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VARIATIONAL PRINCIPLE FOR NONLINEAR SCHRÖDINGER EQUATION WITH HIGH NONLINEARITY

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This paper is dedicated to Professor Ji-Huan He

Abstract. It is well-known that the Schrödinger equation plays an important role in physics and applied mathematics as well. Variational formulations have been one of the hottest topics. This paper suggests a simple but effective method called the semi-inverse method proposed by Ji-Huan He to construct a variational principle for the nonlinear Schrödinger equation with high nonlinearity.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation with high nonlinearity:

$$i\Psi_t + \alpha\Psi_{xx} + \beta|\Psi|^2\Psi + \gamma|\Psi|^4\Psi = 0, \tag{1}$$

where $\Psi = \Psi(x,t)$ is a complex function of x and t.

This equation can be solved by the homotopy perturbation method[2,4,5,17], the variational iteration method[3,6,12,13,15,18,19] and the exp-function method [1,14,21,26,27]. In this paper we will establish a variational formulation using the semi-inverse method[7].

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2. Variational formulation

On substituting $\Psi(x,t) = u(x,t) + iv(x,t)$, where $u(x,t)$ and $v(x,t)$ are real functions of x and t, in Eq.(1), we get:

$$[-v_t+\alpha u_{xx}+\beta u(u^2+v^2)+\gamma u(u^2+v^2)^2]+i[u_t+\alpha v_{xx}+\beta v(u^2+v^2)+\gamma v(u^2+v^2)^2] = 0, \tag{2}$$

which leads to a system of two second-order equations expressed as

$$-v_t + \alpha u_{xx} + \beta u(u^2 + v^2) + \gamma u(u^2 + v^2)^2 = 0, \tag{3}$$

$$u_t + \alpha v_{xx} + \beta v(u^2 + v^2) + \gamma v(u^2 + v^2)^2 = 0. \tag{4}$$

In order to search for a variational principle for system (3) and (4), according to the semi-inverse method[7], we can construct a trial-functional in the form:

$$J(u,v) = \int [u_t v - \frac{\alpha}{2}v_x^2 + \frac{\beta}{4}(2u^2v^2 + v^4) + \frac{\gamma}{6}(3u^4v^2 + 3u^2v^4 + v^6) + F(u)]d\Omega, \tag{5}$$

where $d\Omega = dxdt$ and F is an unknown function of u and/or their derivatives.

There exist various alternative approach to the construction of the trial functional, illustrative examples can be found details in Refs.[8,20,24,25]. The advantage of the above trial-functional lies on the fact that stationary condition with respect to v, noting that F is a absence of u and its derivatives, is Eq.(4). Now calculating the variation of Eq.(5) with respect u results in the following Euler-Lagrange equation:

$$-v_t + \beta uv^2 + 2\gamma u^3v^2 + \gamma uv^4 + \frac{\delta F}{\delta u} = 0, \tag{6}$$

where $\frac{\delta F}{\delta u}$ is called He's variational differential with respect to u, defined as[7]

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}(\frac{\partial F}{\partial u_t}) - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) + \frac{\partial^2}{\partial t^2}(\frac{\partial F}{\partial u_{tt}}) + \frac{\partial^2}{\partial x^2}(\frac{\partial F}{\partial u_{xx}}) + \cdots . \tag{7}$$

We search for such an F that Eq.(6) turns out to Eq.(3). We therefore, set

$$\frac{\delta F}{\delta u} = v_t - \beta uv^2 - 2\gamma u^3v^2 - \gamma uv^4 = \alpha u_{xx} + \beta u^3 + \gamma u^5, \tag{8}$$

from which we identify F in the form:

$$F = -\frac{\alpha}{2}u_x^2 + \frac{\beta}{4}u^4 + \frac{\gamma}{6}u^6. \tag{9}$$

We, therefore, obtain the final variational principle for the discussed problem, which reads

$$J(u,v) = \int [u_t v - \frac{\alpha}{2}(u_x^2 + v_x^2) + \frac{\beta}{4}(u^2 + v^2)^2 + \frac{\gamma}{6}(u^2 + v^2)^3]d\Omega. \tag{10}$$

On substituting $u = \frac{\Psi+\Psi^*}{2}$,$v = i\frac{\Psi^*-\Psi}{2}$, where $\Psi^* = u - iv$, the following variational principle can be obtained

$$J(\Psi) = \int \{\frac{i}{4}[(\Psi^* - \Psi)(\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t})] - \frac{\alpha}{2}|\frac{\partial \Psi}{\partial x}|^2 + \frac{\beta}{4}|\Psi|^4 + \frac{\gamma}{6}|\Psi|^6]\}d\Omega. \tag{11}$$

3. Discussion and Conclusion

We obtain a variational principle for the discussed problem by the semi-inverse method[7] which is proven to be a promising method for the search for variational principles directly from field equations without the use of Lagrange multiplier. Ji-Huan He first suggested a variational approach to solitary solutions[9,10] and periodic solutions[11], He's variational method has been caught much attention recently. Ozis and Yidirim[16] considered the following equation

$$i\Psi_t + \Psi_{xx} + \gamma|\Psi|^2\Psi = 0. \tag{12}$$

Using the semi-inverse method, a variational principle is established, and the following solitary solution is obtained via the Ritz method.Zhang established a variational formulation of the generalized Zakharov equation using the semi-inverse method, and find a solitary wave solution[23]. Xu established a variational formulation for coupled nonlinear Schrödinger equations[22]. Applying Ritz method, we can easily obtain solitary solutions and periodic solutions, the solution procedure is illustrated in detailed in Refs. [9,10].

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References

1. A. Bekir and A. Boz, *Exact solutions for a class of nonlinear partial differential equations using exp-function method*, Int. J. Nonlinear Sci. **8** (2007), 505–512.
2. J. Biazar and H. Ghazvini, *Exact solutions for non-linear Schrödinger equations by He's homotopy perturbation method*, Phys. Lett. A **366** (2008), 79–84.
3. N. Bildik and A. Konuralp, *The use of Variational Iteration Method, Differential Transform Method and Adomian Decomposition Method for solving different types of nonlinear partial differential equations*, Int. J. Nonlinear Sci. **7** (2006), 65–70.
4. M. Gorji, D.D. Ganji and S. Soleimani,*New application of He's homotopy perturbation method*, Int. J. Nonlinear Sci. **8** (2007), 319–328.
5. J.H. He, *New interpretation of homotopy perturbation method*, Int. J. Modern Phys. B **20** (2006), 2561–2568.
6. J.H. He, *Variational iteration method - Some recent results and new interpretations*, J. Comput. and Appl. Math. **207** (2007), 3–17.
7. J.H. He, *Variational principles for some nonlinear partial differential equations with variable coefficients*, Chaos, Solitons, Fractals **19** (2004), 847–851.
8. J.H. He, *Variational approach to (2+1)-dimensional dispersive long water equations*, Phys. Lett. A **335** (2005), 182–184.
9. J.H. He, *Some Asymptotic Methods for Strongly Nonlinear Equations*, Int. J. Modern Phys. B **20** (2006), 1141–1199.
10. J.H. He, *Non-perturbative methods for strongly nonlinear problems*, Berlin: dissertation.de-Verlag im Internet GmbH, 2006.
11. J.H. He, *New interpretation of homotopy perturbation method*, Int. J. Modern Phys. B **20** (2006), 2561–2568.
12. J.H. He and X.H. Wu, *Variational iteration method: New development and applications*, Comput. & Math. with Appl. **54** (2007), 881–894.

13. J.H. He and X.H. Wu, *Construction of solitary solution and compacton-like solution by variational iteration method*, Chaos Solitons, Fractals. **29** (2006), 108–113.
14. J.H. He and X.H. Wu, *Exp-function method for nonlinear wave equations*, Chaos Solitons, Fractals **30** (2006), 700–708.
15. Z.M. Odibat and S. Momani, *Application of variational iteration method to Nonlinear differential equations of fractional order*, Int. J. Nonlinear Sci. **7** (2006), 27–34.
16. T. Ozis and A. Yidirim, *Application of He's semi-inverse method to the nonlinear Schrodinger equation*, Comput. & Math. with Appl. **54** (2007), 1039–1042.
17. A. Sadighi and D.D. Ganji , *Analytic treatment of linear and nonlinear Schrodinger equations: A study with homotopy-perturbation and Adomian decomposition methods*, Phys. Lett. A **372** (2008), 465–469.
18. N.H. Sweilam, *Variational iteration method for solving cubic nonlinear Schrödinger equation*, J. of Comput. and Appl. Math. **207** (2007), 155–163.
19. N.H. Sweilam, R.F.Al-Bar , *Variational iteration method for coupled nonlinear Schrodinger equations*, Comput. & Math. with Appl. **54** (2007), 993–999.
20. Z.L. Tao, *Variational approach to the inviscid compressible fluid*, Acta Appl. Math. **100** (2008), 291–294.
21. X.H. Wu and J.H. He, *Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method*, Comput. & Math. with Appl. **54** (2007), 966–986.
22. L. Xu, *Variational principles for coupled nonlinear Schrödinger equations*, Phys. Lett. A **359** (2007), 627–629.
23. J. Zhang, *Variational approach to solitary wave solution of the generalized Zakharov equation*, Comput. & Math. with Appl. **54** (2007), 1043–1046.
24. X.W. Zhou , *Variational theory for physiological flow*, Comput. & Math. with Appl. **54** (2007), 1000–1002.
25. X.W. Zhou, *Variational approach to the Broer-Kaup-Kupershmidt equation*, Phys. Lett. A **363** (2007), 108–109.
26. S.D. Zhu, *Exp-function method for the Hybrid-Lattice system*, Int. J. Nonlinear Sci. **8** (2007), 461–464.
27. S.D. Zhu, *Exp-function method for the discrete mKdV lattice*, Int. J. Nonlinear Sci. **8** (2007), 465–468.

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