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KANNAN FIXED POINT THEOREM ON GENERALIZED METRIC SPACES

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ABSTRACT. We obtain sufficient conditions for existence of unique fixed point of Kannan type mappings defined on a generalized metric space.

1. INTRODUCTION AND PRELIMINARIES

The fixed point theorem most frequently cited in literature is Banach contraction mapping principle (see [2]), which asserts that if X is a complete metric space and $T : X \to X$ is a contractive mapping i.e., there exists $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \lambda d(x, y). \tag{1}$$

Then T has a unique fixed point. The contractive definition (1) implies that T is uniformly continuous. It is natural to ask if there is a contractive definition which do not force T to be continuous. It was answered in affirmative by Kannan [3], who established a fixed point theorem for mappings satisfying:

$$d(Tx, Ty) \leq \lambda\,[d(x, Tx) + d(y, Ty)] \tag{2}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$.

Kannan's paper [3] was followed by a spate of papers containing a variety of contractive definitions in metric spaces. Rhoades [4] considered 250 type of contractive definitions and analyzed the relationship among them.

Recently Branciari [1] introduced a class of generalized metric spaces by replacing triangular inequality by similar ones which involve four or more points instead

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of three and improved Banach contraction mapping principle. In the present paper we continue this investigation for the mappings introduced by Kannan [3].

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{R}$, satisfies:

(1) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \leq d(x, w)+ d(w, z)+ d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$[rectangular property].

Then d is called a *generalized metric* and (X, d) is a *generalized metric space.* Let x_n be a sequence in X and $x \in X$. If for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$,for all $n > n_0$ then $\{x_n\}$ is said to be *convergent*, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$ .We denote this by $\lim_n x_n = x$, or $x_n \longrightarrow x$, as $n \to \infty$. If for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) < \epsilon$ for all $n > n_0$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is called a *complete generalized metric space.*

Let us remark [1] that

(i) $d(a_n, y) \to d(a, y)$ and $d(x, a_n) \to d(x, a)$ whenever a_n is a sequence in X with $a_n \to a \in X$
(ii) X becomes a Hausdorff topological space with neighborhood basis given by:
$$B = \{B(x, r) : x \in X, r \in (0, \infty)\},$$
where,
$$B(x, r) = \{y \in X : \ d(x, y) < r\}.$$

Example 1.2. [1] Let $X = \mathbb{R}$ and $0 \neq \alpha \in \mathbb{R}$. Define $d : X \times X \to \mathbb{R}$ as follow:

$$d(x,y) = \begin{cases} 0 & if\ x = y, \\ 3\alpha & \text{if } x \text{ and } y \text{ are in } \{1,2\},\ x \neq y. \\ \alpha & \text{if } x \text{ and } y \text{ can not both at a time in } \{1,2\},\ x \neq y. \end{cases}$$

Then it is easy to see that (X, d) is a generalized metric space but (X, d) is not a standard metric space because it lacks the triangular property:

$$3\alpha = \ d(1, 2) > \ d(1, 3) + \ d(3, 2) = \alpha + \alpha.$$

2. Main Result

Theorem 2.1. *Let (X, d) be a complete generalized metric space, and the mapping $T : X \to X$ satisfies (2). Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. Let $x_1 = T(x_0)$, If $x_1 = x_0$ then $x_0 = T(x_0)$ this means x_0 is a fixed point of T and there is nothing to prove. Assume that $x_1 \neq x_0$, let $x_2 = T(x_1)$. In this way we can define a sequence of points in X as follows:

$$x_{n+1} = Tx_n = T^{\ n+1}x_0, x_n \neq x_{n+1}\ n = 0, 1, 2, ...$$

Using the inequality (2), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \lambda \ [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\leq \lambda \ [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n). \end{aligned}$$

We can also suppose that x_0 is not a periodic point , in fact if $x_n = x_0$, then,

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) = d(T^{\ n}x_0, T^{\ n+1}x_0) \leq \frac{\lambda}{1-\lambda} d(T^{\ n-1}x_0, T^{\ n}x_0) \\ &\leq \left[\frac{\lambda}{1-\lambda}\right]^2 d(T^{\ n-2}x_0, T^{\ n-1}x_0) \leq ... \leq \left[\frac{\lambda}{1-\lambda}\right]^n d(x_0, Tx_0). \end{aligned}$$

Put $h = \left[\frac{\lambda}{1-\lambda}\right]$, then $h < 1$ and

$$[1 - h^n] \ d(x_0, Tx_0) \leq 0.$$

It follows that x_0 is a fixed point of T.Thus in the sequel of proof we can suppose $T^{\ n}x_0 \neq x_0$ for $n = 1, 2, 3, ...$ Now inequality (2) impies that

$$\begin{aligned} d(T^{\ n}x_0, T^{\ n+m}x_0) &\leq \lambda \left[d(T^{\ n-1}x_0, T^{\ n}x_0) + d(T^{\ n+m-1}x_0, T^{\ n+m}x_0)\right]. \\ & \lambda \left[h^{\ n-1}d(x_0, Tx_0) + h^{\ n+m-1}d(x_0, Tx_0)\right]. \end{aligned}$$

Therefore, $d(x_n, x_{n+m}) \to 0$ as $n \to \infty$. It implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a $u \in X$ such that $x_n \to u$. By rectangular property we have

$$\begin{aligned} d(Tu, u) &\leq d(Tu, T^{\ n}x_0) + d(T^{\ n}x_0, T^{\ n+1}x_0) + d(T^{\ n+1}x_0, u) \\ &\leq \lambda \left[d(u, Tu) + d(T^{\ n-1}x_0, T^{\ n}x_0)\right] + h^n d(x_0, Tx_0) + d(T^{\ n+1}x_0, u) \\ &\leq hd(T^{\ n-1}x_0, T^{\ n}x_0) + \frac{h^n}{1-\lambda} d(x_0, Tx_0) + \frac{1}{1-\lambda} d(T^{\ n+1}x_0, u) \\ &\leq h^{\ n}d(x_0, Tx_0) + \frac{h^n}{1-\lambda} d(x_0, Tx_0) + \frac{1}{1-\lambda} d(T^{\ n+1}x_0, u). \end{aligned}$$

Letting $n \to \infty$ and using the fact that, $d(a_n, y) \to d(a, y)$ and $d(x, a_n) \to d(x, a)$ whenever a_n is a sequence in X with $a_n \to a \in X$, we have $u = Tu$. Now we show that T has a unique fixed point . For this, assume that there exists another point v in X such that $v = Tv$. Now,

$$\begin{aligned} d(v, u) &= d(Tv, Tu) \\ &\leq \lambda \ d(v, Tv) + d(u, Tu) \\ &\leq \lambda \ d(v, v) + d(u, u) = 0. \end{aligned}$$

Hence, $u = v$. □

Example 2.2. Let $X = \{1, 2, 3, 4\}$. Define $d : X \times X \to \mathbb{R}$ as follows:

$$\begin{aligned} d(1, 2) &= d(2, 1) = 3 \\ d(2, 3) &= d(3, 2) = d(1, 3) = d(3, 1) = 1 \\ d(1, 4) &= d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 4. \end{aligned}$$

Then $(X, d\ )$ is a complete generalized metric space but $(X, d\ )$ is not a metric space because it lacks the triangular property:

$$3 = d(1,2) > d(1,3) + d(3,2) = 1 + 1 = 2.$$

Now define a mapping $T : X \to X$ as follows:

$$Tx = \begin{cases} 3 & if\ x \neq 4, \\ 1 & if\ x = 4. \end{cases}$$

Note that

$$d(T(1), T(2)) = d(T(1), T(3)) = d(T(2), T(3)) = 0$$

and in all other cases

$$d(Tx, Ty) = 1,\ [d(x, Tx) + d(y, Ty)] \geq 4.$$

Hence, for $\lambda\ = \frac{1}{3}$, all conditions of Theorem 3 are satisfied to obtain a unique fixed point 3 of T .

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