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THE ROLE OF DELAY IN DIGESTION OF PLANKTON BY FISH POPULATION: A FISHERY MODEL

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Abstract. In this Paper we have developed a model in which the revenue is generated from fishing and the growth of fish depends upon the plankton which in turn grows logistically. The conditions for the persistence of system around non zero equilibrium have been found out using average Liapnouv function after establishing existence and boundedness of the solution. Then we formulated a model with delay in digestion of plankton by fish. Further the the threshold value of conversional parameter has been found out for hopf-bifurcation. The phenomena of hopf-bifurcation is demonstrated using graphs.

1. Introduction

Many researchers have studied the fishery dynamics with or without considering plankton growth [1, 2, 6, 7, 8, 9, 10, 12]. Again, the delay induced bifurcation in population dynamics shown by many researchers, for example [3, 4, 5, 11, 13, 14]. The first model in the economic theory of open access fishery is as follows [5]:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - qEx \tag{1.1}$$

$$TR - TC = pqEx - cE \tag{1.2}$$

Where $x(t)$ is the population of fish at time t and they are growing logistcally with constant rate 'r' and 'K' is the carrying capacity.'q' is the rate of fishing when effort 'E' is applied for fishing. Total sustainable revenue(TR) is equal to $pqEx$ Where 'p' is the per unit cost of harvested biomass. cE is the total cost(TC) Where c is the cost per unit effort. Sustainable economic rent is the

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difference of TR and TC, i.e., sustainable economic rent is $TR - TC$. 4 From the above model he found out the bio economic equilibrium and concluded that equilibrium level varies around the MSY (maximum sustainable yield). As a function of cost prise ratio.But the knowledge of bio economic equilibrium is insufficient as it does not gives the answers to the question (1) and what is the optimum effort level; (2) what the optimum sustainable yield and how these can be achieved. However,in order to maximize the sustainable economic rent($TR - TC$)to achieve the optimum level of fishing effort, he has studied away with the essential ingredient namely the dynamic of economic, which this is a crucial omission. After some time Scheffer [12] gave the autonomous temporal model as:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - qEx \tag{1.3}$$

$$\frac{dE}{dt} = kE(pqx - c) \tag{1.4}$$

Though the above model contains the dynamics of economy and biological process. But it does not contain the dynamic of plankton species which provide the necessary nutrients for the growth of fish, as well as the above model is not having the equation of economic rent. The above mentioned omission in the model have forced us to formulate a model which include the dynamics of plankton which grow logistically. Moreover we shall be using the dynamic of economic rent in term of effort.

2. The Mathematical Model

Model is assumed to be closed in which plankton species are growing logistically with a growth rate a and has the carrying capacity k. Again, α is the rate of harvesting of plankton spices by the fish population and the interaction between the plankton and fish is assumed to follow law of mass action. Conversion rate from plankton to fish is denoted by α_1, takeing $\beta = \alpha\alpha_1$ is the conversion rate of the fish population. The self decay of fish population is denoted by c_1. The rate of catchabliety of fish when effort E is applied is denoted by q_1 . The cost per unit fish b_1 and c is the cost per unit effort All the parameters are assumed to be positive. The rate of change of economic rent (E_R) is equal to the difference of total revenue from fish sale and total cost of fishing. now $P(t)$, $F(T)$ and $E(T)$ represent plankton population, fish population and effort respectively at any time t. Hence, we can write the mathematical model for the above system as follows:

$$\frac{dP}{dt} = aP\left(1 - \frac{P}{K}\right) - \alpha PF \tag{2.1}$$

$$\frac{dF}{dt} = \beta PF - c_1F - q_1EF \tag{2.2}$$

$$\frac{dE_R}{dt} = q_1b_1EF - cE \tag{2.3}$$

Taking $E_R = \rho E$ and non-dimensionalized the above system and choosing the following new variables:

$$x \equiv \frac{P}{K}, \; y \equiv \frac{F}{F_0}, \; z \equiv \frac{E}{E_0} \; at \equiv \tau.$$

The above system reduces to

$$\frac{dx}{dt} = x(1-x) - bxy \tag{2.4}$$

$$\frac{dy}{dt} = \beta_1 xy - c_0 y - yz \tag{2.5}$$

$$\frac{dz}{dt} = \theta yz - dz \tag{2.6}$$

Where

$$b = \frac{\alpha F_0}{a}; \; \beta_1 = \frac{\beta k}{a}; \; c_0 = \frac{c_1}{a}; \theta = \frac{bF_0}{\rho E_0}; \; d = \frac{c}{\rho a}.$$

3. Existence of Equilibrium Points and Boundedness

There are four feasible equilibria of the system (2.4)-(2.6), namely,

(1) $E_0 = (0,0,0)$ is the trivial steady state,
(2) $E_1 = (1,0,0)$, here only plankton population exists,
(3) $E_2 = (\frac{c_0}{\beta_1}, \frac{1}{b}\left(1 - \frac{c_0}{\beta_1}\right), 0)$, here no fishing take place only plankton and fish are living together and
(4) (iv) $E^* = (x^*, y^*, z^*)$, all three population co-exists, where $x^* = 1 - \frac{bd}{\theta}$; $y^* = \frac{d}{\theta}$; $z^* = \beta_1 - c_0 - \frac{\beta_1 bd}{\theta}$.

Again, E_2 is feasible if $\beta_1 > c_0$ and E^* is feasible if

$$\theta > max\left\{bd, \frac{\beta_1 bd}{\beta_1 - c_0}\right\} \text{ and } \beta_1 > c_0.$$

Now we will show that all the solutions of the system (2.4)-(2.6) are bounded in a region $B \subset R^3_+$. We consider the following function

$$w(\tau) = x(\tau) + y(\tau) + z(\tau).$$

Then the time derivative of the above function after substituting the values from (2.4)-(2.6), we get

$$\frac{dw}{d\tau} = x(1-x) - (d - \beta_1)xy - (1-\theta)yz - dz - c_0 y.$$

$$\frac{dw}{d\tau} \le x(1-x) - dz - c_0 y.$$

$$\frac{dw}{d\tau} + \eta w(\tau) \le (1+\eta)x - x^2 = f(x).$$

where

$$\eta = min\{d, c_0\} \text{ and } f(x) = (1+\eta)x - x^2.$$

Hence

$$\frac{dw}{d\tau} + \eta w(\tau) \le \frac{1+\eta^2}{4} = M(say).$$

Now, using comparison theorem, as $\tau \to \infty$, then

$$sup\ w(\tau) \le \frac{M}{\eta}.$$

4. Dynamical Behavior

The dynamical behavior of the equilibrium can be studied by computing the variational matrix at various equilibrium points and Using the Routh-Hurwitz criterion. We can note the following points:

(1) The equilibrium point E_0 is a saddle point with locally stable manifold in Y-Z plane and with unstable manifold in x direction.
(2) If $\beta_1 < c_0$,then the equilibrium point $E_1(1, 0, 0)$ is locally asymptotically stable in X-Y-Z space,as E_2 and E_3 does not exit for $\beta_1 < c_0$, but if $\beta_1 > c_0$, then E_1 is saddle point with local stable manifold in X-Z direction and with unstable manifold in Y-direction.
(3) E_2 is saddle point but with stable manifold in X-Y plane when $\beta_1 > c_0$.

Lemma 4.1. *If $\beta_1 > c_0$, then E_2 is globally asymptotically stable in the interior of positive quadrant of X-Y plane.*

Proof. Taking $H = \frac{1}{xy}$, where is $H > 0$ in the interior of positive quadrant and

$$f_1(xy) = x\,(1-x) - bxy,$$
$$f_2(xy) = \beta_1 xy - c_0 y.$$

Clearly

$$\frac{df_1 H}{dx} + \frac{df_2 H}{dy} < 0$$

and it does not changes sign in positive quadrant. Therefore, using Bendixon-Dulec criterion there does not exist any limit cycle in X-Y plane. □

Now we will study the uniform persistence of the system using average Lyapunove function [15].

Theorem 4.2. *The system (2.4)-(2.6) is uniformly persistent if $l_1 > c_0 l_2 + dl_3$, $l_2 > \frac{dl_3}{\beta_1 - c_0}$ and $\beta_1 > c_0$, where l_1, l_2, l_3 are all positive.*

Proof. Take average Lyapunove function for the system as $\rho(X) = x_1^l y_2^l z_3^l$. Clearly $\rho(x)$ is non negative function defined in R_+^3 and X is a function of x, y, z. After differentiating we have

$$\begin{aligned}\psi(X) &= \tfrac{\dot\rho(X)}{\rho(X)} = l_1\tfrac{\dot x}{x} + l_2\tfrac{\dot y}{y} + l_3\tfrac{\dot z}{z}\\ &= l_1(1 - x - by) + l_2(\beta_1 x - c_0 - z) + l_3(\theta y - d)\end{aligned}$$

Further from above theorem, the system has no periodic orbit in the interior of X-Y plane. Thus, the uniform persistent exists, if there exists l_1, l_2 and l_3, such that $\psi(X)$ is positive at E_0, E_1 and E_2. Now

(1) $\psi(E_0) > 0$ if $l_1 > c_0 l_2 + d l_3$
(2) $\psi(E_1) > 0$ if $l_2 > \frac{d l_3}{\beta_1 - c_0}$
(3) $\psi(E_2) > 0$ if $\theta\left(1 - \frac{c_0}{\beta_1}\right) > d \Rightarrow \beta_1 > c_0\left(1 - \frac{d}{\theta}\right)$, which is always true for any set of positive values of l_1, l_2, l_3, since in the existence of E_2, $\beta_1 > c_0$ and all the parameter of the systems are positive.
(4) $\psi(E_3) > 0$, always true for any set of positive values of l_1, l_2, l_3.

Hence, the system is persistent if

$$l_1 > c_0 l_2 + d l_3,\ l_2 > \frac{d l_3}{\beta_1 - c_0} \text{ and } \beta_1 > c_0 \tag{4.1}$$

are satisfied. □

Example 4.3. Let us choose suitable values of the parameters: $b = 0.5$, $c_0 = 0.01$, $\theta = 0.5$, $d = 0.1$, $l_1 = 0.5$, $l_2 = 0.3$ and $l_3 = 0.5$, which will clearly satisfy the conditions (4.1). Hence the system always stable around E^*. The numerical solution of the system (2.4)-(2.6), taking the same set of values for the parameters as mentioned above, with $\beta_1 = 0.02$, 0.03 and 0.2 respectively.

5. The Model with Delay

Here we assume that fishes takes time to digest the plankton and grow proportionally.

$$\frac{dx}{dt} = x(1-x) - bxy \tag{5.1}$$

$$\frac{dy}{dt} = \beta_1 y \int_{-\infty}^{t} \beta \exp(-\beta(t-s) f(s) - c_0 y - yz \tag{5.2}$$

Where $f(s) = f(x) = x$

$$\frac{dz}{dt} = \theta yz - dz \tag{5.3}$$

Put

$$R(t) = \int_{-\infty}^{t} \beta \exp(-\beta(t-s) f(s) ds$$

Therefore

$$dR/dt = \beta(x - R)$$

$$\frac{dx}{dt} = x(1-x) - bxy \tag{5.4}$$

$$\frac{dy}{dt} = \beta_1 y R - c_0 y - yz \tag{5.5}$$

$$\frac{dz}{dt} = \theta yz - dz \tag{5.6}$$

There are three steady state of the system with delay, namely,

(1) $E_0(0, 0, 0, 0)$ is trivial equilibrium,
(2) $E_1(1, 0, 0, 0)$ here only plankton population exists

(3) $E^*(x^*, y^*, z^*, R^*)$ is the non-trivial equilibrium, where$x^* = (1 - bd/\theta)$, $y^* = b/d$, $z^* = (\beta_1 - c_0 - b\beta_1 d/\theta)$ and $R^* = (1 - bd/\theta)$.

The non-trivial equilibrium is non negative if $(1 - bd/\theta)\beta_1 > c_0$ and $\theta) > bd$ the characteristic equation of the delayed system at $E -^* (x^*, y^*, z^*, R^*)$ is given by

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \tag{5.7}$$

Where $A_1 = \beta + x^*$, $A_2 = \theta y^* z^* + \beta x^*$, $A_3 = (\beta + x^*)\theta y^* z^* + \beta\beta_1 b x^* y^*$ and $A_4 = \theta\beta x^* y^* z^*$ on substitution the values of x^*, y^*, z^* and R^*, it can be easily verified that $A_i > 0$, for $i = 1, 2, 3, 4$. Now, from Routh-Hurwitz criterion a set of necessary conditions for all the roots of the equation (5.7) having negative real part are $A_i > 0$, $i = 1, 2, 3, 4$. Now we shell diagnose the hopf bifurcation of the given system for β_1 variable which represent the conversional rate from plankton to fish population.

We know that the necessary and sufficient conditions for Hopf-Bifurcation, that there exist $\beta_1 = \beta_0$ such that (i) $A_i(\beta_0) > 0$ for $i = 1, 2, 3, 4$, (ii) $H_2(\beta_0) = A_1A_2 - A_3 \neq 0$, (iii) $H_3(\beta_0) = A_1A_2A_3 - A_1^2A_2 - A_3^2 = 0$ and (iv) $\frac{dH_3}{d\beta_1}(\beta_0) \neq 0$.

The condition (i) is true for all values of β_1 established earlier. Now, assume there exist $\beta_0 > 0$ such that $H_3(\beta_0) = 0$, which implies

$$a_1 + a_2\beta_0 + a_3\beta_0^2 = 0 \tag{5.8}$$

where $a_1 = (\beta + x^*)L_1L_3 - (\beta + x^*)^2L_5 - L - 3^2$, $a_2 = (\beta + x^*)(L_1L_4 - L_2L_3) - L_6(\beta + x^*)^2 - 2L_3L_4$, $a_3 = (\beta + x^*)L_2L_4 - L_4^2$, $L_1 = \beta x^* - \theta y^* c_0$, $L_2 = \theta x^* y^*$ $L_3 = -(\beta + x^*)\theta y^* c_0$, $L_4 = \beta b x^* y^* + (\beta + x^*)\theta x^* y^*$ $L_5 = -\theta c_0 \beta b x^* y^*$, $L_6 = \theta\beta b(x^*)^2 y^*$. By taking $c_0 = 0.01, b = 0.5, \theta = 0.5, d = 0.1$ from (5.8), we get $\beta_0 = 0.0216$. Further, $H_2(\beta_0) = \beta x^*(\beta + 1 - bd/\theta) - \beta_1 b^2 d) \neq 0$ and $\frac{dH_3}{d\beta_1}(\beta_0) \neq 0$. Hence the system start bifurcating at β_0. The results are shown graphically, taking $b = 0.5$, $c_0 = 0.01$, $\theta = 0.5$, $d = 0.1$, $\beta = 0.01$ with same set of values of β_1 as in the previous section. When $\beta_1 = 0.02 < \beta_0 = 0.0216$, the solution is converging to E^* and as β_1 crosses the value β_0, the system converges in a limit cycle, which is shown in Figure 2(b) and 2(c). On comparing the figures 1(a)-(c) with 2(a)-(c) respectively, it can be established that the delay induces Hopf-bifurcation in the system.

Remark 5.1. In this model we have established using average Liapounove function that the model without delay in conversion rate of fish persist uniformly under some conditions. Again, with the introduction of delay in conversion rate of fish, the system starts oscillating when β_1 crosses a threshold value as shown in figures.

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