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Separating hyperplane theorems in convex metric spaces

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Abstract

The purpose of this note is to establish a theorem akin to Mazur's theorem concerning separating hyperplanes within convex metric spaces.

Keywords: Hahn Banach theorem, Mazur theorem, separating hyperplane, convexity.

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1. Introduction

One of the most profound and widely applicable results in analysis is the renowned Hahn-Banach extension theorem. Its geometric counterpart, the hyperplane separation theorem, adds even more utility from an applications standpoint. The study of convexity is crucial for tackling extremum problems, further underscoring the significance of these theorems. For comprehensive exploration, we direct readers to [1, 7]. Pioneering contributions by Klee [6], Bair [2], van de Vel [10], and Chepoi [5] have delved into various facets of convexity and its practical implications.

Initiated by Takahashi [9], the conceptualization of convexity in metric spaces has opened new avenues for research. Subsequently, Shimizu and Takahashi [8] and Beg [3], have delved into the geometric properties of convex metric spaces, advancing our understanding in this domain. This paper is in continuation of these studies and here we study separating hyperplanes in convex metric spaces.

This paper is structured as follows. Section 2 provides the definition of convex metric spaces and introduces several additional notions essential for our proposed framework. In Section 3, we introduce and analyze novel concepts of separating hyperplanes, exploring their properties and implications. Finally, Section 4 offers concluding remarks to summarize the findings and contributions of this paper.

2. Preliminaries

This section is dedicated to establishing the notation relevant to convex metric spaces and discussing pertinent prior research that will serve as a foundation for subsequent discussions.

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Definition 2.1 ([9]). Let $I = [0, 1]$, (X, d) be a metric space and $W : X \times X \times I \to X$. Mapping W is called a convex structure on X if

$$d(u, W(x, y, \alpha)) \leqslant \alpha d(u, x) + (1 - \alpha) d(u, y),$$

for each $(x, y, \alpha) \in X \times X \times I$ and $u \in X$.

Space (X, d, W) is called a convex metric space. A nonempty subset M of X is called convex if for all $(x, y, \alpha) \in M \times M \times I$, $W(x, y, \alpha) \in M$.

Remark 2.2 ([9]). Any convex metric space X possesses the following properties.

(i) $W(a, a, \lambda) = a, W(a, b, 0) = b, W(a, b, 1) = a$.
(ii) If $\{K_\alpha : \alpha \in A\}$ is a family of convex subsets of X, then $\bigcap_{\alpha \in A} K_\alpha$ is convex.

Definition 2.3 ([3]). A function $h : (X, d, W) \to [0, 1]$ satisfying $h(W(x, y, \alpha)) = \alpha h(x) + (1 - \alpha) h(y)$ is called convexity preserving (CP). $CP(X)$ denotes set of all real valued CP functions on X.

Remark 2.4. For a continuous function $h \in CP(X)$, $f^{-1}[t_1, t_2]$ is a closed and convex set for any $t_1 \leqslant t_2$.

Definition 2.5 ([4]). Let M be a subset of (X, d, W). A point $x \in X$ is an outer point M, if there exists u in M and v in X such that $W(x, u, \alpha)$ is in M and $W(x, v, \alpha)$ is not in M for every α in $(0, 1)$.

Definition 2.6 ([4]). A point of M which is not an outer point of M is called an interior point. $\mathrm{int}(M)$ denotes set of all interior points of M.

Example 2.7 ([4]). Let (R^2, d, W) be the convex Euclidean metric space, and $M = \{(x, y); y < -x + 1$ and $x, y \geqslant 0\}$, then$\mathrm{int}(M)) = \{(x, y) : x, y > 0$ and $y < -x + 1\}$.

3. Separating hyperplane

Let (X, d, W) be a convex metric space. The *convex hull* of a subset M of X is the intersection of all convex sets containing M and it is the smallest convex set containing M. The convex hull of M is denoted by $co(M)$.The convex hull of a finite subset F of a convex metric space X is called a *polytope* and points of F are called the *vertices* of the polytope $co(F)$. In X, a subset M is convex if and only if M contains any polytope with all the vertices from M. The closed convex hull of a set M is denoted by $\overline{co}(M)$ and it is the smallest closed and convex set containing M.

Remark 3.1.

(i) Let A be a subset of (X, d, W), then $co(A) = \cup\{co(F) : F$ is a finite subset of $A\}$.
(ii) In (X, d, W), convex hull of each finite subset of X is closed.

Lemma 3.2. *Let $\mathcal{F}(M)$ be any increasingly directed family of convex subsets M of a convex metric space X, then $\bigcup_{M \in \mathcal{F}} M$ is convex.*

Proof. To show that $\bigcup_{M \in \mathcal{F}} M$ is convex. Let $u, v \in \bigcup_{M \in \mathcal{F}} M$, then there exists M_u, M_v in $\mathcal{F}$ such that $u \in M_u$ and $v \in M_v$. As $\mathcal{F}$ is increasingly directed there exists M_w in $\mathcal{F}(M)$ such that $M_u \subset M_w$ and $M_v \subset M_w$. Thus, $u, v \in M_v \cup M_v \subset M_w$. Therefore, $W(x, y, \alpha) \in M_w \subset \bigcup_{M \in \mathcal{F}} M$. Hence $\bigcup_{M \in \mathcal{F}} M$ is a convex subset of X. □

Definition 3.3. A *half space* H is a convex subset of X , whose complement H^c is also convex.

Definition 3.4. Two disjoint subsets E, F of X are *separated* by a half space H if $E \subset H$ and $F \subset H^c$.

Definition 3.5. If A, B are any two subsets of (X, d, W), and if $f : (X, d, W) \to [0, 1]$ is any function, then f separates A and B if $f(A) \subset \{0\}, f(B) \subset \{1\}$.

Definition 3.6. A convex metric space X is said to have property (S) if any pair of disjoint convex subsets E, F can be separated by a half space H.

Lemma 3.7. *Each closed convex subset of* (X, d, W) *is intersection of a family of closed half spaces.*

Proof. Let $f : (X, d, W) \to [0, 1]$ is a continuous CP function, then each set $H = f^{-1}[0, t]$ or $f^{-1}[t, 1]$ is a closed half space of (X, d, W). Now for each compact set $K \subset H^c$, $f(K)$ is a compact set disjoint with a set of type $[0, t]$ or $[t, 1]$. Hence $f(K)$ is contained in some closed interval $[a, b]$ disjoint with $[0, t]$ or $[t, 1]$. It further implies $\overline{\overline{co}}(K) \subset f^{-1}[a, b] \subset H^c$. □

Proposition 3.8. *Let* E, F *be any two convex subsets of a convex metric space* (X, d, W) *with property (S) and* z *be any arbitrary point in* X. *If* $x \in co(E \cup z)$, $y \in co(F \cup z)$, *then* $co(E \cup y) \cap co(F \cup x) \neq \emptyset$.

Proof. Assume that $co(E \cup y) \cap co(F \cup x) = \emptyset$, and let $x \in co(E \cup z)$, $y \in co(F \cup z)$ be the two points for which $co(E \cup y) \cap co(F \cup x) = \emptyset$. Then by using Definition 3.6 there exists a half space H of X such that $co(E \cup y) \subset H$ and $co(F \cup x) \subset H^c$. Suppose that $z \in H$. Since $x \in co(E \cup z)$ and H is convex, therefore it implies that $x \in H$. Hence a contradiction. □

Definition 3.9. For any two subsets E, F of X define $\nabla(E, F) = \{x \in X : (co(E \cup x)) \cap F \neq \emptyset\}$.

Clearly, $F \subset \nabla(E, F)$.

Proposition 3.10. *Let* E *and* F *be any two convex subsets of* X *with property (S), then* $\nabla(E, F)$ *is a convex set.*

Proof. Let $z \in (co(\nabla(E, F))) \backslash \nabla(E, F)$. Since $F \subset \nabla(E, F)$, therefore the convex subsets $co(E \cup z)$ and F are disjoint. Definition 3.6 further implies that there exists a half space H of X with $co(E \cup z) \subset H$ and $F \subset H^c$. Thus $\nabla(E, F) \subset H^c$. Therefore $co(\nabla(E, F)) \subset H^c$. Thus there exist $z \in H \cap co(\nabla(E, F)) \subset H \cap H^c$. Hence a contradiction. □

Corollary 3.11. *Two disjoint convex polytopes* Ω, Φ *of a convex metric space* (X, d, W) *with property (S) are always separated by some half space* H.

Corollary 3.12. *Let* Ω *and* Φ *be any two convex polytopes of* X *with property* $(\bigstar)$, *then* $\nabla(\Omega, \Phi)$ *is a convex subset of* X.

Proposition 3.13. *Let* (X, d, W) *be a convex metric space,* E, F *be any two convex subsets of* X, *and* $z \in X$. *If* $x \in co(E \cup z)$, $y \in co(F \cup z)$ *implies* $co(E \cup y) \cap co(F \cup x) \neq \emptyset$, *then* X *has property (S).*

Proof. Let E_0, F_0 be two disjoint convex subsets in X. Since W is convex structure, therefore by Zorn's lemma there exists maximal convex sets E, F with $E_0 \subset E, F_0 \subset F$, and $E \cap F = \emptyset$. Next we claim that $E \cup F = X$. If otherwise, let $z \in co(X \backslash (E \cup F))$. By maximality of E, F, there exist points $x \in co(E \cup z) \cap F$ and $y \in co(F \cup z) \cap E$. Therefore $co(E \cup y) \cap co(F \cup x) \neq \emptyset$. As $co(E \cup y) = E$, and $co(F \cup x) = F$, hence a contradiction to $E \cap F = \emptyset$. □

Theorem 3.14. *In a convex metric space* (X, d, W) *the following are equivalent.*

(i) *For any two convex subsets* E, F *and* $z \in X$, *if* $x \in co(E \cup z)$, $y \in co(F \cup z)$, *then* $co(E \cup y) \cap co(F \cup x) \neq \emptyset$.
(ii) *For any two convex subsets* E *and* F *of* X, $\nabla(E, F)$ *is a convex set.*

Proof.

(i)⇒(ii) Propositions 3.13 and 3.10 imply the result.

(ii)⇒(i) Without loss of generality it is sufficient to prove that any two maximal disjoint convex sets $E(= H)$ and $F(= H^c)$ are half spaces. Assume otherwise and let $z \in \nabla(X, (E \cup F))$. By maximality of E, F there exist points $x \in co(F \cup z) \cap E$ and $y \in co(E \cup z) \cap F$. Therefore $z \in \nabla(E, F) \cap \nabla(F, E)$. As $E \subset \nabla(F, E)$ and $F \subset \nabla(E, F)$, it further implies

$$y \in co(E \cup z) \subset co(\nabla(F, E)) = \nabla(F, E), \quad x \in co(F \cup z) \subset co(\nabla(E, F)) = \nabla(E, F).$$

Also by using Definition 3.9 and the fact that $x \in E$ and $y \in F$ and E, F are convex subsets we have $E \cap F \neq \emptyset$. Hence a contradiction. Therefore X has property (S), which further implies (i). □

Theorem 3.15. *Let (X, d, W) be a convex metric space with property (S) in which closure of a convex set is convex. If E, F are disjoint convex subsets with E open and F closed, then there exists a closed half space H with $E \cap H = \emptyset, F \subset H$.*

Proof. Let H be a maximal convex subsets of (X, d, W) with $E \cap H = \emptyset, F \subset H$. Then $\overline{H}$ is convex and closed, and disjoint with E. Definition 3.6 further implies that H is a half space. □

Theorem 3.16 (Separating hyperplane theorem)**.** *Let (X, d, W) be a convex metric space with property (S) in which closure of a convex set is convex. If E, F are disjoint convex subsets with E closed and F compact, then there exists a closed half space H with $E \cap H = \emptyset, F \subset H$.*

Proof. For each $x \in F$, there exists a closed half space $H(x)$ with $x \in int(H(x))$, $(int(H(X))) \cap E = \emptyset$. Now select a finite cover $\{H(x_1), H(x_2), \ldots, H(x_n)\}$ of F. Then

$$C = \bigcap_{n}^{i=1} \nabla(X, H(x_i),$$

is an open convex subset of X and $E \subset C, C \cap F = \emptyset$. Now Theorem 3.15 implies the result. □

4. Conclusion

This paper discusses under what conditions two disjoint convex subsets of a convex metric space can be separated by a hyperplane. A Mazur's type theorem for separating hyperplane on convex metric spaces is proven. Other kinds of separation, i.e., strict separation, proper separation, and strong separation are to be investigated in future.

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