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VISCOSITY APPROXIMATION METHOD FOR NONEXPANSIVE NONSELF-MAPPING AND VARIATIONAL INEQUALITY

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ABSTRACT. Let E be a real reflexive Banach space which has uniformly Gâteaux differentiable norm. Let K be aclosed convex subset of E which is also a sunny nonexpansive retract of E, and $T : K \to E$ be nonexpansive mapping satisfying the weakly inward condition and $F(T) = \{x \in K, Tx = x\} \neq \emptyset$, and $f : K \to K$ be a contractive mapping. Suppose that $x_0 \in K$, $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)((1-\delta)x_n + \delta y_n), \\ y_n = P(\beta_n x_n + (1-\beta_n)Tx_n), \quad n \geq 0, \end{cases}$$

where $\delta \in (0, 1)$, $\alpha_n, \beta_n \in [0, 1]$, P is sunny nonexpansive retractive from E into K. Under appropriate conditions, it is shown that $\{x_n\}$ converges strongly to a fixed point T and the fixed point solutes some variational inequalities. The results in this paper extend and improve the corresponding results of [2] and some others.

1. INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space and E^* its dual space. Let J denote the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, where $\langle \cdot, \cdot\rangle$ denote the generalized duality pairing between E and E^*. It is well-known that if E^* is strictly convex then J is sing-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j.

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We first recall some definitions and conclusions:

Definition 1.1. T is a mapping with domain $D(T)$ and $R(T)$ in E. T is said to be a $L-$*Lipschitz* mapping, if $\forall\, x, y \in D(T)$, $\|Tx-Ty\| \leq L\|x-y\|$. Especially, if $L = 1$, i.e. $\|Tx-Ty\| \leq \|x-y\|$, then T is said to non-expansive; if $0 < L < 1$, then T is said to contraction mapping.

Definition 1.2. Let K be a nonempty closed convex subsets of a Banach E. A mapping $P : E \to K$ is called a *retraction* from E into K if P is continuous with $F(P) = \{x \in E : Px = x\} = K$. A mapping $P : E \to K$ is called *sunny* if

$$P(Px + t(x - Px)) = Px, \ \forall\, x \in E$$

whenever $Px + t(x - Px) \in E$ and $\forall \in t > 0$. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction from E into K. For more details, see [4].

Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of the real Banach space E. E is said to have a *Gâteaux differentiable norm* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; and E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit is attained uniformly for $x \in S$. Furthermore, if E has a *uniformly Gâteaux differentiable norm*, then the duality map j is *norm-to-weak* uniformly continuous* on bounded subsets of E(see, p.111 of [4]). Let E be a normed space with dim $E \geq 2$, the modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\}.$$

The space E is called uniformly smooth if and only if $\lim_{\tau\to 0^+} \rho_E\tau/\tau = 0$.

Let $F(T)$ denote a fixed point set of mapping T.

Let K be a nonempty convex subset of a Banach space E. Then for $x \in K$, set $I_K(x)$ is called inward set [2,7], where

$$I_K(x) = \{y \in E : y = x + \lambda(z - x), z \in K \text{ and } \lambda \geq 0\}.$$

A mapping $T : K \to E$ is said to be satisfying the inward condition if $Tx \in I_K(x)$ for all $x \in K$. T is also said to be satisfying the weakly inward condition if for each $x \in K$, $Tx \in \overline{I_K(x)}$ ($\overline{I_K(x)}$ is the closure of $I_K(x)$). Clearly $K \subset I_K(x)$ and it is not hard to show that $I_K(x)$ is a convex set as K does.

Let K be a close convex subset of a uniformly smooth Banach space E, $f : K \to K$ a contraction, $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then for any $t \in (0, 1)$, the mapping

$$T_t^f : \ x \mapsto tf(x) + (1 - t)Tx$$

is also contraction. Let x_t denote the unique fixed point of T_t^f. In [6], H.K.Xu proved that as $t \downarrow 0$, $\{x_t\}$ converges to a fixed point u of T that is the unique solution of the variational inequality

$$\langle (I - f)u, j(u - p)\rangle \leq 0 \quad \text{for all } p \in F(T).$$

H.K. Xu also proved the following explicit iterative process $\{x_n\}$ given by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n$$

converges strongly to a fixed point p of T.

Let K be a close convex subset of a real Banach space E which is also a sunny nonexpansive retract of E. $f : K \to K$ is a contraction. $T : K \to E$ is a nonexpansive nonself-mapping. Inspired by Xu [6], in 2006, Y.Song and R.Chen [2] considered the following algorithm,

$$x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n), n \geq 0, \tag{1.1}$$

where $x_0 \in K$, P is a sunny nonexpansive retractive from E into K, $\alpha_n \in (0,1)$. Then Y.Song and R. Chen [2] obtained the following results:

Theorem 1.3. *(Theorem 2.4 of [2]). Let E be a reflexive Banach space which admits a weakly sequentially continuous J from E to E^*. Suppose K is a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E, and $T : K \to E$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.1), where P is a sunny nonexpansive retract from E into K, and $\alpha_n \in (0,1)$ satisy the following conditions:*
(i) $\alpha_n \to 0$, as $n \to \infty$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$
(ii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Then $\{x_n\}$ converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)p, j(p-u)\rangle \leq 0, \ \ \forall\, u \in F(T).$$

Motivated by Song and Chen's work, in this paper, we introduce a new composite iterative scheme as follows:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)((1-\delta)x_n + \delta y_n), \\ y_n = P(\beta_n x_n + (1-\beta_n)Tx_n), \quad n \geq 0, \end{cases} \tag{1.2}$$

where $\alpha_n, \beta_n \in (0,1)$, $\sigma \in (0,1)$ is arbitrary (but fixed). Under appropriate conditions, the $\{x_n\}$ defined by (1.2) converges strongly to a fixed point q of T such that q is a solution of some variational inequalities. The results obtained in this paper extend and improve the corresponding that of [2] and some others. At the same time, this paper provides a new approach for the construction of a fixed point of nonexpansive mapping.

In what follows, we shall make use of the following Lemmas.

Lemma 1.4. *([1]).Let E be a real normed linear space and J the normalized duality mapping on E, then for each $x, y \in E$ and $j(x+y) \in J(x+y)$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 1.5. *(Suzuki, [3]).Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 1.6. *([8]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

if (i) $\alpha_n \in [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$, $\sum \gamma_n < \infty$, then $a_n \to 0$, as $n \to \infty$.

Let μ be a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n; n \in N\} \leq \mu(a) \leq \sup\{a_n; n \in N\}$$

for every $a = (a_1, a_2, ...) \in l^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a Banach limit if $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, ...) \in l^\infty$. Furthermore, we know the following result [5, Lemma 1] and[4, Lemma4.5.4].

Lemma 1.7. *([5], Lemma 1). Let K be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on N.Let $z \in K$. Then*

$$\mu_n\|x_n - z\| = \min_{y \in K} \mu_n\|x_n - y\|$$

if and only if

$$\mu_n\langle y - z, j(x_n - z)\rangle \leq 0, \ \ \forall\, y \in K,$$

where j is the duality mapping of E.

Lemma 1.8. *(Lemma 1.2 of [2]). Let E be a smooth Banach space, and K be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E, and $T : K \to E$ be mapping satisfying the weakly inward condition, and P be a sunny nonexpansive retraction from E into K. Then $F(T) = F(PT)$.*

2. Main results

Throughout this paper, suppose that
(a) E is a real reflexive Banach space E which has a uniformly Gâteaux differentiable norms;
(b) K is a nonempty closed convex subset of E;
(c) every nonempty closed bounded convex subset of E has the fixed point property for nonexpansive mappings.

Lemma 2.1. *Let $T : K \to K$ be a nonexpansive mapping with $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Let $f : K \to K$ be a contraction with contraction constant$\alpha \in (0,1)$, then there exists $x_t \in K$ such that*

$$x_t = tf(x_t) + (1-t)Tx_t, \tag{2.1}$$

where $t \in (0,1)$. Further, as $t \to 0^+$, x_t converges strongly a fixed point $q \in F(T)$ which solutes the following variational inequality:

$$\langle q - f(q), j(q-p)\rangle \leq 0, \ \ \forall\, p \in F(T). \tag{2.2}$$

Proof. Firstly, let H_t^f denote a mapping defined by

$$H_t^f x = tf(x) + (1-t)Tx, \quad \forall\, t \in (0,1), \ \forall\, x \in E.$$

Obviously, H_t^f is contraction, then by Banach contraction mapping principle there exists $x_t \in K$ such that

$$x_t = tf(x_t) + (1-t)Tx_t.$$

Now, let $q \in F(T)$, then

$$\|x_t - q\| = \|t(f(x_t) - q) + (1-t)(Tx_t - q)\| \le (1 - t + t\alpha)\|x_t - q\| + t\|f(q) - q\|,$$

i.e.,

$$\|x_t - q\| \le \frac{\|f(q) - q\|}{1-\alpha}.$$

Hence $\{x_t\}$ is bounded. Assume that $t_n \to 0^+$ as $n \to \infty$. Set $x_n := x_{t_n}$, define a function g on K by

$$g(x) = \mu_n\|x_n - x\|^2.$$

Let

$$C = \{x \in K; g(x) = \min_{y\in E} \mu_n\|x_n - y\|^2\}.$$

It is easy to see that C is a closed convex bounded subset of K. Since $\|x_n - Tx_n\| \to 0 (n \to \infty)$, hence

$$g(Tx) = \mu_n\|x_n - Tx\|^2 = \mu_n\|Tx_n - Tx\|^2 \le \mu_n\|x_n - x\|^2 = g(x),$$

it follows that $T(C) \subset C$, that is C is invariant under T. By assumption (c), non-expansive mapping T has fixed point $q \in C$. Using Lemma 1.7 we obtain

$$\mu_n\langle x - q, j(x_n - q)\rangle \le 0.$$

Taking $x = f(q)$, then

$$\mu_n\langle f(q) - q, j(x_n - q)\rangle \le 0. \tag{2.3}$$

Since

$$x_t - q = t(f(x_t) - q) + (1-t)(Tx_t - q),$$

then

$$\begin{aligned} \|x_t - q\|^2 &= t\langle f(x_t) - q, j(x_t - q)\rangle + (1-t)\langle Tx_t - q, j(x_t - q)\rangle \\ &\le t\langle f(x_t) - q, j(x_t - q)\rangle + (1-t)\|x_t - q\|^2 \end{aligned}$$

Further,

$$\begin{aligned} \|x_t - q\|^2 &\le \langle f(x_t) - q, j(x_t - q)\rangle \\ &= \langle f(x_t) - f(q), j(x_t - q)\rangle + \langle f(q) - q, j(x_t - q)\rangle. \end{aligned}$$

Thus,

$$\mu_n\|x_n - q\|^2 \le \mu_n\alpha\|x_n - q\|^2 + \mu_n\langle f(q) - q, j(x_n - q)\rangle.$$

it follows from (2.3) that

$$\mu_n\|x_n - q\|^2 = 0.$$

Hence there exists a subsequence of $\{x_n\}$ which is still denoted by $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n - q\| = 0$. Now assume that another subsequence $\{x_m\}$ of $\{x_n\}$ converge strongly to $\bar{q} \in F(T)$. Since j is *norm-to-weak* uniformly continuous* on bounded subsets of E, then for any $p \in F(T)$,we have

$$\begin{aligned}
&|\langle x_m - f(x_m), j(x_m - p)\rangle - \langle \bar{q} - f(\bar{q}), j(\bar{q} - p)\rangle| \\
= \ & |\langle x_m - f(x_m) - (\bar{q} - f(\bar{q})), j(x_m - p)\rangle \\
& + \langle (\bar{q} - f(\bar{q})), j(x_m - p)\rangle - \langle \bar{q} - f(\bar{q}), j(\bar{q} - q)\rangle| \\
\leq \ & \|(I - f)x_m - (I - f)\bar{q}\| \|x_m - p\| \\
& + |\langle \bar{q} - f(\bar{q}), j(x_m - p) - j(\bar{q} - p)\rangle| \to 0 \ (m \to \infty),
\end{aligned} \tag{2.4}$$

i.e.,

$$\langle \bar{q} - f(\bar{q}), j(\bar{q} - p)\rangle = \lim_{n\to\infty} \langle x_m - f(x_m), j(x_m - p)\rangle. \tag{2.5}$$

Since $x_m = tf(x_m) + (1 - t)Tx_m$, we have

$$(I - f)x_m = -\frac{1 - t}{t}(I - T)x_m,$$

hence for any $p \in F(T)$,

$$\langle (I - f)x_m, j(x_m - p)\rangle = -\frac{1 - t}{t}\langle (I - T)x_m - (I - T)p, j(x_m - p)\rangle \leq 0, \tag{2.6}$$

it follows from (2.5) and (2.6) that

$$\langle \bar{q} - f(\bar{q}), j(\bar{q} - p)\rangle \leq 0. \tag{2.7}$$

Interchange p and q to obtain

$$\langle \bar{q} - f(\bar{q}), j(\bar{q} - q)\rangle \leq 0, \tag{2.8}$$

i.e.,

$$\langle \bar{q} - q + q - f(\bar{q}), j(\bar{q} - q)\rangle \leq 0, \tag{2.9}$$

hence

$$\|\bar{q} - q\|^2 \leq \langle f(\bar{q}) - q, j(\bar{q} - q)\rangle. \tag{2.10}$$

Interchange q and $\bar{q}$ to obtain

$$\|\bar{q} - q\|^2 \leq \langle f(q) - \bar{q}, j(q - \bar{q})\rangle. \tag{2.11}$$

Adding up (2.10) and (2.11) yields that

$$2\|\bar{q} - q\|^2 \leq (1 + \alpha)\|\bar{q} - q\|, \tag{2.12}$$

this implies that $q = \bar{q}$. Hence $x_t \to q$ as $t \to 0^+$ and q is a solution of the following variational inequality

$$\langle q - f(q), j(q - p)\rangle \leq 0, \quad \forall\, p \in \mathrm{F(T)}.$$

This completes the proof of Lemma 2.1. □

Theorem 2.2. *Let K be a sunny nonexpansive retract of E. $T : K \to E$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$. $f : K \to K$ is contractive with constant $\alpha \in (0, 1)$. Let P be a sunny nonexpansive retraction from E into K. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)((1 - \delta)x_n + \delta y_n), \\ y_n = P(\beta_n x_n + (1 - \beta_n)Tx_n), \quad n \geq 0, \end{cases} \tag{2.13}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. $\delta \in (0, 1)$ is arbitrary (but fixed). Suppose that $\{\alpha_n\}, \{\beta_n\}$ satisfy the following conditions:

(i) $\alpha_n \to 0$ as $n \to \infty$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 \le \beta_n < a$, $|\beta_{n+1} - \beta_n| \to 0$ *as* $n \to \infty$, *where* $a \in (0,1)$.
Then $\{x_n\}$ *converges strongly to a fixed point* $q \in F(T)$, *where* $q = \lim_{t\to 0^+} x_t$ *is a solution of variational inequality (2.2).*

Proof. We splits four steps to prove it.

Step 1. $\{x_n\}$ is bounded. In deed, by (2.13), it is easy to see that

$$\begin{aligned}\|y_n - x^*\| &= \|P(\beta_n x_n + (1-\beta_n)Tx_n) - x^*\| = \|P(\beta_n x_n + (1-\beta_n)Tx_n) - Px^*\| \\ &\le \|\beta_n(x_n - x^*) + (1-\beta_n)(Tx_n - x^*)\| \le \|x_n - x^*\| \qquad (2.14)\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)((1-\delta)(x_n - x^*) + \delta(y_n - x^*))\| \\ &\le (1-\alpha_n)(1-\delta)\|x_n - x^*\| + \alpha_n\alpha\|x_n - x^*\| \\ &\quad + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\delta\|y_n - x^*\|, \qquad (2.15)\end{aligned}$$

where $x^* \in F(T)$. It follows from (2.14) and (2.15) that

$$\|x_{n+1} - x^*\| \le (1-(1-\alpha)\alpha_n)(\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\|. \qquad (2.16)$$

By simplicity deducing, from (2.16) we have

$$\|x_n - x^*\| \le \max\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\alpha}\}, \quad n \ge 0.$$

Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. In deed, let $M > 0$ be a constant such that

$$\max\{\|x_{n+1}\|, \|x_n\|, \|y_{n+1}\|, \|Tx_{n+1}\|, \|Tx_n\|, \|f(x_{n+1})\|, \|f(x_n)\|\} \le M.$$

It follows from (2.13) that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P(\beta_{n+1}x_{n+1} + (1-\beta_{n+1})Tx_{n+1}) - P(\beta_n x_n + (1-\beta_n)Tx_n)\| \\ &\le \|\beta_{n+1}x_{n+1} - \beta_n x_n\| + \|(1-\beta_{n+1})Tx_{n+1} - (1-\beta_n)Tx_n\| \\ &\le 2|\beta_{n+1} - \beta_n|M + \|x_{n+1} - x_n\|. \qquad (2.17)\end{aligned}$$

Now, let $\gamma_n = \delta + \alpha_n(1-\delta)$, $\bar{y}_n = \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n} = \frac{\alpha_n f(x_n) + (1-\alpha_n)\delta y_n}{\gamma_n}$, then

$$\begin{aligned}&\overline{y}_{n+1} - \overline{y}_n \\ &= \frac{\alpha_{n+1}}{\gamma_{n+1}} f(x_{n+1}) - \frac{\alpha_n}{\gamma_n} f(x_n) + \frac{(1-\alpha_{n+1})\delta y_{n+1}}{\gamma_{n+1}} - \frac{(1-\alpha_n)\delta y_n}{\gamma_n} \\ &= \frac{\alpha_{n+1}}{\gamma_{n+1}} f(x_{n+1}) - \frac{\alpha_n}{\gamma_n} f(x_n) + \frac{(1-\alpha_n)\delta}{\gamma_n}(y_{n+1} - y_n) + \left(\frac{1-\alpha_{n+1}}{\gamma_{n+1}} - \frac{1-\alpha_n}{\gamma_n}\right)\delta y_{n+1},\end{aligned}$$

which yields that

$$\|\overline{y}_{n+1} - \overline{y}_n\| \le 2\frac{\alpha_{n+1} + \alpha_n}{\gamma_{n+1}\gamma_n} M + \frac{(1-\alpha_n)\delta}{\gamma_n}\|y_{n+1} - y_n\|. \qquad (2.18)$$

It follows from (2.17) and (2.18) that

$$\|\overline{y}_{n+1} - \overline{y}_n\| \le 2\frac{\alpha_{n+1} + \alpha_n}{\gamma_{n+1}\gamma_n} M + \frac{2|\beta_{n+1} - \beta_n|M}{\gamma_n} + \frac{(1-\alpha_n)\delta}{\gamma_n}\|x_{n+1} - x_n\|. \qquad (2.19)$$

Using the conditions (i-ii), from (2.19) we get that

$$\limsup_{n\to\infty}\{\|\overline{y}_{n+1} - \overline{y}_n\| - \|x_{n+1} - x_n\|\} \le 0. \qquad (2.20)$$

Based on Lemma 1.5 and (2.20), we obtain $\lim_{n\to\infty}\|\overline{y}_n - x_n\| = 0$, which implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Step3. $\|x_n - PTx_n\| \to 0$ as $n \to \infty$. Since

$$\|x_{n+1} - ((1-\delta)x_n + \delta y_n)\| = \alpha_n\|f(x_n) - ((1-\delta)x_n + \delta y_n)\| \to 0 (n \to \infty)$$

and

$$\delta\|x_n - y_n\| - \|x_{n+1} - x_n\| \le \|x_{n+1} - x_n - \delta(y_n - x_n)\| = \|x_{n+1} - ((1-\delta)x_n + \delta y_n)\|,$$

hence,

$$\|x_n - y_n\| \le \frac{\|x_{n+1} - x_n\| + \|x_{n+1} - ((1-\delta)x_n + \delta y_n)\|}{\delta} \to 0(n \to \infty).$$

Further,

$$\|x_n - PTx_n\| \le \|x_n - y_n\| + \|y_n - PTx_n\| \le \|x_n - y_n\| + a\|x_n - PTx_n\|,$$

which yields that

$$\|x_n - PTx_n\| \to 0(n \to \infty). \tag{2.21}$$

Step 4. $\|x_n - x^*\| \to 0$ as $n \to \infty$, where $x^* \in F(T)$ and x^* satisfies the variational inequality (2.2).

Since PT is nonexpansive mapping, then by Lemma 2.1 there exists x_t such that

$$x_t = tf(x_t) + (1-t)PTx_t, \quad \forall\ t \in (0,1),$$

Then, using Lemma 1.4, we have

$$\begin{aligned}
&\|x_t - x_n\|^2 = \|t(f(x_t) - x_n) + (1-t)(PTx_t - x_n)\|^2\\
&\le (1-t)^2\|PTx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle\\
&\le (1-t)^2(\|PTx_t - PTx_n\| + \|PTx_n - x_n\|)^2 + 2t\langle f(x_t) - x_t + x_t - x_n, j(x_t - x_n)\rangle\\
&\le (1+t^2)\|x_t - x_n\|^2 + \|PTx_n - x_n\|(2\|x_t - x_n\| + \|PTx_n - x_n\|)\\
&\quad +2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle,
\end{aligned}$$

hence,

$$\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{\|PTx_n - x_n\|}{2t}(2\|x_t - x_n\| + \|PTx_n - x_n\|),$$

let $n \to \infty$ in the last inequality, then we obtain

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le \frac{t}{2}M',$$

where $M' \ge 0$ is a constant such that $\|x_t - x_n\|^2 \le M'$ for all $t \in (0,1)$ and $n \ge 0$. Now letting $t \to 0^+$, then we have that

$$\limsup_{t\to 0^+}\limsup_{n\to\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le 0.$$

Thus , for $\forall\,\varepsilon > 0$, there exists a positive number δ' such that for any $t \in (0, \delta')$,

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le \frac{\varepsilon}{2}.$$

On the other hand, By Lemma 1.8 and Lemma 2.1 we have $x_t \to x^* \in F(PT) = F(T)$ as $t \to 0^+$. In addition, j is norm-to-weak* uniformly continuous on bounded subsets of E, so there exists $\delta'' > 0$ such that, for any $t \in (0, \delta'')$, we have

$$
\begin{aligned}
&|\langle (f(x^*)-x^*, j(x_n-x^*)\rangle - \langle f(x_t)-x_t, j(x_n-x_t)\rangle|\\
&\le |\langle f(x^*)-x^*, j(x_n-x^*)-j(x_n-x_t)\rangle + |\langle f(x^*)-x^*, j(x_n-x_t)\rangle - \langle f(x_t)-x_t, j(x_n-x_t)\rangle|\\
&\le \|f(x^*)-x^*\|\|j(x_n-x^*)-j(x_n-x_t)\| + (1+\alpha)\|\ x_t-x^*\|\|x_n-x_t\|\\
&< \frac{\varepsilon}{2}.
\end{aligned}
$$

Taking $\delta = \min\{\delta', \delta''\}$, for $t \in (0, \delta)$, we have that

$$
\langle f(x^*)-x^*, j(x_n-x^*)\rangle \le \langle f(x_t)-x_t, j(x_n-x_t)\rangle + \frac{\varepsilon}{2}.
$$

Hence,

$$
\limsup_{n\to\infty} \langle f(x^*)-x^*, j(x_n-x^*)\rangle \le \varepsilon, \quad \text{where } \varepsilon > 0 \text{ is arbitrary,}
$$

which yields that

$$
\limsup_{n\to\infty} \langle f(x^*)-x^*, j(x_n-x^*)\rangle \le 0. \tag{2.22}
$$

Now we prove that $\{x_n\}$ converges strongly to x^*. It follows from Lemma 1.4 and (2.13) that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 = \|\alpha_n(f(x_n)-x^*) + (1-\alpha_n)((1-\delta)(x_n-x^*)+\delta(y_n-x^*))\|^2\\
&\le (1-\alpha_n)^2\|(1-\delta)(x_n-x^*)+\delta(y_n-x^*)\|^2 + 2\alpha_n\langle f(x_n)-x^*, j(x_{n+1}-x^*)\rangle\\
&= (1-\alpha_n)^2\|x_n-x^*\|^2 + 2\alpha_n\langle f(x_n)-f(x^*)+f(x^*)-x^*, j(x_{n+1}-x^*)\rangle\\
&\le (1-\alpha_n)^2\|x_n-x^*\|^2 + 2\alpha_n\alpha\|x_n-x^*\|\|x_{n+1}-x^*\| + 2\alpha_n\langle f(x^*)-x^*, j(x_{n+1}-x^*)\rangle\\
&\le (1-\alpha_n)^2\|x_n-x^*\|^2 + \alpha_n\alpha(\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2)\\
&\qquad + 2\alpha_n\langle f(x^*)-x^*, j(x_{n+1}-x^*)\rangle,
\end{aligned} \tag{2.23}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n-x^*\|^2 + \frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-x^*\|^2\\
&\quad + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(x^*)-x^*, j(x_{n+1}-x^*)\rangle\\
&= (1-\bar{\alpha}_n)\|x_n-x^*\|^2 + \frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-x^*\|^2\\
&\quad + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(x^*)-x^*, j(x_{n+1}-x^*)\rangle,
\end{aligned} \tag{2.24}
$$

where $\bar{\alpha}_n = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$. By boundness of $\{x_n\}$ and condition (i) and Lemma 1.6, $\{x_n\}$ converges strongly to x^*. This completes the proof of Theorem 2.2. □

Remark 2.3. Theorem 2.2 is obtained under the coefficient α_n satisfying $\lim \alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$. In addition, this paper omits the request that space E admits a weakly sequentially continuous duality mapping from E into E^*. Hence it is an improvement of Theorem 2.4 of [2].

Remark 2.4. If E is uniformly smooth then E is reflexive and has a uniformly Gâteaux differentiable norm with the property that every nonempty closed and bounded subset of E has the fixed point property for nonexpansive mappings(see, remark 3.5 of [9]). Thus, if E is a real uniformly smooth Banach space, then the results in this paper are true, too.

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