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WHEN IS A QUASI-P-PROJECTIVE MODULE DISCRETE?

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Abstract. It is well-known that every quasi-projective module has D_2-condition. In this note it is shown that for a quasi-p-projective module M which is self-generator, duo, then M is discrete.

1. Introduction and preliminaries

Throughout, R is an associative ring with identity and right R-modules are unitary. Let M be a right R-module. A module N is called *M-generated* if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. In particular, N is called *M-cyclic* if it is isomorphism to M/L for submodule $L \subseteq M$. Following [3] a module M is called *self-generate* if it generates all its submodules. For standard notation and terminologies, we refer to [4], [3].

Let M be a right R-module. A right R-module N is called *M-p- projective* if every homomorphism from N to an M-cyclic submodule of M can be lifted to an R-homomorphism from N to M. A right R-module M is called *quasi-p-projective* , if it is M-p-projective. A submodule A of M is said to be a *small submodule* of M (denoted by $A \ll M$) if for any $B \subseteq M$, $A + B = M$ implies $B = M$. A module M is called *hollow* if every its submodule is small.

In [2], S.Chotchaisthit showed that a quasi-p-injective module M is continouse, if M is duo and semiprefect. Here we study, when a quasi-p-projective module is discrete.

Consider the following conditions for a module M which have studied in [3] :

D_1: For every submodules N of M there exist submodules K, L of M such that $M = K \oplus L$ and $K \leq N$ and $N \cap L \ll L$.

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D_2: If N is a submodule of M such that M/N is isomorphism to a direct summand of M, then N is a direct summand of M.

D_3: For every direct summands K, L of M with $M = K + L$, $K \cap L$ is a direct summand of M.

If the module M satisfies D_1 and D_2 then it is called a *discrete* module.

It is clear that if M is hollow, then it has D_1 and D_2 conditions, since hollow module is indecomposition.

2. Main results

Recall that a submodule N of M is called a *fully invariant* submodule if $s(N) \subseteq N$, for any endomorphism s of M. A right R-module is called a *duo* module if every submudole is fully invariant. A ring R is right duo if every right ideal is two sided. The proof of the following Lemma is routine.

Lemma 2.1. *Let M be a duo right R-module and A its direct summand. Then:*
(1) A is itself a duo module;
(2) If M is a self-generator, then A is also a self- generator.

Proof. (1) Let $f \in End(A)$, $\pi : M \to A$, $i : A \to M$ be the projection and inclusion maps. Then $g = if\pi \in End(M)$. It follows that for any submodule X of A, $f(X) = g(X) \subset X$, proving our Lemma.

(2) Let $M = A \oplus B$. Then $f(M) = f(A) + f(B)$ for any $f \in End(M)$. Let X be a submodule of A. Since M is a self-generator, we can write $X = \sum_{f \in I} f(M) = \sum_{f \in I}(f(A) + f(B))$, for some subset I of $End(M)$. Since $f(B) \subset B$, it follows that $f(B) = 0$ for all $f \in I$. Hence $X = \sum_{f \in I} f(A)$. Moreover, f can be considered as an endomorphism of A, since $f(A) \subset A$. This shows that A is a self-generator. □

Lemma 2.2. *Let M be a quasi-p-projective. If $S = End(M_R)$ is local, then for any non-traivial fully invariant M-cyclic submodules A and B of M, $A + B \neq M$.*

Proof. let $0 \neq s(M) = A$ and $0 \neq t(M) = B$, $s, t \in S$ and $A + B = M$. Difine the map $f : M = (s+t)(M) \to M/(A \cap B)$ such that $f(s+t)(m) = s(m) + (A \cap B)$. For any $m, m' \in M$, $(s+t)(m) = (s+t)(m')$ implies $s(m-m') = t(m'-m) \in A \cap B$. So $s(m) + (A \cap B) = s(m') + (A \cap B)$. Clearly f is an R-homomorphism. By quasi-p-projective, there exist $g \in S$ such that $\pi \circ g = f$ and $\pi : M \to M/(A \cap B)$ is natural epimorphism . It follows $\pi \circ g(s+t))(m) = \pi(s(m))$. Then $((1-g) \circ s - g \circ t)(M) \subseteq (A \cap B)$. Since S is local, g or $1 - g$ is invertible. If $1 - g$ be invertible we have $(s - (1-g)^{-1} \circ g \circ t)(M) \subseteq (A \cap B)$. $A \subseteq (s - (1-g)^{-1} \circ g \circ t)(M) \subseteq (1-g)^{-1}(A \cap B) \subseteq (A \cap B)$. Then $A \subseteq (A \cap B)$, that is contradiction. If g be invertible we have $B \subseteq (g^{-1} \circ (1-g) \circ s - t) \subseteq g^{-1}(A \cap B) \subseteq (A \cap B)$. Then $B \subseteq (A \cap B)$, that is contradiction. □

Corollary 2.3. *If M is qasi-p-projective duo module which is a self-generator with local endomorphism ring, then M is hollow, hence it is discrete.*

Proof. It is clear by Lemma 2.2 □

Lemma 2.4. *Let $M = \oplus_{i \in I} B_i$ be duo module. Then for any submodule A of M we have $A = \oplus_{i \in I}(A \cap B_i)$.*

Proof. See [1]. □

Corollary 2.5. *Let M be a duo module. If A and B are direct summands of M, then so $A \cap B$.*

Proof. Let $M = A \oplus A_1 = B \oplus B_1$, then by lemma 2.4 $B = B \cap (A \oplus A_1 = (A \cap B) \oplus (B \cap A_1)$. hence $M = (A \cap B) \oplus (B \cap A_1) \oplus B_1$. So $A \cap B$ is a direct summand of M.

□

Theorem 2.6. *Let $M = \oplus_{i \in I} M_i$ be qusi-p-projective module where each $M_i$is hollow. If M is duo module, $Rad(M) \ll M$ then M is discrete.*

Proof. By Lemma 2.4 every submodule A of M can be written in the form $A = \oplus_{j \in J}(A \cap M_j)$ where $J \subseteq I$ and $A \cap M_j \neq 0$. Since $A \cap M_j$ is small in M_j we see that A is small in M. Thus we have proved.

□

Theorem 2.7. *Suppose that M is semisimple quasi-p-projective duo module and $Rad(M) \ll M$. If M is self-generator, then M is discrete.*

Proof. We have $M = \oplus_{i \in I} M_i$ such that M_i is simple, then $End(M_i)$ is local. By Lemma 2.1 each M_i is duo and self-generator. Since any direct summand of a quasi-p-projective is again quasi-p-projective , it follows from Corollary 2.3 that each M_i is discrete. From Theorem 2.6 that M is discrete, proving our Theorem. □

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