau\}$. Then

$$\mathbf{K}(s,\varkappa)\subseteq\big\{(r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q)\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)>1-\varkappa_0 \text{ or } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)<\varkappa_0,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)<\varkappa_0\big\}.$$

Consequently,

$$d_q\big(((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)>1-\varkappa_0 \text{ and } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)<\varkappa_0,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)<\varkappa_0\big)=1,$$

which gives that

$$d_q\big(((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)\leqslant 1-\varkappa_0 \text{ and } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)\geqslant\varkappa_0,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t_0,s\right)\geqslant\varkappa_0\big)=0.$$

As a result, $(t_{\alpha\beta})$ is statistically bounded. □

Corollary 3.8. *In an NFGMS $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$, every statistically convergent sequence is statistically bounded.*

Proof. The result follows from Theorems 3.2 and 3.7. □

In addition to Theorem 2.16, we can also assert the following theorem.

Theorem 3.9. *Considering $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ as an NFGMS, the following statements are equivalent for a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$.*

(a) *The sequence $(t_{\alpha\beta})$ is statistically Cauchy.*
(b) *There is a statistically dense subsequence $(t_{\alpha_m\beta_n})$ of $(t_{\alpha\beta})$ and $(t_{\alpha_m\beta_n})$ is Cauchy in $\mathcal{U}$.*

4. Statistical limit points and statistical cluster points

In this segment, we expand upon the concepts of thin subsequence, nonthin subsequence, statistical limit points, and statistical clustering points within the context of NFGMS.

Definition 4.1. Consider an NFGMS denoted by $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$. A point $t\in\mathcal{U}$ be identified as a limit point of a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ w.r.t. (Θ,Ω,Ξ), if there exists a subsequence $(t_{\alpha_m\beta_n})$ of $(t_{\alpha\beta})$ that converges to t. The notation ${}^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})$ denotes the collection of all limit points of the sequence $(t_{\alpha\beta})$.

Definition 4.2. Consider an NFGMS represented by $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$, with $(t_{\alpha_m\beta_n})$ being a subsequence of a sequence $(t_{\alpha\beta})$ in $\mathcal{U}$. Let $\mathbf{K}=\{(\alpha_m,\beta_n):m,n\in\mathbb{N}\}\subset\mathbb{N}^2$. We define $(t_{\alpha_m\beta_n})$ as a "thin subsequence" of $(t_{\alpha\beta})$ if $d_q(\mathbf{K})=0$. If $d_q(\mathbf{K})\neq 0$, then term $(t_{\alpha_m\beta_n})$ as a "nonthin subsequence" of $(t_{\alpha\beta})$.

Definition 4.3. Consider an NFGMS denoted as $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$. Then $t\in\mathcal{U}$ is termed a statistical limit point of the sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ w.r.t. (Θ,Ω,Ξ), if there exists a nonthin subsequence $(t_{\alpha_m\beta_n})$ of $(t_{\alpha\beta})$ that converges to t. We denote $\Lambda^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})$ as the set comprising all statistical limit points of $(t_{\alpha\beta})$.

Definition 4.4. In an NFGMS represented by $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$, $t\in\mathcal{U}$ is termed a statistical cluster point of the sequence $(t_{\alpha\beta})$ in $\mathcal{U}$ w.r.t. (Θ,Ω,Ξ) if for every $\varkappa\in(0,1)$ and $s>0$,

$$d_q\big(((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)>1-\varkappa \text{ and } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)<\varkappa,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)<\varkappa\big)\neq 0.$$

We denote by $\Gamma^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})$, the set containing all statistical cluster points of $(t_{\alpha\beta})$.

Theorem 4.5. *Let $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ be an NFGMS and $(t_{\alpha\beta})$ be a sequence in $\mathcal{U}$. Then*

$$\Lambda^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})\subseteq\Gamma^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})\subseteq{}^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta}).$$

Proof. Consider the scenario where $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ represents an NFGMS, and $(t_{\alpha\beta})$ is a sequence in $\mathcal{U}$. When $t\in\Lambda^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})$, there exists a subsequence $(t_{\alpha_m\beta_n})$ of $(t_{\alpha\beta})$ such that the index set $A=\{(\alpha_m,\beta_n):m,n\in\mathbb{N}\}\subset\mathbb{N}^2$ possesses a non zero q-dimensional asymptotic density, denoted as:

$$d_q(A)=\lim_{\alpha,\beta\to\infty}\frac{q!}{(\alpha\beta)^q}\big|\{((\alpha_1,\alpha_2,\dots,\alpha_q),(\beta_1,\beta_2,\dots,\beta_q))\in A^q\times A^q: r_1,r_2,\dots,r_q\leqslant\alpha,\ u_1,u_2,\dots,u_q\leqslant\beta\}\big|=y>0$$

and $(t_{\alpha_m\beta_n})$ converges to t. Due to the properties

$$\begin{aligned}&\big\{((\alpha_1,\alpha_2,\dots,\alpha_q),(\beta_1,\beta_2,\dots,\beta_q))\in A^q\times A^q:\Theta\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\dots,t_{\alpha_q\beta_q},t,s\right)>1-\varkappa\\&\quad\text{and } \Omega\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\dots,t_{\alpha_q\beta_q},t,s\right)<\varkappa,\ \Xi\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\dots,t_{\alpha_q\beta_q},t,s\right)<\varkappa\big\}\\&\subseteq\big\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)>1-\varkappa\\&\quad\text{and } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)<\varkappa,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)<\varkappa\big\},\end{aligned}$$

which hold for all $\varkappa\in(0,1)$ and $s>0$, we can conclude

$$\begin{aligned}&\{((\alpha_1,\alpha_2,\dots,\alpha_q),(\beta_1,\beta_2,\dots,\beta_q))\in A^q\times A^q:\alpha_x,\beta_y\in\mathbb{N},x,y=1,2,\dots,q\}\\&\quad\setminus\big\{((\alpha_1,\alpha_2,\dots,\alpha_q),(\beta_1,\beta_2,\dots,\beta_q))\in A^q\times A^q:\Theta\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\dots,t_{\alpha_q\beta_q},t,s\right)\leqslant1-\varkappa\\&\quad\text{or } \Omega\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\dots,t_{\alpha_q\beta_q},t,s\right)\geqslant\varkappa,\ \Xi\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\dots,t_{\alpha_q\beta_q},t,s\right)\geqslant\varkappa\big\}\\&\subseteq\big\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)>1-\varkappa\\&\quad\text{and } \Omega\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)<\varkappa,\ \Xi\left(t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},t,s\right)<\varkappa\big\}.\end{aligned}$$

Additionally, since the subsequence $\left(t_{\alpha_m\beta_n}\right)$ converges to t, we have

$$\left\{((\alpha_1,\alpha_2,\ldots,\alpha_q),(\beta_1,\beta_2,\ldots,\beta_q))\in A^q\times A^q:\Theta\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\ldots,t_{\alpha_q\beta_q},t,s\right)\leqslant 1-\varkappa\right.$$
$$\left.\text{or }\Omega\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\ldots,t_{\alpha_q\beta_q},t,s\right)\geqslant\varkappa,\Xi\left(t_{\alpha_1\beta_1},t_{\alpha_2\beta_2},\ldots,t_{\alpha_q\beta_q},t,s\right)\geqslant\varkappa\right\}$$

being a finite subset of $\mathbb{N}^q$. Consequently, we obtain

$$\begin{aligned}
d_q&\left(((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)>1-\varkappa\text{ and}\right.\\
&\quad\left.\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)<\varkappa,\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)<\varkappa\right)\\
&\geqslant d_q\left(((\alpha_1,\alpha_2,\ldots,\alpha_q),(\beta_1,\beta_2,\ldots,\beta_q))\in A^q\times A^q:\alpha_x,\beta_y\in\mathbb{N},x,y=1,2,\ldots,q\right)\\
&\qquad-d_q\left(((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in A^q\times A^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\leqslant 1-\varkappa\text{ or}\right.\\
&\qquad\left.\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\geqslant\varkappa,\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\geqslant\varkappa\right)\\
&\geqslant d_q\left((\alpha_1,\alpha_2,\ldots,\alpha_q),(\beta_1,\beta_2,\ldots,\beta_q)\in A^q\times A^q:\alpha_x,\beta_y\in\mathbb{N},x,y=1,2,\ldots,q\right)-0\quad y>0.
\end{aligned}$$

Therefore, $t\in\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$ and so $\Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)\subseteq\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. Now, let $y\in\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. Then, for each $s>0$ and $\varkappa\in(0,1)$, we have

$$\begin{aligned}
d_q(\mathbf{H}(s,\varkappa))=d_q&\left(((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)>1-\varkappa\right.\\
&\left.\text{and }\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)<\varkappa,\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)<\varkappa\right)>0.
\end{aligned}$$

Let $(\alpha_1,\alpha_2,\ldots,\alpha_q),(\beta_1,\beta_2,\ldots,\beta_q)\in\mathbf{H}(s,\varkappa)$. Then, by Proposition 1.24, we have $t_{\alpha_m\beta_n}\in\mathbf{B}_y^{(\Theta,\Omega,\Xi)}(s,\varkappa)$ for each $m,n\in\{0,1,\ldots,q\}$. Set

$$B=\left\{(\alpha_m,\beta_n)\in\mathbb{N}^2:t_{\alpha_m\beta_n}\in\mathbf{B}_y^{(\Theta,\Omega,\Xi)}(s,\varkappa)\right\}.$$

We can observe that $d_q(B)=d_q(\mathbf{H}(s,\varkappa))>0$, indicating that $\left(t_{\alpha_m\beta_n}\right)$ forms a nonthin subsequence of $\left(t_{\alpha\beta}\right)$ along B. As B contains an infinite number of positive integers, it follows that $y\in{}^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. Consequently, $\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)\subseteq{}^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. □

Theorem 4.6. *Given an NFGMS* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$, *let* $\left(t_{\alpha\beta}\right)$ *be a sequence in* $\mathcal{U}$ *such that* $st_2-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$. *Then* $\Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\{t\}$.

Proof. Given that $st_2-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$, and considering Definitions 2.8 and 4.4, we can conclude that $t\in\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. Let's assume there exists $y\in\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$, where $t\neq y$. For any $\varkappa\in(0,1)$ and $s>0$, we have

$$\begin{aligned}
d_q(\mathbf{A}(s,\varkappa))=d_q&\left(((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)>1-\varkappa\right.\\
&\left.\text{and }\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)<\varkappa,\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)<\varkappa\right)\neq 0,
\end{aligned}$$

and

$$\begin{aligned}
d_q(\mathbf{B}(s,\varkappa))=d_q&\left(((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)>1-\varkappa\right.\\
&\left.\text{and }\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)<\varkappa,\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)<\varkappa\right)\neq 0.
\end{aligned}$$

Since $t\neq y$, we get $\mathbf{A}(s,\varkappa)\cap\mathbf{B}(s,\varkappa)=\emptyset$. So, $\mathbf{A}(s,\varkappa)^c\supseteq\mathbf{B}(s,\varkappa)$, i.e.,

$$\begin{aligned}
&\left\{((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\leqslant 1-\varkappa\right.\\
&\quad\left.\text{or }\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\geqslant\varkappa,\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},t,s\right)\geqslant\varkappa\right\}\\
&\supseteq\left\{((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:\Theta\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)>1-\varkappa\right.\\
&\quad\left.\text{and }\Omega\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)<\varkappa,\Xi\left(t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},y,s\right)<\varkappa\right\}.
\end{aligned}$$

Hence

$$\begin{aligned} & d_q\left(\left\{\left(\left(r_1, r_2, \ldots, r_q\right),\left(u_1, u_2, \ldots, u_q\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, s\right) \leqslant 1-\varkappa\right.\right. \\ & \quad \left.\left.\text{or } \Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, s\right) \geqslant \varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, s\right) \geqslant \varkappa\right\}\right) \\ & \geqslant d_q\left(\left\{\left(\left(r_1, r_2, \ldots, r_q\right),\left(u_1, u_2, \ldots, u_q\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, y, s\right)>1-\varkappa\right.\right. \\ & \quad \left.\left.\text{and } \Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, y, s\right)<\varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, y, s\right)<\varkappa\right\}\right). \end{aligned} \tag{4.1}$$

As $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$, we have

$$\begin{aligned} & d_q\left(\left\{\left(\left(r_1, r_2, \ldots, r_q\right),\left(u_1, u_2, \ldots, u_q\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, s\right) \leqslant 1-\varkappa\right.\right. \\ & \quad \left.\left.\text{or } \Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, s\right) \geqslant \varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t, s\right) \geqslant \varkappa\right\}\right)=0. \end{aligned}$$

From (4.1), it follows that

$$\begin{aligned} & d_q\left(\left\{\left(\left(r_1, r_2, \ldots, r_q\right),\left(u_1, u_2, \ldots, u_q\right)\right) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, y, s\right)>1-\varkappa\right.\right. \\ & \quad \left.\left.\text{and } \Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, y, s\right)<\varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, y, s\right)<\varkappa\right\}\right)=0. \end{aligned}$$

This contradicts the assertion that $y \in \Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. Hence, we conclude that $\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\{t\}$. Let's consider $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$. Then, according to Definition 4.3 and Theorem 2.16, it follows that $t \in \Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. Therefore, by Theorem 4.5, we conclude that $\Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\{t\}$. □

The converse of Theorem 4.6 does not hold true. In other words, there exists a sequence $\left(t_{\alpha\beta}\right)$ in $\mathcal{U}$, where $\Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\{t\}$, yet $\left(t_{\alpha\beta}\right)$ does not statistically converge to t.

Example 4.7. Let's consider $(\mathbb{R},\Theta,\Omega,\Xi,*,\circledast)$ as an NFGMS defined in Example 1.15. We define the sequence $\left(t_{\alpha\beta}\right)$ as

$$t_{\alpha\beta}=\begin{cases}1, & \text{if } \alpha=2m, \beta=2n,\\ 2, & \text{otherwise},\end{cases}$$

where $m, n \in \mathbb{N}$. Then, we observe that $\Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\{1,2\}$ but the sequence $\left(t_{\alpha\beta}\right)$ does not statistically converge.

Theorem 4.8. *Consider $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ to be an NFGMS. Suppose $\left(t_{\alpha\beta}\right)$ and $\left(z_{\alpha\beta}\right)$ are two sequences in $\mathcal{U}$ such that $t_{\alpha\beta}=z_{\alpha\beta}$ for almost all α,β. Then $\Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\Lambda^{(\Theta,\Omega,\Xi)}\left(z_{\alpha\beta}\right)$ and $\Gamma^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)=\Gamma^{(\Theta,\Omega,\Xi)}\left(z_{\alpha\beta}\right)$.*

Proof. Let $t_{\alpha\beta}=z_{\alpha\beta}$ for almost all α,β. Consequently, the set $T_1=\left\{(\alpha,\beta)\in\mathbb{N}^2 : t_{\alpha\beta}\neq z_{\alpha\beta}\right\}$ has a zero q-dimensional asymptotic density, denoted by $d_q\left(T_1\right)=0$. Let $t \in \Lambda^{(\Theta,\Omega,\Xi)}\left(t_{\alpha\beta}\right)$. Then, there exists a subsequence $\left(t_{\alpha_m\beta_n}\right)$ of $\left(t_{\alpha\beta}\right)$ converging to t and the index set $T_2=\{(\alpha_m,\beta_n): m,n\in\mathbb{N}\}$ possesses a non-zero q-dimensional asymptotic density, i.e., $d_q\left(T_2\right)\neq 0$. Establish the set

$$T_3=\left\{\left(\alpha_m,\beta_n\right)\in T_2 : m,n\in\mathbb{N},\, t_{\alpha_i\beta_j}\neq z_{\alpha_i\beta_j}\right\}.$$

It's evident that $d_q\left(T_3\right)=0$, i.e.,

$$\begin{aligned} & \lim_{\alpha,\beta\to\infty}\frac{q!}{(\alpha\beta)^q}\left|\left\{\left(\left(\alpha_1,\alpha_2,\ldots,\alpha_q\right),\left(\beta_1,\beta_2,\ldots,\beta_q\right)\right)\in T_2^q\times T_2^q :\right.\right. \\ & \qquad\qquad \left.\left.\alpha_1,\alpha_2,\ldots,\alpha_q\leqslant\alpha,\ \beta_1,\beta_2,\ldots,\beta_q\leqslant\beta,\ t_{\alpha_i\beta_j}\neq z_{\alpha_i\beta_j}\right\}\right|=0. \end{aligned}$$

This yields that

$$\lim_{\alpha,\beta\to\infty}\frac{q!}{(\alpha\beta)^q}\left|\left\{\left(\left(\alpha_1,\alpha_2,\ldots,\alpha_q\right),\left(\beta_1,\beta_2,\ldots,\beta_q\right)\right)\in T_2^q\times T_2^q :\right.\right.$$

$$\alpha_1, \alpha_2, \ldots, \alpha_q \leqslant \alpha,\ \beta_1, \beta_2, \ldots, \beta_q \leqslant \beta,\ t_{\alpha_i \beta_j} = z_{\alpha_i \beta_j} \}\Big| > 0,$$

which is equivalent to

$$d_q(Q) = d_q\left(\{(\alpha_m, \beta_n) \in T_2 : t_{\alpha_m \beta_n} = z_{\alpha_m \beta_n}\}\right) > 0.$$

Now, let's examine the sequence $(z_{\alpha_m \beta_n})$ along Q. This sequence is a non-thin subsequence of $(z_{\alpha\beta})$ that converges to t, and so $t \in \Lambda^{(\Theta,\Omega,\Xi)}(z_{\alpha\beta})$. This implies that $\Lambda^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta}) \subseteq \Lambda^{(\Theta,\Omega,\Xi)}(z_{\alpha\beta})$. Symmetrically, we also observe that $\Lambda^{(\Theta,\Omega,\Xi)}(z_{\alpha\beta}) \subseteq \Lambda^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})$. Therefore, we conclude that $\Lambda^{(\Theta,\Omega,\Xi)}(z_{\alpha\beta}) = \Lambda^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})$. Similarly, we can establish that $\Gamma^{(\Theta,\Omega,\Xi)}(z_{\alpha\beta}) = \Gamma^{(\Theta,\Omega,\Xi)}(t_{\alpha\beta})$. □

5. Lacunary statistical convergence of double sequences in NFGMS

After defining statistical convergence and examining its characteristics within these contexts, we delve into lacunary statistical convergence and strongly lacunary convergence of double sequences in NFGMS, along with exploring their interconnections.

Definition 5.1. Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ be an NFGMS and $(t_{\alpha\beta})$ be a double sequence in $\mathcal{U}$. The sequence $(t_{\alpha\beta})$ is said to be strongly $[C,1,1]$-statistically convergent to $t \in \mathcal{U}$ w.r.t. (Θ, Ω, Ξ) if, for each $\varkappa \in (0,1)$ and $s > 0$,

$$\frac{q!}{(\alpha\beta)^q} \sum_{r_1,\cdots,r_q=1}^{\alpha} \sum_{u_1,\cdots,u_q=1}^{\beta} \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) > 1 - \varkappa$$

and

$$\frac{q!}{(\alpha\beta)^q} \sum_{r_1,\cdots,r_q=1}^{\alpha} \sum_{u_1,\cdots,u_q=1}^{\beta} \Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa,$$

$$\frac{q!}{(\alpha\beta)^q} \sum_{r_1,\cdots,r_q=1}^{\alpha} \sum_{u_1,\cdots,u_q=1}^{\beta} \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) < \varkappa.$$

This is denoted by $[C,1,1] - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta,\Omega,\Xi)$ or $t_{\alpha\beta} \overset{[C,1,1]-(\Theta,\Omega,\Xi)}{\longrightarrow} t$.

Theorem 5.2. *Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ represent an NFGMS and $(t_{\alpha\beta})$ be a double sequence in $\mathcal{U}$. Then*

(i) $t_{\alpha\beta} \overset{[C,1,1]-(\Theta,\Omega,\Xi)}{\longrightarrow} t$ *implies* $t_{\alpha\beta} \overset{st_2-(\Theta,\Omega,\Xi)}{\rightarrow} t$;

(ii) *if $(t_{\alpha\beta})$ is a bounded sequence,* $t_{\alpha\beta} \overset{st_2-(\Theta,\Omega,\Xi)}{\rightarrow} t$ *implies* $t_{\alpha\beta} \overset{[C,1,1]-(\Theta,\Omega,\Xi)}{\longrightarrow} t$.

Definition 5.3. Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ represent an NFGMS. The sequence $(t_{\alpha\beta})$ is said to be lacunary statistically convergent to $t \in \mathcal{U}$ w.r.t. (Θ, Ω, Ξ) if, for each $\varkappa \in (0,1)$ and $s > 0$,

$$\lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} |\{(r_w, u_w) \in I_{rs}, 1 \leqslant w \leqslant q : \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s\right) \leqslant 1 - \varkappa \text{ or}$$
$$\Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s\right) \geqslant \varkappa, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s\right) \geqslant \varkappa\}| = 0,$$

or equivalently

$$d_q^{\theta_2}\left(((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s\right) \leqslant 1 - \varkappa\right.$$
$$\left.\text{or } \Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s\right) \geqslant \varkappa, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s\right) \geqslant \varkappa\right) = 0,$$

and is indicated by $S_{\theta_2} - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta,\Omega,\Xi)$ or $t_{\alpha\beta} \overset{S_{\theta_2}-(\Theta,\Omega,\Xi)}{\longrightarrow} t$. The collection of all lacunary statistical convergent sequences in NFGMS is symbolized as $S_{\theta_2}^{(\Theta,\Omega,\Xi)}$.

Definition 5.4. Let $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ represent an NFGMS. The sequence $\left(t_{\alpha\beta}\right)$ is called to be strongly lacunary convergent to $t\in\mathcal{U}$ w.r.t. (Θ,Ω,Ξ) if, for each $\varkappa\in(0,1)$ and $s>0$,

$$\frac{q!}{(h_{rs})^q}\sum_{(j_w,k_w)\in I_{rs},\,1\leqslant w\leqslant q}\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)>1-\varkappa$$

and

$$\frac{q!}{(h_{rs})^q}\sum_{(j_w,k_w)\in I_{rs},\,1\leqslant w\leqslant q}\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)<\varkappa,$$

$$\frac{q!}{(h_{rs})^q}\sum_{(j_w,k_w)\in I_{rs},\,1\leqslant w\leqslant q}\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)<\varkappa,$$

where $1\leqslant w\leqslant q$, and is indicated by $[N_{\theta_2}]-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$ or $t_{\alpha\beta}\overset{[N_{\theta_2}]-(\Theta,\Omega,\Xi)}{\longrightarrow}t$.

Theorem 5.5. *Let $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ represent an NFGMS and $\left(t_{\alpha\beta}\right)$ be a double sequence in $\mathcal{U}$. Then, the following statements hold:*

(i) $t_{\alpha\beta}\overset{[N_{\theta_2}]-(\Theta,\Omega,\Xi)}{\longrightarrow}t$ *implies* $t_{\alpha\beta}\overset{S_{\theta_2}-(\Theta,\Omega,\Xi)}{\longrightarrow}t$;

(ii) *if $\left(t_{\alpha\beta}\right)$ is a bounded sequence,* $t_{\alpha\beta}\overset{S_{\theta_2}-(\Theta,\Omega,\Xi)}{\longrightarrow}t$ *implies* $t_{\alpha\beta}\overset{[N_{\theta_2}]-(\Theta,\Omega,\Xi)}{\longrightarrow}t$.

Next, we establish connections between statistical convergence and lacunary statistical convergence of double sequences in an NFGMS.

Theorem 5.6. *Let $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ represent an NFGMS and $\left(t_{\alpha\beta}\right)$ be a double sequence in $\mathcal{U}$. For a double lacunary sequence $\theta_2=\theta_{r,s}$, if $\liminf_r q_r>1$ and $\liminf_s q_s>1$, then $st_2-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$ implies $S_{\theta_2}-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$.*

Proof. Assume first that $\liminf_r q_r>1$ and $\liminf_s q_s>1$. Then, there exist $\xi,\sigma>0$ such that $q_r>1+\xi$ and $q_s>1+\sigma$ for sufficiently large r, s which means that

$$\frac{h_{rs}}{k_{rs}}\geqslant\frac{\xi\sigma}{(1+\xi)\,(1+\sigma)}.$$

If $st_2-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$, then for all $\varkappa\in(0,1)$, $s>0$ and for sufficiently large r,s we get

$$\begin{aligned}
&\frac{q!}{(k_{rs})^q}\left|\{((r_1,r_2,\ldots,r_q),(u_1,u_2,\ldots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:r_1,r_2,\ldots,r_q\leqslant k_r,\right.\\
&\qquad u_1,u_2,\ldots,u_q\leqslant l_s,\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},s\right)\leqslant1-\varkappa\\
&\qquad \left.\text{or }\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},s\right)\geqslant\varkappa,\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_qu_q},s\right)\geqslant\varkappa\}\right|\\
&\geqslant\frac{q!}{(k_{rs})^q}\left|\{(r_w,u_w)\in I_{rs},1\leqslant w\leqslant q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\leqslant1-\varkappa\right.\\
&\qquad\left.\text{or }\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\geqslant\varkappa,\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\geqslant\varkappa\}\right|\\
&\geqslant\left(\frac{h_{rs}}{k_{rs}}\right)^q\frac{q!}{(h_{rs})^q}\left|\{(r_w,u_w)\in I_{rs},1\leqslant w\leqslant q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\leqslant1-\varkappa\right.\\
&\qquad\left.\text{or }\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\geqslant\varkappa,\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\geqslant\varkappa\}\right|\\
&\geqslant\left(\frac{\xi\sigma}{(1+\xi)\,(1+\sigma)}\right)^q\frac{q!}{(h_{rs})^q}\left|\{(r_w,u_w)\in I_{rs},1\leqslant w\leqslant q:\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\leqslant1-\varkappa\right.\\
&\qquad\left.\text{or }\Omega\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\geqslant\varkappa,\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\ldots,t_{r_wu_w},s\right)\geqslant\varkappa\}\right|.
\end{aligned}$$

As a result $S_{\theta_2}-\lim_{\alpha,\beta\to\infty}t_{\alpha\beta}=t\,(\Theta,\Omega,\Xi)$. □

Theorem 5.7. *Let* $(\mathcal{U},\Theta,\Omega,\Xi,*,\circledast)$ *represent an NFGMS and* $(t_{\alpha\beta})$ *be a double sequence in* $\mathcal{U}$. *For a double lacunary sequence* $\theta_2=\theta_{r,s}$, *if* $\limsup_r q_r<\infty$, $\limsup_s q_s<\infty$, *then* $S_{\theta_2}-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t(\Theta,\Omega,\Xi)$ *implies* $st_2-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t(\Theta,\Omega,\Xi)$.

Proof. Assume $\limsup_r q_r<\infty$, $\limsup_s q_s<\infty$. Then there exist $Q,R>0$ such that $q_r<Q$ and $q_s<R$ for all $r,s\geqslant 1$. Assume that $S_{\theta_2}-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t(\Theta,\Omega,\Xi)$ and

$$\begin{aligned}N_{r,s}:=|\{(j_w,k_w)\in I_{rs},\,1\leqslant w\leqslant q:\Theta(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_wu_w},s)\leqslant 1-\varkappa \text{ or}\\ \Omega(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_wu_w},s)\geqslant\varkappa,\,\Xi(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_wu_w},s)\geqslant\varkappa\}|.\end{aligned}$$

By the definition of $S_{\theta_2}-\lim_{\alpha,\beta\to\infty} t_{\alpha\beta}=t(\Theta,\Omega,\Xi)$, given $\psi>0$, there exists $r_0\in\mathbb{N}$ such that $\frac{N_{r,s}}{h_{rs}}<\psi$ for all $r,s\geqslant r_0$. Take

$$U=\max\{N_{r,s}:1\leqslant r\leqslant r_0,\,1\leqslant s\leqslant r_0\}.$$

Assume α,β be such that $k_{r-1}<\alpha\leqslant k_r$ and $l_{s-1}<\beta\leqslant l_s$. As a result, we obtain

$$\begin{aligned}
&\frac{q!}{(\alpha\beta)^q}|\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:r_1,r_2,\dots,r_q\leqslant\alpha,\\
&\qquad u_1,u_2,\dots,u_q\leqslant\beta,\,\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\leqslant 1-\varkappa\\
&\qquad \text{or } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa,\,\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa\}|\\
&\leqslant\frac{q!}{(k_{r-1}l_{s-1})^q}|\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:r_1,r_2,\dots,r_q\leqslant k_r,\\
&\qquad u_1,u_2,\dots,u_q\leqslant l_s,\,\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\leqslant 1-\varkappa\\
&\qquad \text{or } \Omega\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa,\,\Xi\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\geqslant\varkappa\}|\\
&=\frac{q!}{(k_{r-1}l_{s-1})^q}\{N_{11}+N_{12}+N_{21}+N_{22}+\cdots+N_{r_0s_0}+\cdots+N_{rs}\}\\
&=\frac{q!}{(k_{r-1}l_{s-1})^q}\left\{\sum_{r_1,\dots,r_w=1}^{r}\sum_{u_1,\dots,u_w=1}^{s}N_{r_1,\dots,r_w,u_1,\dots,u_w}\right\}\\
&\leqslant\frac{Ur_0^2q!}{(k_{r-1}l_{s-1})^q}+\frac{q!}{(k_{r-1}l_{s-1})^q}\left\{\sum_{r_1,\dots,r_w=1}^{r}\sum_{u_1,\dots,u_w=1}^{s}N_{r_1,\dots,r_w,u_1,\dots,u_w}\right\}\\
&\leqslant\frac{Ur_0^2q!}{(k_{r-1}l_{s-1})^q}+\frac{q!}{(k_{r-1}l_{s-1})^q}\left\{\sum_{r_1,\dots,r_w=r_0+1}^{r}\sum_{u_1,\dots,u_w=r_0+1}^{s}\frac{N_{r_1,\dots,r_w,u_1,\dots,u_w}h_{r_1,\dots,r_w,u_1,\dots,u_w}}{h_{r_1,\dots,r_w,u_1,\dots,u_w}}\right\}\\
&\leqslant\frac{Ur_0^2q!}{(k_{r-1}l_{s-1})^q}+\frac{q!}{(k_{r-1}l_{s-1})^q}\left(\sup_{r_1,\dots,r_w,u_1,\dots,u_w\geqslant r_0}\frac{N_{r_1,\dots,r_w,u_1,\dots,u_w}}{h_{r_1,\dots,r_w,u_1,\dots,u_w}}\right)\\
&\qquad\times\left\{\sum_{r_1,\dots,r_w=r_0+1}^{r}\sum_{u_1,\dots,u_w=r_0+1}^{s}h_{r_1,\dots,r_w,u_1,\dots,u_w}\right\}\\
&\leqslant\frac{Ur_0^2q!}{(k_{r-1}l_{s-1})^q}+\psi\left\{\sum_{r_1,\dots,r_w=r_0+1}^{r}\sum_{u_1,\dots,u_w=r_0+1}^{s}h_{r_1,\dots,r_w,u_1,\dots,u_w}\right\}\leqslant\frac{Ur_0^2q!}{(k_{r-1}l_{s-1})^q}+\psi QR.
\end{aligned}$$

Since $k_{r-1}l_{s-1}\to\infty$ as $\alpha,\beta\to\infty$, it concludes that for all $\varkappa\in(0,1)$ and $s>0$,

$$\begin{aligned}&\frac{q!}{(\alpha\beta)^q}|\{((r_1,r_2,\dots,r_q),(u_1,u_2,\dots,u_q))\in\mathbb{N}^q\times\mathbb{N}^q:r_1,r_2,\dots,r_q\leqslant\alpha,\\
&\qquad u_1,u_2,\dots,u_q\leqslant\beta,\,\Theta\left(t,t_{r_1u_1},t_{r_2u_2},\dots,t_{r_qu_q},s\right)\leqslant 1-\varkappa\end{aligned}$$

$$\text{or } \Omega\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa, \Xi\left(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa\}\big| \to 0.$$

It gives that $st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta, \Omega, \Xi)$. □

Theorem 5.8. *Let $\theta_2 = \theta_{r,s}$ be a lacunary double sequence. If*

$$1 < \liminf_r q_r \leqslant \limsup_r q_r < \infty \text{ and } 1 < \liminf_s q_s \leqslant \limsup_s q_s < \infty,$$

then $S_{\theta_2}^{(\Theta,\Omega,\Xi)} = st_2^{(\Theta,\Omega,\Xi)}$.

Theorem 5.9. *If $\left(t_{\alpha\beta}\right) \in st_2^{(\Theta,\Omega,\Xi)} \cap S_{\theta_2}^{(\Theta,\Omega,\Xi)}$, then $S_{\theta_2} - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta}$.*

Proof. Assume $st_2 - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t_0\,(\Theta, \Omega, \Xi)$ and $S_{\theta_2} - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t_1\,(\Theta, \Omega, \Xi)$ such that $t_0 \neq t_1$. Let

$$\begin{aligned}\lim_{\alpha,\beta\to\infty} \frac{q!}{(\alpha\beta)^q} \big|\{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \ldots, r_q \leqslant \alpha,\\ u_1, u_2, \ldots, u_q \leqslant \beta, \Theta\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \leqslant 1 - \varkappa\\ \text{or } \Omega\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa, \Xi\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa\}\big| = 1,\end{aligned}$$

for $\varkappa < \frac{1}{2}|t_0 - t_1|$. Let us now consider the $k_p l_\nu$-th term of the following expression:

$$\begin{aligned}&\frac{q!}{(\alpha\beta)^q} \big|\{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \ldots, r_q \leqslant \alpha,\\ &\quad u_1, u_2, \ldots, u_q \leqslant \beta, \Theta\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \leqslant 1 - \varkappa\\ &\quad \text{or } \Omega\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa, \Xi\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa\}\big|\\ &= \frac{q!}{(k_p l_\nu)^q} \left|\left\{(r_q, u_q) \in \bigcup_{r,s=1,1}^{p,\nu} I_{r,s} : \Theta\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \leqslant 1 - \varkappa\right.\right.\\ &\quad \text{or } \Omega\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa, \Xi\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa\}\big|\\ &\leqslant \frac{1}{\sum_{r,s}^{p,\nu} h_{r,s}} \sum_{r,s=1,1}^{p,\nu} h_{r,s}\rho_{r,s}\\ &= \frac{q!}{(k_p l_\nu)^q} \sum_{r,s=1,1}^{p,\nu} \big|\{(r_q, u_q) \in I_{r,s} : \Theta\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \leqslant 1 - \varkappa\\ &\quad \text{or } \Omega\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa, \Xi\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa\},\end{aligned}$$

where

$$\begin{aligned}\rho_{r,s} = \frac{q!}{(h_{rs})^q} \big|\{(r_q, u_q) \in I_{r,s} : \Theta\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \leqslant 1 - \varkappa\\ \text{or } \Omega\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa, \Xi\left(t_1, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, s\right) \geqslant \varkappa\}\end{aligned}$$

is a Pringsheim null sequence, since $S_{\theta_2} - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t_1\,(\Theta, \Omega, \Xi)$. As the double lacunary sequence $\theta_{r,s}$ satisfies all conditions for a four-dimensional matrix transformation to map Pringsheim null sequence into another Pringsheim null sequence, the last equation consequently tends to zero in the Pringsheim sense. Moreover, it constitutes a double sequence of the form:

$$\frac{q!}{(\alpha\beta)^q} \big|\{((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : r_1, r_2, \ldots, r_q \leqslant \alpha,$$

$$u_1, u_2, \ldots, u_q \leqslant \beta, \Theta\left(t_1, t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, s\right) \leqslant 1-\varkappa$$
$$\text{or } \Omega\left(t_1, t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, s\right) \geqslant \varkappa, \Xi\left(t_1, t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, s\right) \geqslant \varkappa\}\Big| .$$

However, this sequence does not tend to 1 in the Pringsheim sense. This contradiction implies that $t_0 = t_1$. □

Definition 5.10. Let $(\mathcal{U}, \Theta, \Omega, \Xi, *, \circledast)$ represent an NFGMS. A sequence $\left(t_{\alpha\beta}\right)$ is said to be lacunary statistically Cauchy (or $S_{\theta_2}^{(\Theta,\Omega,\Xi)}$-Cauchy) w.r.t. (Θ, Ω, Ξ) if there is a subsequence $\left(t_{\alpha_r \beta_s}\right)$ of $\left(t_{\alpha\beta}\right)$ such that $(\alpha_r, \beta_s) \in I_{rs}$ for each (Θ, Ω, Ξ)-$\lim_{r,s\to\infty} t_{\alpha_r\beta_s} = t$ and for each $\varkappa \in (0,1)$ and $s > 0$,

$$d_q^{\theta_2}\left(((r_1, r_2, \ldots, r_q), (u_1, u_2, \ldots, u_q)) \in \mathbb{N}^q \times \mathbb{N}^q : \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{\alpha_r\beta_s}, s\right) \leqslant 1-\varkappa\right.$$
$$\left.\text{or } \Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \varkappa,\ \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \varkappa\right) = 0,$$

or equivalently

$$\lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} \Big|\{(r_w, u_w) \in I_{rs}, 1 \leqslant w \leqslant q,\ \Theta\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{\alpha_r\beta_s}, s\right) \leqslant 1-\varkappa$$
$$\text{or } \Omega\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \varkappa, \Xi\left(t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_q u_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \varkappa\}\Big| .$$

Theorem 5.11. *If a sequence $\left(t_{\alpha\beta}\right)$ is lacunary statistically convergent w.r.t. (Θ, Ω, Ξ), then it is lacunary statistically Cauchy w.r.t. (Θ, Ω, Ξ).*

Proof. We first suppose that $S_{\theta_2} - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta, \Omega, \Xi)$. For $s > 0$ and $i, j > 0$, let

$$K(i,j) := \left\{(r_w, u_w) \in I_{rs}, 1 \leqslant w \leqslant q : \Theta(t, t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_w u_w}, s) > 1 - \frac{1}{ij} \text{ and}\right.$$
$$\left.\Omega(t, t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_w u_w}, s) < \frac{1}{ij}, \Xi(t, t_{r_1 u_1}, t_{r_2 u_2}, \ldots, t_{r_w u_w}, s) < \frac{1}{ij}\right\}.$$

Then, we have the following.

(a) $K(i+1, j+1) \subset K(i,j)$; and
(b) $\frac{|K(i,j)\cap I_{rs}|}{h_{rs}} \to 1$ as $r, s \to \infty$.

This implies that we can choose positive integers $u(1)$ and $v(1)$ such that for $r \geqslant u(1)$ and $s \geqslant v(1)$ we have $\frac{|K(1,1)\cap I_{rs}|}{h_{rs}} > 0$, i.e., $K(1,1) \cap I_{rs} \neq \emptyset$. Next, we can choose $u(2) > u(1)$ and $v(2) > v(1)$ such that $r \geqslant u(2)$ and $s \geqslant v(2)$ imply that $K(2,2) \cap I_{rs} \neq \emptyset$. Thus, for each pair (r,s) satisfying $u(1) \leqslant r \leqslant u(2)$ and $v(1) \leqslant s \leqslant v(2)$, we can choose $(\alpha_r, \beta_s) \in I_{rs}$ such that $(\alpha_r, \beta_s) \in K(1,1) \cap I_{rs}$, i.e.,

$$\Theta\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_r\beta_s}, s\right) > 1,\ \Omega\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_r\beta_s}, s\right) < 0,\ \Xi\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_r\beta_s}, s\right) < 0.$$

In general, we can choose $u(i+1) > u(i)$ and $v(j+1) > v(j)$ such that $r > u(i+1)$ and $s > v(j+1)$ imply that $K(i+1, j+1) \cap I_{rs} \neq \emptyset$. Then, for each pair (r,s) satisfying $u(i) \leqslant r \leqslant u(i+1)$ and $v(j) \leqslant s \leqslant v(j+1)$ we can choose $(\alpha_r, \beta_s) \in I_{rs}$, i.e.,

$$\Theta\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_r\beta_s}, s\right) > 1 - \frac{1}{ij}$$

and

$$\Omega\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_r\beta_s}, s\right) < \frac{1}{ij},$$
$$\Xi\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_r\beta_s}, s\right) < \frac{1}{ij}. \tag{5.1}$$

Thus, $(\alpha_r, \beta_s) \in I_{rs}$ for each r, s. Together with (5.1), this implies that $(\Theta, \Omega, \Xi) - \lim t_{\alpha_r\beta_s} = t$. For $\varkappa > 0$, choose $s > 0$ such that $(1-m) * (1-m) > 1-\varkappa$ and $s \circledast s < m$. For $s > 0$, if we take

$$\begin{aligned}
A := &\left\{(r_w, u_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) > 1-\varkappa\right.\\
&\left.\text{or } \Omega\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) < \varkappa,\ \Xi\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) < \varkappa\right\},\\
B := &\{(r_w, u_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) > 1-m\\
&\text{or } \Theta(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) < m,\ \Theta(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) < m\},\\
C := &\left\{(\alpha_w, \beta_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) > 1-s\right.\\
&\left.\text{or } \Theta\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) < m,\ \Theta\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) < m\right\},
\end{aligned}$$

then we obtain that $(B \cap C) \subset A$ and therefore $A^c \subset (B^c \cup C^c)$. Thus, we have

$$\begin{aligned}
\lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} &\left|\left\{(r_w, u_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \leqslant 1-\varkappa\right.\right.\\
&\left.\left.\text{or } \Omega\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \varkappa,\ \Xi\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \varkappa\right\}\right|\\
\leqslant \lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} &|\{(r_w, u_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) \leqslant 1-m\\
&\text{or } \Omega(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) \geqslant m,\ \Xi(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) \geqslant m\}|\\
+ \lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} &\left|\left\{(\alpha_w, \beta_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) \leqslant 1-m\right.\right.\\
&\left.\left.\text{or } \Omega\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) \geqslant m,\ \Xi\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) \geqslant m\right\}\right|.
\end{aligned}$$

Since $S_{\theta_2} - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta, \Omega, \Xi)$ and $S_{\theta_2} - \lim_{r,s\to\infty} t_{\alpha_r\beta_s} = t\,(\Theta, \Omega, \Xi)$, it follows that $\left(t_{\alpha\beta}\right)$ is $S_{\theta_2}^{(\Theta,\Omega,\Xi)}$-Cauchy. Conversely, assume that $\left(t_{\alpha\beta}\right)$ is $S_{\theta_2}^{(\Theta,\Omega,\Xi)}$-Cauchy w.r.t. (Θ, Ω, Ξ). By definition, there is a subsequence $\left(t_{\alpha_r\beta_s}\right)$ of $\left(t_{\alpha\beta}\right)$ such that $(\alpha_r, \beta_s) \in I_{rs}$ for each (Θ, Ω, Ξ)-$\lim_{r,s\to\infty} t_{\alpha_r\beta_s} = t$ and for each $\varkappa \in (0,1)$ and $s > 0$,

$$\begin{aligned}
\lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} &\left|\left\{(r_w, u_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \leqslant 1-\frac{\varkappa}{2}\right.\right.\\
&\left.\left.\text{or } \Omega\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \frac{\varkappa}{2},\ \Xi\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \frac{\varkappa}{2}\right\}\right| = 0.
\end{aligned} \tag{5.2}$$

As before, we have the following inequality:

$$\begin{aligned}
\lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} &|\{(r_w, u_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) \leqslant 1-\varkappa\\
&\text{or } \Omega(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) \geqslant \varkappa,\ \Xi(t, t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_wu_w}, s) \geqslant \varkappa\}|\\
\leqslant \lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} &\left|\left\{(r_w, u_w) \in I_{rs},\, 1 \leqslant w \leqslant q, \Theta\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \leqslant 1-\frac{\varkappa}{2}\right.\right.\\
&\left.\left.\text{or } \Omega\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \frac{\varkappa}{2},\ \Xi\left(t_{r_1u_1}, t_{r_2u_2}, \ldots, t_{r_qu_q}, t_{\alpha_r\beta_s}, s\right) \geqslant \frac{\varkappa}{2}\right\}\right|\\
+ \lim_{r,s\to\infty} \frac{q!}{(h_{rs})^q} &\left|\left\{(\alpha_w, \beta_w) \in I_{rs},\, 1 \leqslant w \leqslant q,\ \Theta\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) \leqslant 1-\frac{\varkappa}{2}\right.\right.\\
&\left.\left.\text{or } \Omega\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) \geqslant \frac{\varkappa}{2},\ \Xi\left(t, t_{\alpha_1\beta_1}, t_{\alpha_2\beta_2}, \ldots, t_{\alpha_w\beta_w}, s\right) \geqslant \frac{\varkappa}{2}\right\}\right|.
\end{aligned}$$

Since (Θ, Ω, Ξ)-$\lim_{r,s\to\infty} t_{\alpha_r\beta_s} = t$, it follows from (5.2) that $S_{\theta_2} - \lim_{\alpha,\beta\to\infty} t_{\alpha\beta} = t\,(\Theta, \Omega, \Xi)$. □

6. Conclusion

This study introduces the concepts of statistical convergence, statistical limit points, and statistical cluster points of double sequences within neutrosophic fuzzy G-metric spaces with order q, thereby extending the framework of neutrosophic fuzzy metric spaces. By substantiating our claims with pertinent theorems and illustrating them with examples, we provide a comprehensive understanding of these notions. Furthermore, after establishing the foundation of statistical convergence and analyzing its characteristics in these spaces, we delve into the concepts of lacunary statistical convergence and strongly lacunary convergence of double sequences, while also exploring their interconnections. Through these investigations, we contribute to a deeper understanding of convergence properties within neutrosophic fuzzy G-metric spaces, paving the way for further research in this area.

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