J. Nonlinear Sci. Appl., 18 (2025), 33–42
ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.isr-publications.com/jnsa

On a subclass of analytic functions defined by Bell distribution series

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Abstract

The Bell distribution is a major and helpful model that may be applied to a wide variety of real-world situations and problems. Bell distributions play a significant role in geometric function theory, particularly in the study of univalent functions and their properties. The importance of Bell distributions in geometric function theory lies in their ability to provide a combinatorial framework for analyzing the properties and behaviors of univalent functions. By leveraging these distributions, mathematicians can gain deeper insights into the geometric and analytic aspects of complex functions, enhancing both theoretical understanding and practical applications. The main purpose of this paper is to introduce a new subclass of analytic functions involving Bell distribution series and obtain coefficient inequalities, distortion theorem, convex linear combination, convolution and neighborhood result for this class.

Keywords: Analytic, starlike, convexity, coefficient estimate, neighborhood.

2020 MSC: 30C45, 30C50.

1. Introduction

Let A specify the category of analytical functions, u represent on the unit disc $U = \{z : |z| < 1\}$ with normalization $u(0) = 0$ and $u'(0) = 1$, such a function has the extension of the Taylor series on the origin in the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Indicated by S, the subclass of A be composed of functions that are univalent in U. Then a $u(z)$ function of A is known as starlike and convex of order α if it delights the pursing

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \alpha,\ (z \in U), \quad \text{and} \quad \Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \alpha,\ (z \in U),$$

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doi: 10.22436/jnsa.018.01.04

Received: 2024-07-31 Revised: 2024-09-22 Accepted: 2024-11-09

for a given α ($0 \leqslant \alpha < 1$), aand we denote by $S^*(\alpha)$ and $K(\alpha)$ the subclass of A corresponding to these functions, respectively. Also, indicate by T the subclass of A made up of functions of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geqslant 0,\ z \in U), \tag{1.2}$$

and let $T^*(\alpha) = T \cap S^*(\alpha)$, $C(\alpha) = T \cap K(\alpha)$. There are interesting properties in the $T^*(\alpha)$ and $C(\alpha)$ classes and were thoroughly studied by Silverman [21] and others.

The class $UCV(\alpha, \sigma)$ consists of uniform σ-convex functions of order α, and $SP(\alpha, \sigma)$ consists of parabolic σ-starlike functions of order α, with $-1 < \alpha \leqslant 1$, $\sigma \geqslant 0$, are defined by

$$UCV(\alpha, \sigma) := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \sigma \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in U \right\},$$

and

$$SP(\alpha, \sigma) := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \sigma \left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in U \right\}. \tag{1.3}$$

These classes generalizes the class UCV and S_p, respectively, studied by Rønning [17, 18] and others [10, 12]. Bell distributions and their relation to univalent functions offer a fascinating intersection of probability and complex analysis. Univalent functions, which are holomorphic and injective in a given domain, play a key role in various areas, including geometric function theory and complex analysis. The Bell distribution, also known as the normal mixture distribution, is a probability distribution that arises in the context of statistical inference, signal processing, and other fields of science. The Bell distribution is a continuous probability distribution that is a mixture of normal distributions.

Connection between Bell distributions and univalent functions are as follows. (i) The coefficients of the Taylor series expansion of univalent functions can be connected to combinatorial objects like Bell numbers. For example, the enumeration of certain classes of univalent functions can be expressed using Bell polynomials. (ii) Univalent functions can be transformed using specific mappings. These transformations may lead to distributions that can be analyzed using the framework of Bell distributions. (iii) When studying the random behavior of univalent functions (e.g., in random conformal mappings), the distributions of certain parameters can exhibit properties analogous to those of Bell distributions. In various complex analysis applications, the growth rates and boundary behaviors of univalent functions may relate to combinatorial structures, offering deeper insights that can be modeled probabilistically using Bell distributions.

Bell numbers can be used to study the coefficients in the Taylor series expansion of univalent functions. Understanding these coefficients helps in analyzing the growth and distortion properties of such functions. The coefficients of univalent functions can be viewed as partitions, where Bell distributions help in counting distinct configurations that arise from these coefficients. When studying families of univalent functions, Bell distributions can help model and analyze the diversity and structure of these families. In probabilistic studies of univalent functions, Bell distributions can be used to model the behavior of random univalent functions, particularly in determining the likelihood of certain properties or outcomes. Researchers can explore the structural properties of univalent functions using combinatorial arguments tied to Bell numbers, particularly in complex function spaces. In studies extending to higher dimensions, the use of Bell distributions can facilitate the exploration of properties of univalent functions in multi-dimensional complex spaces, helping to understand their mappings and behaviors.

The importance of Bell distributions in geometric function theory lies in their ability to provide a combinatorial framework for analyzing the properties and behaviors of univalent functions. By leveraging these distributions, mathematicians can gain deeper insights into the geometric and analytic aspects of complex functions, enhancing both theoretical understanding and practical applications. Implementing Bell distribution-based methods in numerical simulations may help in solving complex problems in applied mathematics, physics, and engineering.

In a Bell distribution, approximately 0.68 of the data falls within one standard deviation of the mean, 0.95 falls within two standard deviations, and 0.997 falls within three standard deviations. The Bell distribution has a symmetric bell-shaped probability density function that resembles a normal distribution but with heavier tails. The mixing parameter p controls the degree of asymmetry of the distribution, with p = 0.5 corresponding to a perfectly symmetric distribution. The Bell distribution has applications in a wide range of fields, including finance, physics, engineering, and biology. It has been used, for example, to model the distribution of stock returns, the properties of noisy signals, and the behavior of biological systems. The Bell curve has many important applications in statistics, such as hypothesis testing, confidence intervals, and regression analysis. It is also used in fields such as finance, economics, and psychology, where it is used to model the behavior of complex systems and to make predictions based on empirical data. The distributions of random variables, which represent the distribution of probabilities over the values of the random variable, serve a fundamental role in the statistics and probability and are widely used to describe and model a variety of real-world occurrences [4]. Geometric function theory has used some of the fundamental distributions, including the Poisson, Pascal, logarithmic, binomial, and Borel distributions, see [1–3, 11, 20, 25].

In 2018, Castellares et al. introduced the Bell distribution [7], which is suitable for count data with over-dispersion. The Bell distribution is an improvement over the Bell numbers [5, 6]. The probability density function of a discrete random variable X, which follows the Bell distribution, is expressed as:

$$\mathcal{P}(X=m)=\frac{\vartheta^m e^{e^{(-\vartheta^2)+1}}\mathcal{B}_m}{m!},\quad m=1,2,3,\dots,$$

where $\mathcal{B}_m=\frac{1}{e}\sum\limits_{k=0}^{\infty}\frac{k^m}{m!}$ are the Bell numbers, $m\geqslant 2$, and $\vartheta>0$. Example of the Bell numbers are $\mathcal{B}_2=2$, $\mathcal{B}_3=5$, $\mathcal{B}_4=15$, and $\mathcal{B}_5=52$. Now, we introduce a new power series whose coefficients represent the probabilities of the Bell distribution

$$\mathcal{B}(\vartheta,z)=z+\sum_{n=2}^{\infty}\frac{\vartheta^{n-1}\,\mathcal{B}_n}{(n-1)!\,e^{\vartheta^2-1}}z^n,\quad (z\in U).$$

Next, we consider the linear operator $\mathbb{K}_\vartheta:\mathcal{A}\to\mathcal{A}$ defined by the convolution (or Hadamard product)

$$\mathbb{K}_\vartheta u(z)=\mathcal{B}(\vartheta,z)*u(z)=z+\sum_{n=2}^{\infty}\Theta(n)a_nz^n,$$

where

$$\Theta(n)=\frac{\vartheta^{n-1}\,\mathcal{B}_n}{(n-1)!\,e^{\vartheta^2-1}}. \tag{1.4}$$

Motivated by the work of [15, 16, 24], we define the new subclass of functions as follows.

Definition 1.1. The function $u(z)$ of the form (1.1) is in the class $\Omega(\mu,\gamma,\ell)$, if it satisfies the inequality

$$\Re\left\{\frac{z(\mathbb{K}_\vartheta u(z))'}{(1-\mu)z+\mu\mathbb{K}_\vartheta u(z)}-\gamma\right\}>\ell\left|\frac{z(\mathbb{K}_\vartheta u(z))'}{(1-\mu)z+\mu\mathbb{K}_\vartheta u(z)}-1\right|$$

for $0\leqslant\mu\leqslant 1$, $0\leqslant\gamma\leqslant 1$, and $\ell\geqslant 0$. Further we define $T\Omega(\mu,\gamma,\ell)=\Omega(\mu,\gamma,\ell)\cap T$.

The aim of present paper is to study the coefficient bounds, radii of close-to-convex, and starlikeness convex linear combinations and integral means inequalities of the $T\Omega(\mu,\gamma,\ell)$.

2. Coefficient bounds

Theorem 2.1. *A function u(z) of the form* (1.1) *is in* $\Omega(\mu,\gamma,\ell)$, *then*

$$\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)|a_n| \leqslant 1-\gamma, \tag{2.1}$$

where $0\leqslant\mu\leqslant 1$, $0\leqslant\gamma\leqslant 1$, $\ell\geqslant 0$, *and* $\Theta(n)$ *is given by* (1.4).

Proof. It suffices to show that

$$\ell\left|\frac{z(\mathbb{K}_{\vartheta}u(z))'}{(1-\mu)z+\mu\mathbb{K}_{\vartheta}u(z)}-1\right|-\Re\left\{\frac{z(\mathbb{K}_{\vartheta}u(z))'}{(1-\mu)z+\mu\mathbb{K}_{\vartheta}u(z)}-1\right\}\leqslant 1-\gamma.$$

We have

$$\begin{aligned}&\ell\left|\frac{z(\mathbb{K}_{\vartheta}u(z))'}{(1-\mu)z+\mu\mathbb{K}_{\vartheta}u(z)}-1\right|-\Re\left\{\frac{z(\mathbb{K}_{\vartheta}u(z))'}{(1-\mu)z+\mu\mathbb{K}_{\vartheta}u(z)}-1\right\}\\ &\leqslant(1+\ell)\left|\frac{z(\mathbb{K}_{\vartheta}u(z))'}{(1-\mu)z+\mu\mathbb{K}_{\vartheta}u(z)}-1\right|\\ &\leqslant\frac{(1+\ell)\sum_{n=2}^{\infty}(n-\mu)\Theta(n)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}\mu\Theta(n)|a_n||z|^{n-1}}\leqslant\frac{(1+\ell)\sum_{n=2}^{\infty}(n-\mu)\Theta(n)|a_n|}{1-\sum_{n=2}^{\infty}\mu\Theta(n)|a_n|}.\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$, if

$$\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)|a_n| \leqslant 1-\gamma,$$

and the proof is complete. □

Theorem 2.2. *Let* $0\leqslant\mu\leqslant 1$, $0\leqslant\gamma\leqslant 1$, $\ell\geqslant 0$, *then a function* u *of the form* (1.2) *is in the class* $T\Omega(\mu,\gamma,\ell)$ *if and only if*

$$\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)|a_n| \leqslant 1-\gamma, \tag{2.2}$$

where $\Theta(n)$ *is given by* (1.4) .

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $u\in T\Omega(\mu,\gamma,\ell)$ and z is real, then

$$\Re\left\{\frac{1-\sum_{n=2}^{\infty}n\Theta(n)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}\mu\Theta(n)a_nz^{n-1}}-\gamma\right\}>\ell\left|\frac{\sum_{n=2}^{\infty}(n-\mu)\Theta(n)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}\mu\Theta(n)a_nz^{n-1}}\right|.$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\ell)-\mu(\gamma+1)]\Theta(n)|a_n| \leqslant 1-\gamma.$$ □

Corollary 2.3. *If* $u(z)\in T\Omega(\mu,\gamma,\ell)$, *then*

$$|a_n|\leqslant\frac{1-\gamma}{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}, \tag{2.3}$$

where $0\leqslant\mu\leqslant 1$, $0\leqslant\gamma\leqslant 1$, $\ell\geqslant 0$, *and* $\Theta(n)$ *is given by* (1.4). *Equality holds for the function*

$$u(z)=z-\frac{1-\gamma}{[n(1+\ell)-\mu(\gamma+\ell)]\Theta(n)}z^n.$$

Theorem 2.4. *Let* $u_1(z) = z$ *and*

$$u_n(z) = z - \frac{1-\gamma}{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)} z^n, \quad n \geqslant 2. \tag{2.4}$$

Then $u(z) \in T\Omega(\mu, \gamma, \ell)$, *if and only if it can be expressed in the form*

$$u(z) = \sum_{n=1}^{\infty} w_n u_n(z), \;\; w_n \geqslant 0, \;\; \sum_{n=1}^{\infty} w_n = 1. \tag{2.5}$$

Proof. Suppose $u(z)$ can be written as in (2.5), then

$$u(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)} z^n.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{(1-\gamma)[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \leqslant 1.$$

Thus $u(z) \in T\Omega(\mu, \gamma, \ell)$. Conversely, let $u(z) \in T\Omega(\mu, \gamma, \ell)$, then using (2.3), we get

$$w_n = \frac{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{(1-\gamma)} a_n, \quad n \geqslant 2,$$

and $w_1 = 1 - \sum_{n=2}^{\infty} w_n$. Then we have $u(z) = \sum_{n=1}^{\infty} w_n u_n(z)$ and hence this completes the proof of Theorem. □

Theorem 2.5. *The class* $T\Omega(\mu, \gamma, \ell)$ *is a convex set.*

Proof. Let the function

$$u_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \;\; a_{n,j} \geqslant 0, \;\; j = 1, 2$$

be in the class $T\Omega(\mu, \gamma, \ell)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi u_1(z) + (1-\xi) u_2(z), \;\; 0 \leqslant \xi < 1,$$

is in the class $T\Omega(\mu, \gamma, \ell)$. Since

$$h(z) = z - \sum_{n=2}^{\infty} [\xi a_{n,1} + (1-\xi) a_{n,2}] z^n,$$

an easy computation with the aid of Theorem 2.2 gives

$$\begin{aligned} &\sum_{n=2}^{\infty} [n(1+\ell) - \mu(\gamma+\ell)] \xi \Theta(n) a_{n,1} + \sum_{n=2}^{\infty} [n(1+\ell) - \mu(\gamma+\ell)](1-\xi)\Theta(n) a_{n,2} \\ &\quad \leqslant \xi(1-\gamma) + (1-\xi)(1-\gamma) \leqslant (1-\gamma), \end{aligned}$$

which implies that $h \in T\Omega(\mu, \gamma, \ell)$. Hence $T\Omega(\mu, \gamma, \ell)$ is convex. □

3. Radii of close-to-convexity, starlikeness, and convexity

In this section, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $T\Omega(\mu,\gamma,\ell)$.

Theorem 3.1. *Let the function $u(z)$ defined by (1.2) belong to the class $T\Omega(\mu,\gamma,\ell)$, then $u(z)$ is close-to-convex of order $\delta (0 \leqslant \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{n \geqslant 2} \left[\frac{(1-\delta) \sum_{n=2}^{\infty} [n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{n(1-\gamma)} \right]^{1/n-1}, \quad n \geqslant 2.$$

The result is sharp, with the external function $u(z)$ given by (2.4).

Proof. Given $u \in T$ and u being close-to-convex of order δ, we have

$$|u'(z) - 1| < 1 - \delta. \tag{3.1}$$

For the left hand side of (3.1), we have

$$|u'(z) - 1| \leqslant \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1-\delta$,

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n |z|^{n-1} \leqslant 1.$$

Using the fact that $u(z) \in T\Omega(\mu,\gamma,\ell)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{1-\gamma} a_n \leqslant 1,$$

we can see that (3.1) is true, if

$$\frac{n}{1-\delta} |z|^{n-1} \leqslant \frac{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{1-\gamma},$$

or, equivalently

$$|z| \leqslant \left\{ \frac{(1-\delta)[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{n(1-\gamma)} \right\}^{1/n-1},$$

which completes the proof. □

Theorem 3.2. *Let the function $u(z)$ defined by (1.2) belong to the class $T\Omega(\mu,\gamma,\ell)$. Then $u(z)$ is starlike of order $\delta (0 \leqslant \delta < 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_{n \geqslant 2} \left[\frac{(1-\delta) \sum_{n=2}^{\infty} [n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{(n-\delta)(1-\gamma)} \right]^{1/n-1}.$$

The result is sharp, with external function $u(z)$ given by (2.4).

Proof. Given $u \in T$ and u being starlike of order δ, we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| < 1 - \delta. \tag{3.2}$$

For the left hand side of (3.2), we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \leqslant \sum_{n=2}^{\infty} \frac{(n-1)a_n|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n|z|^{n-1}}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n|z|^{n-1} < 1.$$

Using the fact that $u(z) \in T\Omega(\mu, \gamma, \ell)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{1-\gamma} a_n \leqslant 1,$$

we can say (3.2) is true, if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta}|z|^{n-1} \leqslant \frac{[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{1-\gamma},$$

or, equivalently

$$|z|^{n-1} \leqslant \frac{(1-\delta)[n(1+\ell) - \mu(\gamma+\ell)]\Theta(n)}{(n-\delta)(1-\gamma)},$$

which yields the starlikeness of the family. □

4. Integral means inequalities

In [21], Silverman found that the function $u_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjunctured [22] and settled in [23], that

$$\int_0^{2\pi} |u(re^{i\varphi})|^\tau d\varphi \leqslant \int_0^{2\pi} |u_2(re^{i\varphi})^n|^\tau d\varphi$$

for all $u \in T$, $\tau > 0$, and $0 < r < 1$. In [23], he also proved his conjuncture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T. Now, we prove Silverman's conjecture for the class of functions $T\Omega(\mu, \gamma, \ell)$. We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [14]. Two functions u and v, which are analytic in E, the function u is said to be subordinate to v in E, if there exists a function w analytic in E with $w(0) = 0$, $|w(z) < 1$, $(z \in E)$ such that $u(z) = v(w(z))$, $(z \in E)$. We denote this subordination by $u(z) \prec v(z)$ ($\prec$ denotes subordination).

Lemma 4.1. *If the function u and v are analytic in E with $u(z) \prec v(z)$, then for $\tau > 0$ and $z = re^{i\varphi}$, $0 < r < 1$,*

$$\int_0^{2\pi} |v(re^{i\varphi})|^\tau d\varphi \leqslant \int_0^{2\pi} |u(re^{i\varphi})|^\tau d\varphi.$$

Now, we discuss the integral means inequalities for functions u in $T\Omega(\mu, \gamma, \ell)$.

Theorem 4.2. $u \in T\Omega(\mu,\gamma,\ell)$, $0 \leqslant \mu < 1$, $0 \leqslant \gamma < 1$, *and* $u_2(z)$ *be defined by*

$$u_2(z) = z - \frac{1-\gamma}{\phi(2)} z^2. \tag{4.1}$$

Proof. For $u(z) = z - \sum_{n=2}^{\infty} a_n z^n$, (4.1) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^{\tau} d\varphi \leqslant \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\phi(2)} z\right|^{\tau} d\varphi.$$

By Lemma 4.1, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\phi(2)} z.$$

Assuming

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\gamma}{\phi(2)} w(z),$$

and using (2.2), we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\phi(n)}{1-\gamma} a_n z^{n-1}\right| \leqslant |z| \sum_{n=2}^{\infty} \frac{\phi(n)}{1-\gamma} a_n \leqslant |z| < 1,$$

where

$$\phi(n) = [n(1+\ell) - \mu(\gamma+1)\Theta(n).$$

This completes the proof. □

5. Neighborhood for the class $T\Omega(\mu,\gamma,\ell)$

Following the earlier investigations by Goodman [9], Ruscheweyh [19], Darwish et al [8], Kazimogulu [13], and others, we define the (n,δ)-neighborhood of a function $u(z) \in T$ by

$$N_\delta(u) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n\,|a_n - b_n| \leqslant \delta \right\}. \tag{5.1}$$

In particular, if $e(z) = z$, we have

$$N_\delta(e) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n\,|b_n| \leqslant \delta \right\}. \tag{5.2}$$

Now we determine the neighborhood for each of the class $T\Omega(\mu,\gamma,\ell)$, which we define as follows. A function $u \in T$ is said to be in the class $T\Omega(\mu,\gamma,\ell,\xi)$ if there exists a function $g \in T\Omega(\mu,\gamma,\ell)$ such that

$$\left|\frac{u(z)}{g(z)} - 1\right| \leqslant 1 - \xi \quad (z \in U, \ \ 0 \leqslant \xi < 1). \tag{5.3}$$

Theorem 5.1. *If* $g \in T\Omega(\mu,\gamma,\ell)$ *and*

$$\xi = 1 - \frac{\delta[2(1+\ell) - \mu(\gamma+1)]\Theta(2)}{2[(2(1+\ell) - \mu(\gamma+\ell)) - (1-\gamma)]}, \tag{5.4}$$

then $N_\delta(g) \subset T\Omega(\mu,\gamma,\ell,\xi)$.

Proof. Suppose $u(z) \in N_\delta(g)$. We find from (2.1) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leqslant \delta,$$

which implies that

$$\sum_{n=2}^{\infty} |a_n - b_n| \leqslant \frac{\delta}{2} \quad (n \in \mathbb{N}). \tag{5.5}$$

Next, since $g(z) \in T\Omega(\mu, \gamma, \ell)$, we have

$$\sum_{n=2}^{\infty} |b_n| \leqslant \frac{1-\gamma}{[2(1+\ell) - \mu(\gamma+\ell)]\Theta(2)}, \tag{5.6}$$

so that

$$\left|\frac{u(z)}{g(z)} - 1\right| < \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} |b_n|} \leqslant \frac{\delta}{2}\left[\frac{[2(1+\ell) - \mu(\gamma+\ell)]\Theta(2)}{[2(1+\ell) - \mu(\gamma+\ell)]\Theta(2) - (1-\gamma)}\right] \leqslant 1 - \xi,$$

provided that ξ is given by (5.1). Thus, $u(z) \in T\Omega(\mu, \gamma, \ell, \xi)$ for ξ given by (5.1). This completes the proof of the Theorem. □

6. Conclusion

The contributions of Bell distributions to geometric function theory are profound, offering combinatorial, analytical, and probabilistic insights that enhance our understanding of univalent functions and their properties. By integrating these distributions into the study of complex analysis, mathematicians can develop more robust theoretical frameworks and practical applications. The future scope of the Bell distributions in the context of analytic functions holds significant potential for advancing both theoretical and practical aspects of mathematics. By exploring new applications, refining existing theories, and fostering interdisciplinary connections, researchers can uncover deeper insights and develop innovative methods in both combinatorial and analytic contexts.

Acknowledgment

The authors record their sincere thanks to the referees for their insightful suggestions to improve the results in present form.

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