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On solvability for two-dimensional non-linear functional integral equations via measure of non-compactness

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Abstract

In this paper, we find out conditions for the existence result of two dimensional generalized functional integral equations in $C([0,c]\times[0,d])$. The main theorem is proved by using the theory of Petryshyn's fixed point theorem and measure of non-compactness. A few examples of equations is provided to show the applicability in various integral equations.

Keywords: Functional integral equation (FIE), fixed point theorem, Banach algebra, measure of non-compactness (MNC).

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1. Introduction

Functional integral equations (FIEs) play an important role in various fields like modeling, physics, engineering, traffic, neutron transport, control theory, population dynamics, optimization, and theory of gases (see [4, 5, 15, 17, 20, 21, 26, 27]). The Darbo's and the Petryshyn's theorem are very useful in the analysis of FIEs. The application of MNC and fixed point theory for solving FIEs can be easily found in [1–3, 6, 31, 35]. To apply MNC for the existence result of several FIEs, many researchers have worked (see [7, 9, 10, 18, 19, 23, 24, 28, 34]) in this area.

First of all, Kazemi and Ezzati [24] used the Petryshyn fixed point theorem to check the solution of nonlinear functional integral equations whether it exists or not. After that many researchers worked on this idea to discuss the existence of it in Banach spaces as well as Banach Algebra (see [8, 11–14, 22, 25, 34, 36] and references therein). In this paper, we study the existence result of the following FIE

$$\begin{aligned} u(\varphi,\zeta) = f\left(\varphi,\zeta,u(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\right) \\ \times q\left(\varphi,\zeta,u(\varphi,\zeta),u(\eta(\varphi,\zeta)),\int_0^{c}\int_0^{d} V(\varphi,\zeta,z,\nu,u(\delta(z,\nu)))d\nu dz\right), \end{aligned} \tag{1.1}$$

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where $(\varphi, \zeta) \in I_0 = [0, c] \times [0, d]$. Das et al. [6] studied the existence of solutions for 2D FIE

$$u(\varphi, \zeta) = p(\varphi, \zeta) + f\left(\varphi, \zeta, u(\varphi, \zeta), \int_0^{\varphi} \int_0^{\zeta} H(\varphi, \zeta, z, \nu, u(z, \nu)) d\nu dz\right) \tag{1.2}$$

for $(\varphi, \zeta) \in [0, 1] \times [0, 1]$. Kazemi et al. [24] studied the following nonlinear 2D FIE

$$u(\varphi, \zeta) = f\left(\varphi, \zeta, u(\varphi, \zeta), \int_0^{\varphi} P(\varphi, \zeta, y, u(\beta(y, \zeta))) dy, \int_0^{\varphi} \int_0^{\zeta} H(\varphi, \zeta, z, \nu, u(z, \nu)) d\nu dz\right) \tag{1.3}$$

for $(\varphi, \zeta) \in I_0$. Kazemi et al. [24] also discussed the existence result for the following nonlinear 2D FIE

$$u(\varphi, \zeta) = p(\varphi, \zeta, u(\varphi, \zeta)) + f\left(\varphi, \zeta, u(\varphi, \zeta), \int_0^{\varphi} P(\varphi, \zeta, y, u(\beta(y, \zeta))) dy, \int_0^{\varphi} \int_0^{\zeta} H(\varphi, \zeta, z, \nu, u(z, \nu)) d\nu dz\right) \tag{1.4}$$

for $(\varphi, \zeta) \in I_0$. Further, a famous 2D FIE of Hammerstein type [32] has the form

$$u(\varphi, \zeta) = p(\varphi, \zeta) + \int_0^{\varphi} \int_0^{\zeta} p_1(\varphi, \zeta, z, \nu) p_2(z, \nu, u(z, \nu)) d\nu dz.$$

Eq. (1.1) is quite general in nature and covers Eqs. (1.2)-(1.4) as particular problems and these are more valuable in real-world applications, for more details (see [4, 5, 15]). The aim of this study is to use Petryshyn's theorem (preferably Darbo's theorem) to examine the solvability of Eq. (1.1). This study will also investigate the conditions given in multiple papers and it will unify the work in this field. The third condition which we have used is the bounded condition and it shows that the "sub-linear condition" that has been considered in literature will not play a significant role here.

The paper consists of four sections involving the introduction. In Section 2, we give some preliminaries and represent the concept of MNC. Section 3 is concerned with an existence theorem for equations connecting condensing operators with the help of Petryshyn's fixed point theorem. Section 4 deals with some examples as an application of this type of nonlinear FIEs.

2. Preliminaries

Let X be a real Banach space and B_r denotes a closed ball center at 0 with radius r and $\partial B_r = \{u \in X : \|u\| = r\}$ for the sphere in X around 0 with radius $r > 0$. MNC's are rich tools in nonlinear analysis for existence in the fixed point theory and operator equations in X.

Definition 2.1 ([29])**.** If $N \in X$, then Kuratowski MNC of N is given by

$$\hat{\beta}(N) = \inf\left\{\rho > 0 : N = \bigcup_{i=1}^{n} N_i \text{ with diam } N_i \leqslant \rho,\ i = 1, 2, \ldots, n\right\}.$$

Definition 2.2 ([16])**.** The Hausdroff MNC is as

$$\mu(N) = \inf\{\rho > 0 : \exists \text{ a finite } \rho\text{-net for } N \text{ in } X\},$$

where a finite ρ-net for N in X denotes a set $\{u_1, u_2, \ldots, u_n\} \subset X$ such that the ball $B_\rho(X, u_1), B_\rho(X, u_2), \ldots, B_\rho(X, u_n)$ over N. These MNC are mutually equivalent in the sense that $\mu(N) \leqslant \hat{\beta}(N) \leqslant 2\mu(N)$ for a bounded set $N \subset X$.

Theorem 2.3. *Let* $N, \hat{N} \in X$ *and* $\lambda \in \mathbb{R}$. *Then*

(i) $\mu(N) = 0$ *if and only if* N *is relatively compact;*

(ii) $N \subseteq \hat{N} \implies \mu(N) \leqslant \mu(\hat{N})$;
(iii) $\mu(\overline{N}) = \mu(\text{Conv } N) = \mu(N)$;
(iv) $\mu(N \cup \hat{N}) = \max\{\mu(N), \mu(\hat{N})\}$;
(v) $\mu(\lambda N) = |\lambda|\mu(N)$;
(vi) $\mu(N + \hat{N}) \leqslant \mu(N) + \mu(\hat{N})$.

Here, $C(I_0) = C([0, c] \times [0, d])$ consists of all real valued continuous functions with the usual norm

$$||u|| = \max\{|u(\varphi, \zeta)| : (\varphi, \zeta) \in I_0\}.$$

The space $C(I_0)$ is also the structure of Banach algebra. Fix a set $N \in C(I_0)$ and $u \in N$, given $\rho > 0$, the modulus of continuity of u is defined as

$$\omega(u, \rho) = \sup\{|u(\varphi, \zeta) - u(\hat{\varphi}, \hat{\zeta})| : \varphi, \hat{\varphi} \in [0, c], \zeta, \hat{\zeta} \in [0, d], |\varphi - \hat{\varphi}| \leqslant \rho, |\zeta - \hat{\zeta}| \leqslant \rho\}.$$

Further, $\omega(N, \rho) = \sup\{\omega(u, \rho) : u \in N\}$, $\omega_0(N) = \lim_{\rho\to 0} \omega(N, \rho)$.

Theorem 2.4 ([24]). *The Hausdorff MNC is similar to*

$$\mu(N) = \lim_{\rho\to 0} \sup_{u\in N} \omega(u, \rho), \tag{2.1}$$

for all bounded sets $N \subset C(I_0)$.

Definition 2.5 ([30]). Assume $T : X \to X$ be a continuous mapping of X. Then T is called k set contraction, if $\forall$ $D \subset X$ with D and $T(D)$ bounded, and $\hat{\beta}(TD) \leqslant k\hat{\beta}(D), k \in (0, 1)$. If $\hat{\beta}(TD) < \hat{\beta}(D), \forall \hat{\beta}(D) > 0$, then T is called densifying or condensing mapping.

Note that a k-set contraction is condensing but converse need not to be true.

Theorem 2.6 ([33]). *Suppose that $T : B_r \to X$ is a condensing mapping, which fulfills the following boundary condition: if $T(u) = ku$, for some $u \in \partial B_r$, then $k \leqslant 1$. Then the set of fixed points of T in B_r is non-empty.*

3. Main results

We study the Eq. (1.1) with the following assumptions.

(1) $f, q \in C(I_1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), P \in C(I_0 \times I_c \times \mathbb{R}, \mathbb{R}), H, V \in C(I_2 \times \mathbb{R}, \mathbb{R})$, where $I_0 = [0, c] \times [0, d]$, $I_1 = \{(\varphi, \zeta, u) : 0 \leqslant \varphi \leqslant c, 0 \leqslant \zeta \leqslant d, u \in \mathbb{R}\}$, $I_2 = \{(\varphi, \zeta, z, v) \in I_0^2 : 0 \leqslant z \leqslant \varphi \leqslant c, 0 \leqslant v \leqslant \zeta \leqslant d\}$, $\alpha, \theta, \eta : I_0 \to I_0$, $\gamma, \delta : I_0 \to I_0$.

(2) There are non-negative constants $k_{\hat{j}}$ $(\hat{j} = 1, 2, \dots, 6), k_1, k_4 + k_5 < 1$ such that

$$|f(\varphi, \zeta, u_1, u_2, u_3) - f(\varphi, \zeta, v_1, v_2, v_3)| \leqslant k_1|u_1 - v_1| + k_2|u_2 - v_2| + k_3|u_3 - v_3|,$$
$$|q(\varphi, \zeta, u_1, u_2, u_3) - q(\varphi, \zeta, u_1, u_2, u_3)| \leqslant k_4|u_1 - v_1| + k_5|u_2 - v_2| + k_6|u_3 - v_3|,$$

for all $(\varphi, \zeta) \in I_0$ and $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}$.

(3) There is a $r > 0$ such that the resulting bounded condition holds $\sup \Big(\{|(f) \times (q)|\}\Big) \leqslant r$, where

$$\begin{aligned}
\sup f &= \sup\{|f(\varphi, \zeta, u_1, u_2, u_3)| : \text{ for all } (\varphi, \zeta) \in I_0, -r \leqslant u_1, \\
&\qquad r - cN_1 \leqslant u_2 \leqslant cN_1, \text{ and } -cdN_2 \leqslant u_3 \leqslant cdN_2\}, \\
N_1 &= \sup\{|P(\varphi, \zeta, y, u(\beta(y, \zeta)))| : \text{ for all } (\varphi, \zeta) \in I_0, y \in I_c, \text{ and } u \in [-r, r]\}, \\
N_2 &= \sup\{|H(\varphi, \zeta, z, v, u(\gamma(z, v)))| : \text{ for all } (\varphi, \zeta, z, v) \in I_2, \text{ and } u \in [-r, r]\}, \\
\sup q &= \sup\{|G(\varphi, \zeta, u_1, u_2, u_3)| : \text{ for all } (\varphi, \zeta) \in I_0, -r \leqslant u_1, u_2 \leqslant r, \text{ and } -cdN_3 \leqslant u_3 \leqslant cdN_3\}, \\
N_3 &= \sup\{|V(\varphi, \zeta, z, v, u(\delta(z, v)))| : \text{ for all } (\varphi, \zeta, z, v) \in I_2, \text{ and } u \in [-r, r]\}.
\end{aligned}$$

Theorem 3.1. *By assumptions* (1)-(3), *Eq.* (1.1) *has at least one solution in* $X = C(I_0)$.

Proof. Let $L, S : B_r \to X$ be the operators defined as

$$(Lu)(\varphi,\zeta) = f\left(\varphi,\zeta,u(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\right),$$

$$(Su)(\varphi,\zeta) = q\left(\varphi,\zeta,u(\varphi,\zeta),u(\eta(\varphi,\zeta)),\int_0^{c}\int_0^{d} V(\varphi,\zeta,z,\nu,u(\delta(z,\nu)))d\nu dz\right),$$

for $(\varphi,\zeta) \in I_0$. Further, define T on the X by setting $Tu = (Lu)(Su)$.

Step 1. In order to show that T is continuous on B_r, let $\rho > 0$ and any $u, w \in B_r$ such that $|u - w| < \rho$. Then,

$$\begin{aligned}
&|(Lu)(\varphi,\zeta) - (Lw)(\varphi,\zeta)| \\
&\quad = \Big|f\Big(\varphi,\zeta,u(\varphi,\zeta),\int_0^{\zeta} P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Big) \\
&\qquad - f\Big(\varphi,\zeta,w(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,w(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,w(\gamma(z,\nu)))d\nu dz\Big)\Big| \\
&\quad \leqslant \Big|f\Big(\varphi,\zeta,u(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Big) \\
&\qquad - f\Big(\varphi,\zeta,w(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Big)\Big| \\
&\qquad + \Big|f\Big(\varphi,\zeta,w(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Big) \\
&\qquad - f\Big(\varphi,\zeta,w(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,w(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Big)\Big| \\
&\qquad + \Big|f\Big(\varphi,\zeta,w(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,w(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Big) \\
&\qquad - f\Big(\kappa,t,w(\kappa,t),\int_0^{\kappa} P(\kappa,t,y,w(\beta(y,t)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,w(\gamma(z,\nu)))d\nu dz\Big)\Big| \\
&\quad \leqslant k_1|u(\varphi,\zeta) - w(\varphi,\zeta)| + k_2\int_0^{\varphi} |P(\varphi,\zeta,y,u(\beta(y,\zeta))) - P(\varphi,\zeta,y,w(\beta(y,\zeta)))|dy \\
&\qquad + k_3\int_0^{\varphi}\int_0^{\zeta} |H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu))) - H(\varphi,\zeta,z,\nu,w(\gamma(z,\nu)))|d\nu dz \\
&\quad \leqslant (k_1)\|u - w\| + k_2 c\omega(P,\rho) + k_3 cd\omega(H,\rho),
\end{aligned}$$

and similarly, conclude that

$$\begin{aligned}
|(Su)(\varphi,\zeta) - (Sw)(\varphi,\zeta)| &= \Big|q\Big(\varphi,\zeta,u(\varphi,\zeta),u(\eta(\varphi,\zeta)),\int_0^{c}\int_0^{d} V(\varphi,\zeta,z,\nu,u(\delta(z,\nu)))d\nu dz\Big) \\
&\qquad - q\Big(\varphi,\zeta,w(\varphi,\zeta),w(\eta(\varphi,\zeta)),\int_0^{c}\int_0^{d} V(\varphi,\zeta,z,\nu,w(\delta(z,\nu)))d\nu dz\Big)\Big| \\
&\leqslant k_4|u(\varphi,\zeta) - w(\varphi,\zeta)| + k_5|u(\eta(\varphi,\zeta)) - w(\eta(\varphi,\zeta))| \\
&\qquad + k_6\int_0^{c}\int_0^{d} |V(\varphi,\zeta,z,\nu,u(\delta(z,\nu))) - V(\varphi,\zeta,z,\nu,w(\delta(z,\nu)))|d\nu dz
\end{aligned}$$

$$\leqslant (k_4 + k_5)\|u - w\| + k_6 c d \omega(V, \rho),$$

where

$$\begin{aligned}
\omega(P, \rho) &= \sup\{|P(\varphi, \zeta, \nu, u) - P(\varphi, \zeta, \nu, w)| : (\varphi, \zeta) \in I_0, \nu \in I_c, u, w \in [-r, r], |u - w| \leqslant \rho\},\\
\omega(H, \rho) &= \sup\{|H(\varphi, \zeta, z, \nu, u) - H(\varphi, \zeta, z, \nu, w)| : (\varphi, \zeta, z, \nu) \in I_2, u, w \in [-r, r], |u - w| \leqslant \rho\},\\
\omega(V, \rho) &= \sup\{|V(\varphi, \zeta, z, \nu, u) - V(\varphi, \zeta, z, \nu, w)| : (\varphi, \zeta, z, \nu) \in I_2, u, w \in [-r, r], |u - w| \leqslant \rho\}.
\end{aligned}$$

Since, $P = P(\varphi, \zeta, y, u)$, $H = H(\varphi, \zeta, z, \nu, u)$, and $V = V(\varphi, \zeta, z, \nu, u)$ are uniformly continuous on $I_0 \times I_c \times [-r, r]$, $I_2 \times [-r, r]$ and $I_2 \times [-r, r]$, respectively, therefore $\omega(P, \rho)$, $\omega(H, \rho)$ and $\omega(V, \rho) \to 0$ as $\rho \to 0$. From above fact we prove that L and S are continuous on B_r. Hence, T is also continuous on B_r.

Step 2. It is to prove that the operators L and S satisfy the condensing condition with μ. So, let $\rho > 0$ be a fixed arbitrary and $u \in N$, where N is bounded subset of $X, (\varphi_1, \zeta_1), (\varphi_2, \zeta_2) \in I_0$ with $\varphi_1 \leqslant \varphi_2, \zeta_1 \leqslant \zeta_2$ and $\varphi_2 - \varphi_1 \leqslant \rho, \zeta_2 - \zeta_1 \leqslant \rho$. Thus,

$$\begin{aligned}
&|(Lu)(\varphi_2, \zeta_2) - (Lu)(\varphi_1, \zeta_1)|\\
&= \Big| f\Big(\varphi_2, \zeta_2, u(\varphi_2, \zeta_2), \int_0^{\varphi_2} P(\varphi_2, \zeta_2, y, u(\beta(y, \zeta_2)))dy, \int_0^{\varphi_2}\int_0^{\zeta_2} H(\varphi_2, \zeta_2, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\\
&\quad - f\Big(\varphi_1, \zeta_1, u(\varphi_1, \zeta_1), \int_0^{\varphi_1} P(\varphi_1, \zeta_1, y, u(\beta(y, \zeta_1)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1 z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\Big|\\
&\leqslant \Big| f\Big(\varphi_2, \zeta_2, u(\varphi_2, \zeta_2), \int_0^{\varphi_2} P(\varphi_2, \zeta_2, y, u(\beta(y, \zeta_2)))dy, \int_0^{\varphi_2}\int_0^{\zeta_2} H(\varphi_2, \zeta_2, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\\
&\quad - f\Big(\varphi_2, \zeta_2, u(\varphi_2, \zeta_2), \int_0^{\varphi_2} P(\varphi_2, \zeta_2, y, u(\beta(y, \zeta_2)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\Big|\\
&\quad + \Big| f\Big(\varphi_2, \zeta_2, u(\varphi_2, \zeta_2), \int_0^{\varphi_2} P(\varphi_2, \zeta_2, y, u(\beta(y, \zeta_2)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\\
&\quad - f\Big(\varphi_2, \zeta_2, u(\varphi_2, \zeta_2), \int_0^{\varphi_1} P(\varphi_1, \zeta_1, y, u(\beta(y, \zeta_1)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\Big|\\
&\quad + \Big| f\Big(\varphi_2, \zeta_2, u(\varphi_2, \zeta_2), \int_0^{\varphi_1} P(\varphi_1, \zeta_1, y, u(\beta(y, \zeta_1)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\\
&\quad - f\Big(\varphi_2, \zeta_2, u(\varphi_1, \zeta_1), \int_0^{\varphi_1} P(\varphi_1, \zeta_1, y, u(\beta(y, \zeta_1)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\Big|\\
&\quad + \Big| f\Big(\varphi_2, \zeta_2, u(\varphi_1, \zeta_1), \int_0^{\varphi_1} P(\varphi_1, \zeta_1, y, u(\beta(y, \zeta_1)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\\
&\quad - f\Big(\varphi_1, \zeta_1, u(\varphi_1, \zeta_1), \int_0^{\varphi_1} P(\varphi_1, \zeta_1, y, u(\beta(y, \zeta_1)))dy, \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big)\Big|\\
&\leqslant \omega_P(I_0, \rho) + k_3\Big|\int_0^{\varphi_2}\int_0^{\zeta_2} H(\varphi_2, \zeta_2, z, \nu, u(\gamma(z, \nu)))d\nu dz - \int_0^{\varphi_1}\int_0^{\zeta_1} H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))d\nu dz\Big|\\
&\quad + k_2\Big|\int_0^{\varphi_2} P(\varphi_2, \zeta_2, y, u(\beta(y, \zeta_2)))dy - \int_0^{\varphi_1} P(\varphi_1, \zeta_1, y, u(\beta(y, \zeta_1)))dy\Big|\\
&\quad + k_1|u(\varphi_2, \zeta_2) - u(\varphi_1, \zeta_1)| + \omega_F(I_0, \rho)\\
&\leqslant k_1\omega(u, \rho) + \omega_P(I_0, \rho) + \omega_F(I_0, \rho) + k_3\int_0^{\varphi_1}\int_0^{\zeta_1} |H(\varphi_2, \zeta_2, z, \nu, u(\gamma(z, \nu))) - H(\varphi_1, \zeta_1, z, \nu, u(\gamma(z, \nu)))|d\nu dz\\
&\quad + k_3\int_{\varphi_1}^{\varphi_2}\int_0^{\zeta_1} |H(\varphi_2, \zeta_2, z, \nu, u(\gamma(z, \nu)))|d\nu dz + k_3\int_0^{\varphi_1}\int_{\zeta_1}^{\zeta_2} |H(\varphi_2, \zeta_2, z, \nu, u(\gamma(z, \nu)))|d\nu dz
\end{aligned}$$

$$+\int_{\varphi_1}^{\varphi_2}\int_{\zeta_1}^{\zeta_2}|H(\varphi_2,\zeta_2,z,\nu,u(\gamma(z,\nu)))|d\nu dz$$
$$+k_2\int_0^{\varphi_1}|P(\varphi_2,\zeta_2,y,u(\beta(y,\zeta_2)))dy-P(\varphi_1,\zeta_1,y,u(\beta(y,\zeta_1)))|dy+k_2\int_{\zeta_1}^{\zeta_2}|P(\varphi_2,\zeta_2,y,u(\beta(y,\zeta_2)))|dy.$$

Again,

$$\begin{aligned}
&|(Su)(\varphi_2,\zeta_2)-(Su)(\varphi_1,\zeta_1)|\\
&=\Bigg|q\left(\varphi_2,\zeta_2,u(\varphi_2,\zeta_2),u(\eta(\varphi_2,\zeta_2)),\int_0^c\int_0^d V(\varphi_2,\zeta_2,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\\
&\quad-q\left(\varphi_1,\zeta_1,u(\varphi_1,\zeta_1),u(\eta(\varphi_1,\zeta_1)),\int_0^c\int_0^d V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\Bigg|\\
&\leqslant\Bigg|q\left(\varphi_2,\zeta_2,u(\varphi_2,\zeta_2),u(\eta(\varphi_2,\zeta_2)),\int_0^c\int_0^d V(\varphi_2,\zeta_2,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\\
&\quad-q\left(\varphi_2,\zeta_2,u(\varphi_2,\zeta_2),u(\eta(\varphi_2,\zeta_2)),\int_0^c\int_0^d V(\varphi_1,\zeta_1 z,\nu,u(\delta(z,\nu)))d\nu dz\right)\Bigg|\\
&\quad+\Bigg|q\left(\varphi_2,\zeta_2,u(\varphi_2,\zeta_2),u(\eta(\varphi_2,\zeta_2)),\int_0^c\int_0^d V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\\
&\quad-q\left(\varphi_2,\zeta_2,u(\varphi_2,\zeta_2),u(\eta(\varphi_1,\zeta_1)),\int_0^c\int_0^d V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\Bigg|\\
&\quad+\Bigg|q\left(\varphi_2,\zeta_2,u(\varphi_2,\zeta_2),u(\eta(\varphi_1,\zeta_1)),\int_0^c\int_0^d V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\\
&\quad-q\left(\varphi_2,\zeta_2,u(\varphi_1,\zeta_1),u(\eta(\varphi_1,\zeta_1)),\int_0^c\int_0^d V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\Bigg|\\
&\quad+\Bigg|q\left(\varphi_2,\zeta_2,u(\varphi_1,\zeta_1),u(\eta(\varphi_1,\zeta_1)),\int_0^c\int_0^d V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\\
&\quad-q\left(\varphi_1,\zeta_1,u(\varphi_1,\zeta_1),u(\eta(\varphi_1,\zeta_1)),\int_0^c\int_0^d V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\Bigg|\\
&\leqslant\omega_q(I_0,\rho)+k_6\int_0^c\int_0^d|V(\varphi_2,\zeta_2,z,\nu,u(\delta(z,\nu)))-V(\varphi_1,\zeta_1,z,\nu,u(\delta(z,\nu)))|d\nu dz\\
&\quad+k_5|u(\eta(\varphi_2,\zeta_2))-u(\eta(\varphi_1,\zeta_1))|+k_4|u(\varphi_2,\zeta_2)-u(\varphi_1,\zeta_1)|+\omega_q(I_0,\rho)\\
&\leqslant k_6cd\omega_V(I_0,\rho)+k_5\omega(u,\omega(\eta,\rho))+k_4\omega(u,\rho)+\omega_q(I_0,\rho)+\omega_G(I_0,\rho),
\end{aligned}$$

where

$$\begin{aligned}
\omega_H(I_0,\rho)&=\sup\{|H(\varphi,\zeta,z,\nu,u)-H(\hat\varphi,\hat\zeta,z,\nu,u)|:|\varphi-\hat\varphi|,|\zeta-\hat\zeta|\leqslant\rho,\ (\varphi,\zeta,z,\nu)\in I_2,\ u\in[-r,r]\},\\
\omega_V(I_0,\rho)&=\sup\{|V(\varphi,\zeta,z,\nu,u)-V(\hat\varphi,\hat\zeta,z,\nu,u)|:|\varphi-\hat\varphi|,|\zeta-\hat\zeta|\leqslant\rho,\ (\varphi,\zeta,z,\nu)\in I_2,\ u\in[-r,r]\},\\
\omega_P(I_0,\rho)&=\sup\{|P(\varphi,\zeta,y,u)-P(\hat\varphi,\hat\zeta,y,u)|:|\varphi-\hat\varphi|,|\zeta-\hat\zeta|\leqslant\rho,\ (\varphi,\zeta)\in I_0,y\in I_c,\ u\in[-r,r]\},\\
\omega_F(I_0,\rho)&=\sup\{|f(\varphi,\zeta,u_1,u_2,u_3)-f(\hat\varphi,\hat\zeta,u_1,u_2,u_3)|:|\varphi-\hat\varphi|,|\zeta-\hat\zeta|\leqslant\rho,\ u_2\in[-cN_1,cN_1],\\
&\qquad u_3\in[-cdN_2,cdN_2],\ u_1\in[-r,r]\},\\
\omega_q(I_0,\rho)&=\sup\{|q(\varphi,\zeta,u_1,u_2,u_3)-q(\hat\varphi,\hat\zeta,u_1,u_2,u_3)|:|\varphi-\hat\varphi|,|\zeta-\hat\zeta|\leqslant\rho,
\end{aligned}$$

$$u_3 \in [-cdN_3, cdN_3],\ u_1, u_2 \in [-r, r]\}.$$

From above relations, we observe that

$$\begin{aligned}|(Lu)(\varphi_2,\zeta_2)-(Lu)(\varphi_1,\zeta_1)| &\leqslant k_3\omega(u,\rho)+\omega_P(I_0,\rho)+\omega_f(I_0,\rho)\\ &\quad + k_3cd\omega_H(I_0,\rho)+k_7\rho dN_2+k_7c\rho N_2+k_7\rho^2N_2+k_6c\omega_P(I_0,\rho)+k_6\rho N_1\\ |(Su)(\kappa_2,t_2)-(Su)(\kappa_1,t_1)| &\leqslant k_{10}cd\omega_V(I_0,\rho)+k_9\omega(N,\omega(\eta,\rho))+k_8\omega(N,\rho)+\omega_q(I_0,\rho)+\omega_G(I_0,\rho).\end{aligned}$$

Taking limit as $\rho \to 0$, note that

$$\mu(LN) \leqslant (k_5)\mu(N) \quad \text{and} \quad \mu(SN) \leqslant (k_8+k_9)\mu(N). \tag{3.1}$$

From Eqs. (3.1), T is a condensing map. Now, assume $u \in \partial B_r$ and if $Tu = ku$, then $\|Tu\| = k\|u\| = kr$ and by assumption (3), note that

$$\begin{aligned}\|Tu(\varphi,\zeta)\| = f\Bigg(&\varphi,\zeta,u(\varphi,\zeta),\int_0^{\varphi} P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Bigg)\\ &\times q\Bigg(\varphi,\zeta,u(\varphi,\zeta),u(\eta(\varphi,\zeta)),\int_0^{c}\int_0^{d} V(\varphi,\zeta,z,\nu,u(\delta(z,\nu)))d\nu dz\Bigg) \leqslant r,\end{aligned}$$

for all $(\varphi,\zeta) \in I_0$, thus $\|Tu\| \leqslant r$, i.e., $k \leqslant 1$. □

Corollary 3.2. *If* $f(\varphi,\zeta,u_1,u_2,u_3) = f(\varphi,\zeta,u_1,u_3)$ *and* $q(\varphi,\zeta,u_1,u_2,u_3) = q(\varphi,\zeta,u_1,u_3)$, *then Eq.* (1.1) *can be written as*

$$\begin{aligned}u(\varphi,\zeta) = f\Bigg(&\varphi,\zeta,u(\varphi,\zeta),\int_0^{\varphi}\int_0^{\zeta} H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\Bigg)\\ &\times q\Bigg(\varphi,\zeta,u(\varphi,\zeta),\int_0^{c}\int_0^{d} V(\varphi,\zeta,z,\nu,u(\delta(z,\nu)))d\nu dz\Bigg)\end{aligned}$$

and has at least one solution in $C(I_0)$, $I_0 = [0,c]\times[0,d]$.

Proof. The proof follows by the Theorem 3.1. □

4. Examples

This section deals with some examples of FIEs in order to illustrate the advantage of given results.

Example 4.1. Putting $f(\varphi,\zeta,u_1,u_2,u_3) = p(\varphi,\zeta)+\hat{k}(u_2+u_3)$ and $q(\varphi,\zeta,u_1,u_2,u_3) = 1$, then Eq. (1.1) can be written as

$$u(\varphi,\zeta) = p(\varphi,\zeta)+\int_0^{\varphi}\hat{k}P(\varphi,\zeta,y,u(\beta(y,\varphi)))dy+\int_0^{\varphi}\int_0^{\zeta}\hat{k}H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz. \tag{4.1}$$

The Eq. (4.1) is studied by many authors, one can see [5, 32].

Example 4.2. In the above Eq. (4.1), if we have $\hat{k} = \frac{1}{3}$, $p(\varphi,\zeta) = \frac{2}{3}e^{\frac{-2}{\varphi^2+\zeta^3+1}}$, $u_2 = 0$, and $H(\varphi,\zeta,z,\nu,u) = \frac{\varphi^2+\zeta+1}{3}z^2\nu^{\frac{3}{2}}\cos(z\nu+u)$, Eq. (1.1) converts to the following equation

$$u(\varphi,\zeta) = \frac{2}{3}e^{\frac{-2}{\varphi^2+\zeta^3+1}}+\frac{\varphi^2+\zeta+1}{9}\int_0^{\varphi}\int_0^{\zeta} z^2\nu^{\frac{3}{2}}\cos(z\nu+u)d\nu dz,$$

where $(\varphi,\zeta) \in [0,1]\times[0,1] = I$. Clearly, assumptions (1) and (2) hold. Now, it is to show that assumption (3) also holds. Choose $r = 1$, then

$$\begin{aligned}&\sup\{|p(\varphi,\zeta)| : (\varphi,\zeta)\in I\}+\sup\{|f(\varphi,\zeta,u_1,u_2,u_3)| : (\varphi,\zeta)\in I, u_1,u_3\in[-1,1]\}\\ &\leqslant \sup\{\Big|\frac{2}{3}e^{\frac{-2}{\varphi^2+\zeta^3+1}}\Big| : (\varphi,\zeta)\in I\}+\sup\{\frac{1}{3}|u_3| : u_3\in[-1,1]\}\leqslant 1.\end{aligned}$$

Example 4.3. Let the following nonlinear FIE:

$$\begin{aligned} u(\varphi,\zeta) = &\left[\frac{\varphi^2}{8(1+\varphi^2\zeta^2)}e^{-\varphi\zeta} + \frac{\varphi^2\zeta^2}{2}\arctan|u(\varphi,\zeta)|\right. \\ &\left. + \frac{1}{4}\int_0^{\varphi}\int_0^{\zeta}\left(\frac{\varphi\zeta\cos(u(z,\nu))}{2} + \ln(1+|u(z,\nu)|)\right)d\nu dz\right] \\ &\times\left[\frac{1}{2}\int_0^1\int_0^1\left(\sin(u(z,\nu)) + \frac{\varphi^2\zeta^2}{2}\arctan\left(\frac{|u(z,\nu)|}{4+|u(z,\nu)|}\right)\right)d\nu dz\right], \end{aligned} \tag{4.2}$$

for $(\varphi,\zeta) \in I = [0,1]\times[0,1]$. Here,

$$\begin{aligned} f(\varphi,\zeta,u_1,u_2,u_3) &= \frac{\varphi^2}{8(1+\varphi^2\zeta^2)}e^{-\varphi\zeta} + \frac{\varphi^2\zeta^2}{2}u_1 + 0u_2 + \frac{1}{4}u_3, \\ q(\varphi,\zeta,u_1,u_2,u_3) &= \frac{1}{2}u_3, \\ H(\varphi,\zeta,z,\nu,u) &= \frac{\varphi\zeta\sin(u(z,\nu))}{4} + \ln(1+|u(z,\nu)|), \\ V(\varphi,\zeta,z,\nu,u) &= \cos(u(z,\nu)) + \frac{\varphi^6\zeta^4}{4}\cos\left(\frac{|u(z,\nu)|}{8+|u(z,\nu)|}\right). \end{aligned}$$

Moreover, for all $(\kappa,t) \in I$, we have

$$\begin{aligned} |f(\varphi,\zeta,u_1,u_2,u_3) - f(\varphi,\zeta,v_1,v_2,v_3)| &\leqslant \frac{1}{2}|u_1-v_1| + 0|u_2-v_2| + \frac{1}{4}|u_3-v_3|, \\ |q(\varphi,\zeta,u_1,u_2,u_3) - q(\varphi,\zeta,v_1,v_2,v_3)| &\leqslant \frac{1}{2}|u_3-v_3|. \end{aligned}$$

Now observe that these functions satisfy assumptions (1) and (2). It is now remains to show that (3) also holds. Choose $r=\frac{1}{4}$, then $k_1=\frac{1}{2}$, $k_2=0$, $k_3=\frac{1}{4}$, $k_4=k_5=0$, $k_6=\frac{1}{2}$,

$$\begin{aligned} \sup\Bigg| &f\left(\varphi,\zeta,u(\varphi,\zeta),\int_0^{\varphi}P(\varphi,\zeta,y,u(\beta(y,\zeta)))dy,\int_0^{\varphi}\int_0^{\zeta}H(\varphi,\zeta,z,\nu,u(\gamma(z,\nu)))d\nu dz\right) \\ &\times q\left(\varphi,\zeta,u(\varphi,\zeta),u(\eta(\varphi,\zeta)),\int_0^c\int_0^d V(\varphi,\zeta,z,\nu,u(\delta(z,\nu)))d\nu dz\right)\Bigg| \leqslant \frac{1}{4}. \end{aligned}$$

Thus, Eq. (4.2) has at least one solution in $C(I)$ by using Theorem 3.1.

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References

[1] J. Banaś, K. Goebel, *Measures of noncompactness in Banach spaces*, Marcel Dekker, New York, (1980). 1
[2] J. Banaś, M. Lecko, *Fixed points of the product of operators in Banach algebra*, PanAmer. Math. J., **12** (2002), 101–109.
[3] J. Caballero, A. B. Mingarelli, K. Sadarangani, *Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer*, Electron. J. Differ. Equ., **2006** (2006), 11 pages. 1
[4] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London, (1950). 1, 1
[5] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, Cambridge, (1991). 1, 1, 4.1

[6] A. Das, B. Hazarika, P. Kumam, *Some new generalization of Darbo's fixed point theorem and its application on integral quations*, Mathematics, **7** (2019), 9 pages. 1, 1
[7] A. Das, B. Hazarika, H. M. Srivastava, M. Rabbani, R. Arab, *Solvability of infinite systems of nonlinear integral equations in two variables by using semi-analytic method*, Filomat, **33** (2019), 5377–5388. 1
[8] A. Deep, Deepmala, R. Ezzati, *Application of Petryshyn's fixed point theorem to solvability for functional integral equations*, Appl. Math. Comput., **395** (2021), 9 pages. 1
[9] A. Deep, Deepmala, M. Rabbani, *A numerical method for solvability of some non-linear functional integral equations*, Appl. Math. Comput., **402** (2021), 12 pages. 1
[10] A. Deep, Deepmala, J. R. Roshan, *Solvability for generalized nonlinear functional integral equations in Banach spaces with applications*, J. Integral Equ. Appl., **33** (2021), 19–30. 1
[11] A. Deep, D. Dhiman, S. Abbas, B. Hazarika, *Solvability for two dimensional functional integral equations via Petryshyn's fixed point theorem*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **115** (2021), 17 pages. 1
[12] A. Deep, A. Kumar, S. Abbas, B. Hazarika, *An existence result for functional integral equations via Petryshyn's fixed point theorem*, J. Int. Equ. Appl., **34** (2022), 165–181.
[13] A. Deep, D. Saini, H. K. Singh, Ü Çakan, *Solvability for fractional integral equations via Petryshyn's fixed-point theorem*, J. Int. Equ. Appl., **35** (2023), 277–289.
[14] A. Deep, M. Kazemi, *Solvability for 2D non-linear fractional integral equations by Petryshyn's fixed point theorem*, J. Comput. Appl. Math., **444** (2024), 11 pages. 1
[15] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, (1985). 1, 1
[16] L. S. Gol'denšteĭn, A. S. Markus, *On the measure of non-compactness of bounded sets and of linear operators*, Izdat. "Karta Moldovenjaske", Kishinev, (1965). 2.2
[17] G. Gripenberg, *On some epidemic models*, Quart. Appl. Math., **39** (1981/82), 317–327. 1
[18] B. Hazarika, R. Arab, H. K. Nashine, *Applications of measure of non-compactness and modified simulation function for solvability of nonlinear functional integral equations*, Filomat, **33** (2019), 5427–5439. 1
[19] B. Hazarika, H. M. Srivastava, R. Arab, M. Rabbani, *Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction of an iteration algorithm to find solution*, Appl. Math. Comput., **360** (2019), 131–146. 1
[20] S. Hu, M. Khavanin, W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal., **34** (1989), 261–266. 1
[21] M. Kazemi, H. Chaudhary, A. Deep, *Existence and approximate solutions for Hadamard fractional integral equations in a Banach space*, J. Integral Equ. Appl., **35** (2023), 27–40. 1
[22] M. Kazemi, A. Deep, J. Nieto, *An existence result with numerical solution of nonlinear fractional integral equations*, Math. Methods Appl. Sci., **46** (2023), 10384–10399. 1
[23] M. Kazemi, A. Deep, A. Yaghoobnia, *Application of fixed point theorem on the study of the existence of solutions in some fractional stochastic functional integral equations*, Math. Sci. (Springer), **18** (2024), 125–136. 1
[24] M. Kazemi, R. Ezzati, *Existence of solution for some nonlinear two-dimensional Volterra integral equations via measures of noncompactness*, Appl. Math. Comput., **275** (2016), 165–171. 1, 1, 1, 2.4
[25] M. Kazemi, R. Ezzati, *Existence of solutions for some nonlinear Volterra integral equations via Petryshyn's fixed point theorem*, Int. J. Nonlinear Anal. Appl., **9** (2018), 1–12. 1
[26] M. Kazemi, R. Ezzati, A. Deep, *On the solvability of non-linear fractional integral equations of product type*, J. Pseudo-Differ. Oper. Appl., **14** (2023), 18 pages. 1
[27] C. T. Kelly, *Approximation of solutions of some quadratic integral equations in transport theory*, J. Integral Equ., **4** (1982), 221–237. 1
[28] R. Kumar, S. Kumar, M. Sajid, B. Singh, *On solvability for some classes of system of non-linear integral equations in two dimensions via measure of non-compactness*, Axioms, **11** (2022), 13 pages. 1
[29] K. Kuratowski, *Sur les espaces complets*, Fund. Math., **15** (1934), 301–309. 2.1
[30] R. D. Nussbaum, *THE FIXED POINT INDEX AND FIXED POINT THEOREMS FOR K-SET-CONTRACTIONS*, Thesis (Ph.D.)–The University of Chicago, ProQuest LLC, Ann Arbor, MI, (1969). 2.5
[31] İ. Özdemir, Ü. Çakan, *The solvability of some nonlinear functional integral equations*, Studia Sci. Math. Hunger., **53** (2016), 7–21. 1
[32] B. G. Pachpatte, *Multidimensional integral equations and inequalities*, Atlantis Press, Paris, (2011). 1, 4.1
[33] W. V. Petryshyn, *Structure of the fixed points sets of* k-*set-contractions*, Arch. Rational Mech. Anal., **40** (1970/1971), 312–328. 2.6
[34] M. Rabbani, A. Deep, Deepmala, *On some generalized non-linear functional integral equations of two variables via measures of noncompactness and numerical method to solve it*, Math. Sci. (Springer), **15** (2021), 317–324. 1
[35] P. Saini, Ü. Çakan, A. Deep, *Existence of solutions for 2D nonlinear fractional Volterra integral equations in Banach space*, Rocky Mountain J. Math., **53** (2023), 1965–1981. 1
[36] S. Singh, B. Watson, P. Srivastava, *Fixed point theory and best approximation: the KKM-map principle*, Kluwer Academic Publishers, Dordrecht, (1997). 1