J. Nonlinear Sci. Appl. 1 (2008), no. 1, 31 –35

THE JOURNAL OF NONLINEAR SCIENCES AND ITS APPLICATIONS
http://www.tjnsa.com

COMMON FIXED POINT THEOREM IN PROBABILISTIC QUASI-METRIC SPACES

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ABSTRACT. In this paper, we consider complete probabilistic quasi-metric space and prove a common fixed point theorem for R-weakly commuting maps in this space.

1. INTRODUCTION

Menger introduced the notion of a probabilistic metric spaces in 1942 and, since then, the theory of probabilistic metric spaces has developed in many directions, especially, in nonlinear analysis and applications [5]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric.

Recently, Pant introduced the notion of R-weak commutativity of mappings in metric spaces and proved some common fixed point theorems.

In this paper, we define R-weak commutativity of mapping in probabilistic quasi-metric spaces and prove the probabilistic version of Pant's theorem. In the sequel, we shall adopt usual terminology, notation and conventions of the theory of probabilistic metric spaces, as in [1, 5].

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) denotes $\Delta^+ = \{F : \mathbf{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1] : F$ is left-continuous and non-decreasing on $\mathbf{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if

Date: Received: 2 March 2008.

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2000 *Mathematics Subject Classification.* Primary 54E70; Secondary 54H25.

Key words and phrases. Probabilistic metric spaces; quasi-metric spaces; fixed point theorem; R-weakly commuting maps; triangle function.

$F(x) \leq G(x)$ for all x in $\mathbf{R}$. The maximal element for Δ^+ in this order is the d.f. given by $\varepsilon_0 = 0$, if $x \leq 0$ and $\varepsilon_0 = 1$, if $x > 0$

We assume that Δ^+ is metrized by the Sibley metric d_S, which is the modified Lévy metric [4, 6]. If F and G are d.f.'s and $h \in (0, 1]$, let $(F, G; h)$ denote the condition

$$F(x - h) - h \leq G(x) \leq F(x + h) + h$$

for all x in $(-1/h, 1/h)$. Then the modified Lévy metric (Sibley metric) is defined by

$$d_S(F, G) := \inf\{h : \textit{both } (F, G; h) \textit{ and } (G, F; h) \textit{ hold}\}.$$

For any F in Δ^+,

$$\begin{aligned} d_S(F, \varepsilon_0) &= \inf\{h : (F, \varepsilon_0; h) \textit{ holds}\} \\ &= \inf\{h : F(h^+) > 1 - h\} \end{aligned}$$

and, for any $t > 0$,

$$F(t) > 1 - t \Longleftrightarrow d_S(F, \varepsilon_0) < t.$$

It follows that, for every F, G in Δ^+,

$$F \leq G \Longrightarrow d_S(G, \varepsilon_0) \leq d_S(F, \varepsilon_0).$$

A *triangle function* is a binary operation on Δ^+, namely, a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, i.e., for all $F, G, H \in \Delta^+$,

$$\begin{array}{rcl} \tau(\tau(F, G), H) &=& \tau(F, \tau(G, H)), \\ \tau(F, G) &=& \tau(G, F), \\ F \leq G &\Longrightarrow& \tau(F, H) \leq \tau(G, H), \\ \tau(F, \varepsilon_0) &=& F. \end{array}$$

Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+. Typical continuous triangle function is $\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t))$. Here T is a continuous t-norm, i.e., a continuous binary operation on $[0, 1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity. Two typical examples of continuous t-norm are $\pi(a, b) = ab$ and $W(a, b) = \max(a + b - 1, 0)$.

Definition 1.1. A Probabilistic Quasi-Metric (briefly, PQM) space is a triple $(X, \mathcal{F}, \tau)$, where X is a nonempty set, τ is a continuous triangle function, and $\mathbf{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{p,q}$ denotes the value of $\mathcal{F}$ at the pair (p, q), the following conditions hold for all p, q, r in X:

(PQM1) $F_{p,q} = F_{q,p} = \varepsilon_0$ if and only if, $p = q$;

(PQM2) $F_{p,q} \geq \tau(F_{p,r}, F_{r,q})$ for all $p, q, r \in X$.

Theorem 1.2. ([5]) *If $(X, \mathcal{F}, \tau)$ is a PQM space with $\tau \geq \tau_W$ and function β is defined by*

$$\beta(p, q) = d_S(F_{p,q}, \varepsilon_0).$$

Then β is a quasi-metric.

Definition 1.3. Let $(X, \mathcal{F}, \tau)$ be a PQM space.

(1) A sequence $\{p_n\}_n$ in X is said to be *strongly convergent* to p in X if, for every $\lambda > 0$, there exists positive integer N such that $F_{p_n,p}(\lambda) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{p_n\}_n$ in X is called *strong right* ($left$) *Cauchy sequence* [3] if, for every $\lambda > 0$, there exists positive integer N such that $F_{p_n,p_m}(\lambda) > 1 - \lambda$ whenever $n \geq m \geq N$ ($m \geq n \geq N$).

(3) A PQM space $(X, \mathcal{F}, \tau)$ is said to be *strong right* ($left$) *complete* in the strong topology if and only if every strong right (left) Cauchy sequence in X is strongly convergent to a point in X.

Theorem 1.4. ([5]) *Let $(X, \mathcal{F}, \tau)$ be a PQM space and let $\{p_n\}$ be a sequence in X. Then $p_n \to p$ if and only if $d_S(F_{p_n,p}, \varepsilon_0) \to 0$ if and only if $\beta(p_n, p) \to 0$. Similarly, $\{p_n\}$ is a strong right ($left$) Cauchy sequence if and only if for every $\epsilon > 0$ there exists positive integer N such that*

$$\beta(p_n, p_m) < \epsilon$$

whenever $n \geq m \geq N$ ($m \geq n \geq N$).

Theorem 1.5. ([4]) *If τ is a continuous triangle function and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$, then $\lim_{n\to\infty} F_{p_n,q_n} = F_{p,q}$.*

2. Main Results

Definition 2.1. Let f and g be maps from a PQM space $(X, \mathcal{F}, \tau)$ into itself. The maps f and g are said to be *weakly commuting* if

$$F_{fgx,gfx} \geq F_{fx,gx}$$

for each x in X.

Definition 2.2. Let f and g be maps from a PQM space $(X, \mathcal{F}, \tau)$ into itself. The maps f and g are said to be *R-weakly commuting of type (A_f)* if there exists a positive real number R such that

$$F_{fgx,ggx}(t) \geq F_{fx,gx}(t/R)$$

for each $x \in X$ and $t > 0$.

Weak commutativity implies R-weak commutativity in PQM space. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Theorem 2.3. *Let $(X, \mathcal{F}, \tau)$ be a left complete PQM space in which $\tau \geq \tau_W$ and let f, g be R-weakly commuting self-mappings of X satisfying the following conditions:*

(i) *$f(X) \subseteq g(X)$;*
(ii) *f or g is continuous;*
(iii) *for all $x, y \in X$,*

$$F_{fx,fy} \geq C(F_{gx,gy}),$$

where $C : D^+ \longrightarrow D^+$ is a continuous function such that $C(F) > F$ for each $F \in D^+$ with $F < \varepsilon_0$.
Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (i), choose a point x_1 in X such that $fx_0 = gx_1$. In general, choose x_{n+1} such that $fx_n = gx_{n+1}$. Then we have

$$\begin{aligned} F_{fx_n,fx_{n+1}} &\geq C(F_{gx_n,gx_{n+1}}) = C(F_{fx_{n-1},fx_n}) \\ &> F_{fx_{n-1},fx_n} \end{aligned}$$

Thus $\{F_{fx_n,fx_{n+1}}\}$ is increasing sequence in D^+. Therefore, it converges to a limit $G \leq \varepsilon_0$. We claim that $G = \varepsilon_0$. For, if $G < \varepsilon_0$ on making $n \longrightarrow \infty$ in the above inequality, we get $G \geq C(G) > G$, which is a contradiction. Hence $G = \varepsilon_0$, i.e.,

$$\lim_{n\to\infty} F_{fx_n,fx_{n+1}} = \varepsilon_0.$$

By Theorem 1.4, there exists $n_0 \in \mathbf{N}$ such that, for every $n \geq n_0$,

$$\beta(fx_n, fx_{n+1}) \leq \frac{1}{2^n}.$$

Now, for every $m, n \in \mathbf{N}$ with $m > n$, we have

$$\beta(fx_n, fx_m) \leq \sum_{j=n}^{m} \beta(fx_j, fx_{j+1}) \leq \sum_{j=n}^{m} \frac{1}{2^j} \longrightarrow 0$$

as $m, n \to \infty$. Thus, by Theorem 1.4, $\{fx_n\}$ is left Cauchy sequence and, by the left completeness of X, $\{fx_n\}$ converges to $z \in X$. Also $\{gx_n\}$ converges to z in X. Let us suppose that the mapping f is continuous. Then $\lim_{n\to\infty} ffx_n = fz$ and $\lim_{n\to\infty} fgx_n = fz$. Further, since f and g are R-weakly commuting, we have

$$F_{fgx_n,gfx_n}(t) \geq F_{fx_n,gx_n}(t/R).$$

On letting $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} gfx_n = fz$ by Theorem 1.5.

We now prove that $z = fz$. Suppose $z \neq fz$. Then $F_{z,fz} < \varepsilon_0$. By (iii),

$$F_{fx_n,ffx_n} \geq C(F_{gx_n,gfx_n}).$$

On letting $n \to \infty$ in the above inequality, we have

$$F_{z,fz} \geq C(F_{z,fz}) > F_{z,fz},$$

which is a contradiction. Therefore, $z = fz$. Since $f(X) \subseteq g(X)$, we can find a point $z_1 \in X$ such that $z = fz = gz_1$. Now,

$$F_{ffx_n,fz_1} \geq C(F_{gfx_n,gz_1}).$$

Taking limit as $n \to \infty$, we have

$$F_{fz,fz_1} \geq C(F_{fz,gz_1}) = \varepsilon_0$$

since $C(\varepsilon_0) = \varepsilon_0$, which implies that $fz = fz_1$, i.e., $z = fz = fz_1 = gz_1$. Also, for any $t > 0$,

$$F_{fz,gz}(t) = F_{fgz_1,gfz_1}(t) \geq F_{fz_1,gz_1}(t/R) = \varepsilon_0(t/R) = 1$$

which again implies that $fz = gz$. Thus z is a common fixed point of f and g.

Now, to prove the uniqueness of the common fixed point z, let $z' \neq z$ be another common fixed point of f and g. Then $F_{z,z'} < \varepsilon_0$ and

$$F_{z,z'} = F_{fz,fz'} \geq C(F_{gz,gz'}) = C(F_{z,z'}) > F_{z,z'},$$

which is contradiction. Therefore, $z = z'$, i.e., z is a unique common fixed point of f and g.

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