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A NOTE ON D_{11}-MODULES

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Abstract. Let M be a right R-module. M is called $D_{11}-$module if every submodule of M has a supplement which is a direct summand of M and M is called a $D_{11}^{+}-$ module if every direct summand of M is a $D_{11}-$ module. In this paper we study some properties of D_{11} modules.

1. Introduction and preliminaries

Throughout this article , all rings are associative and have an identity, and all modules are unitary right $R-$modules. Let M be an $R-$ module. An $R-$ module N is said to be subgenerated by M, if N is isomorphic to a submodule of an $M-$ generated module. We denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are all $R-$ modules subgenerated by M (see [6]). The injective hull of any module $N \in \sigma[M]$ is denoted by $\hat{N}$. The module $N \in \sigma[M]$ is said to be $M-$ small if $N \ll \hat{N}$. Talebi and Vanaja in [4] defined:

$\overline{Z}_M(N) = Re(N, S) = \bigcap\{\ker(g) | g \in Hom(N, L), L \in S\}$ where S denoted the class of all $M-$small modules. They call N, $M-$cosingular if $\overline{Z}_M(N) = 0$ and N non-M-cosingular if $\overline{Z}_M(N) = N$. Clearly, every $M-$ small module in $\sigma[M]$ is $M-$cosingular. A submodule L is called small in M (denoted by $L \ll M$), if for every proper submodule K of M, $L + K = M$ implies $K = M$ see [1]. In [2] complement of a submodule which is direct summand studied, but in this note we show when supplement submodule is direct summand. For two submodules N and K of M, N is called a *supplement* of K in M if, N is minimal with the property $M = K + N$, equivalently $M = K + N$ and $N \cap L \ll N$. A module M is called *supplemented* if, every submodule of M has a supplement in M. A module M is called a D_{11} -module if every submodule of M has a supplement which is

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direct summand of M. M is called D_{11}^{+}- module if any direct summand of M is D_{11}. A module M is called amply supplemented if for any two submodules A and B of M with $M = A + B$,there exists a supplement P of A in M which is contained in B.

2. Main results

Definition 2.1. We call N lies above K iff $N/K \ll M/K$. A submodule N of M is coclosed in M iff N has no proper submodule K such that N lies above K.

Definition 2.2. A module M is called $FI-$lifting if every fully invariant submodule of M lies above a direct summand.

Theorem 2.3. *Let $M = M_1 \oplus M_2$, where M_1 is a fully invariant coclosed submodule of M. If the intersection of M_2 with any direct summand of M (such K_2)is a direct summand of K_2, then M has D_{11} if and only if both M_1 and M_2 have D_{11}.*

Proof. Assume that M_1 and M_2 have D_{11}, by [5, 2.5], any finite direct sum of modules with D_{11} is a $D_{11}-$module. So M is D_{11} -module. Now suppose that M has D_{11}. Since M_1 is a fully invariant coclosed submodule of M, by [5, 2.4], M_1 has D_{11}. Let $Y \leq M_2$. Since M has D_{11}, there exists a decomposition $M = K_1 \oplus K_2$ such that K_2 is a supplement of Y in M, that is, $K_2 + Y = M$ and $K_2 \cap Y \ll K_2$. Thus $M_2 = M_2 \cap M = M_2 \cap (K_2 + Y) = Y + (K_2 \cap M_2)$. And $(K_2 \cap M_2) \cap Y = K_2 \cap Y \ll K_2$. By assumption, $K_2 \cap M_2$ is a direct summand of K_2. So $K_2 \cap Y \ll K_2 \cap M_2$. Hence M_2 has D_{11}. □

Theorem 2.4. *Let M be a $D_{11}-$ module and X be a fully invariant coclosed submodule of M and $\bar{M} = M/X$.Then $\bar{M}$ has D_{11}.*

Proof. Let $\bar{A} \leq \bar{M}$. Then $\bar{A} = A/X$ for some $A \leq M_R$. Since M has D_{11}, there exists a decomposition $M = M_1 \oplus M_2$ such that M_1 is a supplement of A in M, that is, $A + M_1 = M$ and $A \cap M_1 \ll M_1$. Thus $M = M_1 \oplus M_2$, $M/X = (M_1 \oplus M_2)/X = (M_1 + X)/X \oplus (M_2 + X)/X$. Hence $(M_1 + X)/X$ is a direct summand of M/X. We have $A + M_1 = M$. It follows that $A/X + (M_1 + X)/X = M/X$. We claim that $((A \cap M_1) + X)/X \ll (M_1 + X)/X$. Let $B/X \subseteq (M_1 + X)/X$ for some $B \subseteq M_1 + X$, such that $((A \cap M_1) + X)/X + B/X = (M_1 + X)/X$. Then $((A \cap M_1) + B + X)/X = (M_1 + X)/X$. Hence $(A \cap M_1) + B = M_1 + X$. Since $(A \cap M_1) \ll M_1$, $(A \cap M_1) \ll M_1 + X$. So $B = M_1 + X$. Hence $B/X = (M_1 + X)/X$. Therefore $\bar{M}$ has D_{11}. □

3. D_{11}- Modules and $\bar{Z}^2(M)$

Let $N \in \sigma[M]$. Note that for every direct summand A of N, $\bar{Z}^2_M(A) = \bar{Z}^2_M(N) \cap A$, [4, 2.1(4)]. M is called amply supplemented if for any two submodules N and L of M with $N + L = M$, N contains a supplement of L in M. Also for each decomposition $N = N_1 \oplus N_2$ of N, we have that $\bar{Z}^2_M(N) = (\bar{Z}^2_M(N) \cap N_1) \oplus (\bar{Z}^2_M(N) \cap N_2)$.

Theorem 3.1. *Let $N \in \sigma[M]$ be an amply supplemented $D_{11}-$ module and X is a fully invariant coclosed submodule of N. Then $N = N_1 \oplus N_2$, where $X/N_1 \ll N/N_1$. Moreover:*
(i)$\bar{Z}^2_M(N_1)$ has D_{11} implies N_1 has D_{11}.
(ii)$\bar{Z}^2_M(N_2)$ has D_{11} implies N_2 has D_{11}.
(iii) $N_1 \leq N$; $(\bar{Z}^2_M(N) \leq N_1)$ implies both N_1 and N_2 have D_{11}.
(iv) $\bar{Z}^2_M(N) \leq N_2$ implies both N_1 and N_2 have D_{11}.

Proof. Since N has D_{11} there exists a decomposition $N = N_1 \oplus N_2$ such that, $X + N_2 = N$ and $X \cap N_2 \ll N_2$. Since X is a fully invariant coclosed submodule of N, $X \cap (N_1 \oplus N_2) = (X \cap N_1) \oplus (X \cap N_2)$. Then by [3, 2.3], $X = (X + N_1) \cap (X + N_2) = X + N_1$. Hence $N_1 \leq X$. Thus we have X lies above a direct summand of N_1. Therefore by definition N is $FI-$lifting. So $N = N_1 \oplus N_2$; $X/N_1 \ll N/N_1$.

(i) We prove first $\bar{Z}^2_M(N)$ is a direct summand of N. Since N has D_{11}, there exists a decomposition $N = K \oplus L$ such that $K + \bar{Z}^2_M(N) = N$ and $K \cap \bar{Z}^2_M(N) = \bar{Z}^2_M(K) \ll K$. Then $\bar{Z}^2_M(K)$ is $M-$small and so, $M-$cosingular. On the other hand, by [4, 3.4], $\bar{Z}^2_M(N)$ is a non-$M-$-cosingular submodule of N. So, by [4, 2.4], $\bar{Z}^2_M(K)$ is non-$M-$cosingular. Hence $\bar{Z}^2_M(K)$=0. Therefore $N = K + \bar{Z}^2_M(N) = K \oplus \bar{Z}^2_M(N)$. Now from Theorem 2.3, both $\bar{Z}^2_M(N)$ and K have D_{11}. As $N = N_1 \oplus N_2$, $\bar{Z}^2_M(N) = \bar{Z}^2_M(N_1) \oplus \bar{Z}^2_M(N_2)$. So $N = N_1 \oplus N_2 = \bar{Z}^2_M(N_1) \oplus T$. Then $N_1 = N_1 \cap N = N_1 \cap [\bar{Z}^2_M(N_1) \oplus T] = \bar{Z}^2_M(N_1) \oplus [N_1 \cap T]$. Hence $\bar{Z}^2_M(N_1)$ is a direct summand of N_1. Suppose that $N_1 = \bar{Z}^2_M(N_1) \oplus K_1$. Similarly, $N_2 = \bar{Z}^2_M(N_2) \oplus K_2$. Thus $N = N_1 \oplus N_2 = \bar{Z}^2_M(N_1) \oplus \bar{Z}^2_M(N_2) \oplus K_1 \oplus K_2 = \bar{Z}^2_M(N) \oplus K$. It follows that $K_1 \oplus K_2 \cong K$. Since K has D_{11}, then K_1 and K_2 have D_{11}. By assumption $\bar{Z}^2_M(N_1)$ has D_{11} and K_1 has D_{11}, hence by Theorem 2.3 $N_1 = \bar{Z}^2_M(N_1) \oplus K_1$ has D_{11}.

(ii)It is similar to part (i).

(iii) It follows from Theorem 2.3.

(iv) $\bar{Z}^2_M(N) \subseteq N_2$ implies that $N_1 \cap \bar{Z}^2_M(N) = \bar{Z}^2_M(N_1)$=0. So from $\bar{Z}^2_M(N) = \bar{Z}^2_M(N_1) \oplus \bar{Z}^2_M(N_2)$, we obtain that $\bar{Z}^2_M(N) = \bar{Z}^2_M(N_2)$. Hence $\bar{Z}^2_M(N_1)$=0 has D_{11} and $\bar{Z}^2_M(N_2)$ has D_{11}. By parts (i) and (ii) N_1 and N_2 are $D_{11}-$modules. □

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