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Qualitative analysis of Caputo fractional delayed difference system: a novel delayed discrete fractional sine and cosine-type function

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Abstract

In this paper, we provide an explicit solution for the homogeneous fractional delay oscillation difference equation with an order 2δ ranging from 1 to 2. This solution is achieved through the construction of discrete sine and cosine-type delayed matrix functions. Subsequently, we employ the discrete Laplace transform technique, a powerful method for handling nonhomogeneous terms, to investigate the solution of the corresponding nonhomogeneous equation. The study then delves into the Ulam-Hyers-type stabilities of the homogeneous equation, leveraging the representation of the solution. To validate the stability theory, we illustrate a numerical example. Finally, we extend our analysis by presenting an exact solution for the nonhomogeneous fractional difference equation with $1 < 2\delta < 2$, utilizing the discrete two-parameter delayed sine and cosine-type function.

Keywords: Linear system, fractional difference, time-delay, nabla sine cosine, discrete delayed perturbation.

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1. Introduction

In the last few decades, the field of fractional calculus, sometimes referred to as non-integer calculus, has become increasingly popular because of its diverse applications. There has been an increasing awareness that fractional differential systems are more suitable for modeling real-world issues in various fields like mathematical physics, biophysics, electrochemistry, and engineering; see [14, 16, 23, 28, 29, 34, 46, 48, 49] and the references listed. The solution to linear fractional-order differential equations can be expressed in terms of Mittag-Leffler functions. Various Mittag-Leffler functions and their properties have been developed to analyze the solution's behavior.

Many nonlinear mathematical models are exclusive to discrete time domains, leading to a rise in interest towards discrete fractional-order systems. It is important to note that discrete fractional-order calculus is a numerical formula in discrete form that does not introduce any numerical errors [51]. As a result, discrete fractional-order calculus has become a popular subject of discussion, with several noteworthy

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results [4, 5, 15, 24, 29, 52, 53] being published. A significant development in recent times has been the formulation of explicit solutions for fractional-order difference systems using various discrete Mittag-Leffler functions [1–3, 6, 7, 11, 14, 50]. These functions act as counterparts to the continuing Mittag-Leffler function.

The inclusion of both present and past states in differential equations facilitates the modeling of systems with memory, enabling accurate representation of phenomena like automatic steering, control, and stabilization; see [8–10, 33, 36, 42, 45]. In order to examine the stability and controllability of fractional-order delay differential systems, different types of delayed Mittag-Leffler matrix functions have been developed as the fractional-order analog of delayed exponential matrix functions.

Despite the extensive research on fractional retarded differential systems in continuous time, covering various aspects such as stability and controllability, there is a lack of studies on fractional delayed difference systems of order $1 < \alpha \leqslant 2$ in discrete-time. Although there is sufficient work available for the case of order $0 < \alpha \leqslant 1$, see [18, 19, 21, 22, 24, 26, 30, 31, 41, 44, 47], the literature in this area is not as extensive. This manuscript endeavors to rectify the insufficiency in this matter. It is widely acknowledged that the sine and cosine functions, as two distinct trigonometric functions, serve as the solutions for second-order differential systems. In the study [32], a solution formula for the Cauchy problem in a second-order linear delayed system is proposed using delayed sine and cosine matrices. These trigonometric functions play a crucial role in investigating the controllability and stability of second-ordered differential systems with time-delay in continuous time [12, 13, 17, 20, 37–40, 43]. From what we know, there are scarce studies in discrete-time that relate to the ones in the works mentioned above. The inspiration from the above discussions has led us to focus on investigating the below Caputo fractional delayed difference system of order $1 < 2\alpha \leqslant 2$ with noncommutative coefficient matrices,

$$\begin{cases} {}^{C}\nabla_{0}^{2\delta} v(\sigma) = \Delta_1 v(\sigma) + \Delta_2 v(\sigma - r) + f(\sigma), & \sigma \in \mathbb{Z}_1, \\ v(\sigma) = \psi(\sigma), \ \nabla v(\sigma) = \nabla\psi(\sigma), & \sigma \in \mathbb{Z}_{1-r}^{0}, \end{cases} \tag{1.1}$$

where ${}^{C}\nabla_{0}^{2\delta}$ is the Caputo fractional difference of order $1 < 2\delta < 2$, $r \in \mathbb{Z}_2$ is a delay, $\psi : \mathbb{Z}_{1-r}^{0} \to \mathbb{R}^n$ is an initial function, $\Delta_1, \Delta_2 \in \mathbb{R}^{n\times n}$ are square matrices, $z : \mathbb{Z}_1 \to \mathbb{R}^n$, and the function $f : \mathbb{Z}_1 \to \mathbb{R}^n$ is continuous.

The paper highlights various significant discoveries and contributions.

- A novel matrix function, known as the discrete delayed Mittag-Leffler matrix function, is presented in this study. This function is derived from two noncommutative matrices and serves as a generalization of the conventional discrete delayed exponential matrix function.
- A discrete delayed Mittag-Leffler matrix function is utilized to derive a representation of an analytical formula for solving linear/semilinear fractional delayed difference systems.
- An established criterion is presented for determining the stability of linear/nonlinear fractional delayed difference systems. This criterion is derived from the exact solution that has been derived.
- The research delves into the investigation of solutions for nonlinear fractional delayed difference equations, specifically exploring their existence and uniqueness.

2. Groundwork

Within this section, we showcase the tools that are currently accessible in the literature. $\mathbb{Z}_a = \{a, a+1, a+2, \ldots\}$, $\mathbb{Z}^a = \{\ldots, a-2, a-1, a\}$, $\mathbb{Z}_a^b = \{a, a+1, a+2, \ldots, b\}$, where $a, b \in \mathbb{R}$ (real numbers) with $b - a \in \mathbb{Z}_1$. $\mathbb{R}^n$ is an n-dimensional real space endowed with the norm $\|.\|$, and $\mathbb{R}^{n\times n}$ is the set of all square matrices whose entries are real numbers. We also use the same symbol $\|.\|$ as an arbitrary matrix norm on $\mathbb{R}^{n\times n}$.

Definition 2.1 ([42]). The determining equation $R(i,j)$ is of the following recursive form

$$R(i+1,j) = \Delta_1 R(i,j) + \Delta_2 R(i,j-1), \tag{2.1}$$

and

$$R(-1,j) = R(i,-1) = \Theta, \ R(1,0) = I,$$

for $i,j \in \mathbb{Z}_0$.

Remark 2.2 ([42]). By employing the above recursive equation, one can easily reach to the explicit form in the following table:

	$j=0$	$j=1$	$j=2$	$j=3$	$\cdots$	$j=p$
$R(1,j)$	I	Θ	Θ	Θ	$\cdots$	Θ
$R(2,j)$	Δ_1	Δ_2	Θ	Θ	$\cdots$	
$R(3,j)$	${\Delta_1}^2$	$\Delta_1\Delta_2+\Delta_2\Delta_1$	Δ_2^2	Θ	$\cdots$	Θ
$R(4,j)$	${\Delta_1}^3$	$\Delta_1(\Delta_1\Delta_2+\Delta_2\Delta_1)+\Delta_2\Delta_1^2$	$\Delta_1\Delta_2^2+\Delta_2(\Delta_1\Delta_2+\Delta_2\Delta_1)$	Δ_2^3	$\cdots$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	Θ
$R(p+1,j)$	Δ_1^p	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Δ_2^p

Lemma 2.3. *It is apparent that* $R(i+1,j) = \Theta$ *provided that* $j \leqslant i+1$.

In the case of constant coefficient matrices that commute, the following observation can be made.

Remark 2.4. In the light of the permutable matrices Δ_1 and Δ_2, we have

$$R(i+1,j) = \binom{i}{j} {\Delta_1}^{i-j} {\Delta_2}^j, \quad i,j \in \mathbb{Z}_0.$$

As stated in the following lemma, for non-permutable matrices Δ_1 and Δ_2 and $r=0$, $R(i+1,j)$ exhibits a generalization of binomial formula.

Lemma 2.5. *The ensuing generalized binomial formula remains true:*

$$\left(\Delta_1 + (1-s)^r \Delta_2\right)^l = \sum_{j=0}^{l} R(l+1,j)(1-s)^{jr}.$$

Proof. It is so easy to prove by using the mathematical induction, so we omit it. □

Lemma 2.6. *We have*

$$\left(s^\delta I - \Delta_1 - (1-s)^r \Delta_2\right)^{-1} = \sum_{l=0}^{\infty} \sum_{j=0}^{l} R(l+1,k)(1-s)^{kr} \frac{1}{s^{l\delta+\delta}},$$

where I *is the identity matrix.*

Proof. We have

$$\begin{aligned}\left(s^\delta I - \Delta_1 - (1-s)^r \Delta_2\right)^{-1} &= s^{-\delta}\left(I - s^{-\delta}\left(\Delta_1 + (1-s)^r \Delta_2\right)\right)^{-1} \\ &= \sum_{l=0}^{\infty} \left(\Delta_1 + (1-s)^r \Delta_2\right)^l \frac{1}{s^{l\delta+\delta}} = \sum_{l=0}^{\infty}\sum_{j=0}^{l} R(l+1,k)(1-s)^{kr} \frac{1}{s^{l\delta+\delta}},\end{aligned}$$

where $\|s^{-\delta}\left(\Delta_1 + (1-s)^r \Delta_2\right)\| < 1$ has been used. □

The present article introduces particular notations and provides an overview of essential findings pertaining to nabla calculus, which is employed in the current study.

Definition 2.7 ([27])**.** The generalized rising function is characterized by

$$\sigma^{\bar{r}} = \frac{\Gamma(\sigma+r)}{\Gamma(\sigma)},$$

whenever the equation is valid for the values of σ and r where the right-hand side makes sense. Specifically, $0^{\bar{r}} = 0$.

Definition 2.8 ([27])**.** The (nabla) fractional Taylor monomial $J_\delta(\sigma, a)$ of order δ can be expressed as follows

$$J_\delta(\sigma, a) = \frac{(\sigma-a)^{\bar{\delta}}}{\Gamma(\delta+1)},$$

where the side on the right of the equation makes sense for $\delta \notin \mathbb{Z}^{-1}$.

Below, we outline the nabla fractional sum using the nabla fractional Taylor monomial, the nabla Riemann-Liouville fractional difference, and the nabla Caputo fractional difference, the (nabla) Leibniz Formula.

Definition 2.9 ([27])**.** The (nabla) fractional sum is defined as follows

$$\nabla_a^{-\delta} v(\sigma) = \int_a^\sigma J_{\delta-1}(\sigma, \rho(s)) v(s) \nabla s = \sum_{s=a+1}^{\sigma} J_{\delta-1}(\sigma, \rho(s)) v(s),$$

the nabla fractional difference is defined as follows

$$\nabla_a^{\delta} v(\sigma) = \int_a^\sigma J_{-\delta-1}(\sigma, \rho(s)) v(s) \nabla s = \sum_{s=a+1}^{\sigma} J_{-\delta-1}(\sigma, \rho(s)) v(s),$$

where $\sigma \in \mathbb{Z}_a$, $\rho(s) = s-1$, $z : \mathbb{Z}_{a+1} \to \mathbb{R}$ and $\delta > 0$. In particular, $\nabla_a^{\delta} v(a) = 0$.

Definition 2.10 ([27])**.** Suppose that $z : \mathbb{Z}_{a+1} \to \mathbb{R}$. For $\sigma \in \mathbb{Z}_a$, the (nabla) Caputo fractional difference of order $1 < \delta < 2$ is defined as follows

$$^{C}\nabla_a^{\delta} v(\sigma) = \nabla_a^{-(2-\delta)} \nabla^2 v(\sigma) = \int_a^\sigma J_{-\delta+1}(\sigma, \rho(s)) \nabla^2 v(s) \nabla s = \sum_{s=a+1}^{\sigma} J_{-\delta+1}(\sigma, \rho(s)) \nabla^2 v(s).$$

Theorem 2.11 ([27])**.** *Let $z : \mathbb{Z}_a \times \mathbb{Z}_{a+1} \to \mathbb{R}$. Then*

$$\nabla \left(\int_a^\sigma v(\sigma, s) \nabla s \right) = \int_a^\sigma \nabla_\sigma v(\sigma, s) \nabla s + v(\rho(\sigma), \sigma), \quad \sigma \in \mathbb{Z}_{a+1}.$$

The formula for the well-known composition rule of distinct fractional sums is provided in the theorem statement below.

Theorem 2.12 ([27])**.** *Suppose that $z : \mathbb{Z}_{a+1} \to \mathbb{R}$ and $\delta, \mu > 0$. Then $\nabla_a^{-\delta} \nabla_a^{-\mu} v(\sigma) = \nabla_a^{-\delta-\mu} v(\sigma)$, $\sigma \in \mathbb{Z}_a$.*

Lemma 2.13 ([27])**.** *Assume that $\mu \in \mathbb{R}$ and $\delta > 0$ with $\mu - \delta - 1 \in \mathbb{Z}_0$. Then*

$$\nabla_a^{-\delta} J_{\mu-1}(\sigma, a) = J_{\mu+\delta-1}(\sigma, a), \quad \sigma \in \mathbb{Z}_a,$$

and

$$\nabla_a^{\delta} J_{\mu-1}(\sigma, a) = J_{\mu-\delta-1}(\sigma, a) = {}^{C}\nabla_a^{\delta} J_{\mu-1}(\sigma, a), \quad \sigma \in \mathbb{Z}_a,$$

and for $0 < \mu < 1$ and $t \in \mathbb{Z}_{a+1}$,

$$\nabla_a^{-\mu} \nabla_{\rho(a)}^{\mu} v(\sigma) = v(\sigma) - J_{\mu-1}(\sigma, \rho(a)) v(a).$$

Moving forward, we shall examine the key application of the Laplace transform. Much like its usage in classical calculus, the Laplace transform presents a refined and elegant way for depicting IVPs associated with fractional difference equations in a sophisticated manner.

Theorem 2.14 ([27]). *The Laplace transform in the discrete sense is given as follows*

$$\mathfrak{L}_a \{\nu\}(s) = \sum_{\sigma=1}^{\infty} (1-s)^{\sigma-1} \nu(a+\sigma).$$

Definition 2.15 ([27]). A function $z : \mathbb{Z}_{a+1} \to \mathbb{R}$ is exponentially ordered $\delta > 0$ whether there are real numbers $T \in \mathbb{Z}_{a+1}$ and $S > 0$ such that the absolute value of $\nu(\sigma)$ is less than or equal to $S\delta^{\sigma}$, for every $\sigma \in \mathbb{Z}_T$.

Lemma 2.16 ([27]). *When dealing with a function $z : \mathbb{Z}_{a+1} \to \mathbb{R}$ that has exponential order δ, the Laplace transform is well-defined as long as $|1-s| < \delta$.*

Definition 2.17 ([27]). Let all $\sigma \in \mathbb{Z}_{a+1}$ and $\nu, y : \mathbb{Z}_{a+1} \to \mathbb{R}$. The convolution product of two functions is given as follows

$$(\nu * y)(\sigma) = \int_a^{\sigma} \nu(\sigma - \rho(k) + a) y(k) \nabla k.$$

Theorem 2.18 ([27]). *We have that*

$$\mathfrak{L}_a \{\nu * y\}(s) = \mathfrak{L}_a \{\nu\}(s) \mathfrak{L}_a \{y\}(s),$$

where $z, y : \mathbb{Z}_a \to \mathbb{R}$.

Theorem 2.19 ([27]). *Assume $\delta > 0$ and the nabla Laplace transform of $\nu : \mathbb{Z}_{a+1} \to \mathbb{R}$ converges for $|1-s| < r$ for some $r > 0$. Then*

$$\mathfrak{L}_a \left\{\nabla_a^{-\delta} \nu\right\}(s) = s^{-\delta} \mathfrak{L}_a \{\nu\}(s)$$

for $|1-s| < \min\{1, r\}$.

Theorem 2.20 ([27]). *Assume $\nu : \mathbb{Z}_{a-n+1} \to \mathbb{R}$ is of exponential order $r > 0$. Then*

$$\mathfrak{L}_a \{\nabla_a^n \nu\}(s) = s^n \mathfrak{L}_a \{\nu\}(s) - \sum_{m=1}^{n} s^{n-m} \nabla_a^{m-1} \nu(a)$$

for $|1-s| < r$, for each $n \in \mathbb{Z}_1$.

Lemma 2.21. *Assume $\delta \in (1, 2)$. If the Laplace transform of $\nu : \mathbb{Z}_{a-1} \to \mathbb{R}$ converges for $|1-s| < r$ for some positive r, then one has*

$$\mathfrak{L}_a \left\{{}^{C}\nabla_a^{\delta} \nu\right\}(s) = s^{\delta} \mathfrak{L}_a \{\nu\}(s) - s^{\delta-1} \nu(a) - s^{\delta-2} \nabla \nu(a).$$

Proof. Follows form the Definition 2.10 and Theorems 2.19 and 2.20. □

Lemma 2.22 ([27]). *Assume that $\delta \in \mathbb{C} \backslash \mathbb{Z}_{-\infty}^{\infty}$. One has*

$$\mathfrak{L}_a \{J_\delta(\cdot, a)\}(s) = \frac{1}{s^{\delta+1}}, \quad |1-s| < 1.$$

The subsequent lemmas will demonstrate various fresh features of the discrete Laplace transform.

Theorem 2.23. *One has*

$$\begin{aligned}\mathfrak{L}_{a}\{\nu(\cdot-r)\}(s) &= \sum_{\sigma=1}^{\infty}(1-s)^{\sigma-1}\nu(a+\sigma-r) = \sum_{\sigma=1-r}^{\infty}(1-s)^{\sigma+r-1}\nu(a+\sigma)\\ &= \sum_{\sigma=1}^{\infty}(1-s)^{\sigma+r-1}\nu(a+\sigma) + \sum_{\sigma=1-r}^{0}(1-s)^{\sigma+r-1}\nu(a+\sigma)\\ &= (1-s)^{r}\mathfrak{L}_{a}\{\nu\}(s) + \sum_{\sigma=1-r}^{0}(1-s)^{\sigma+r-1}\nu(a+\sigma).\end{aligned}$$

Lemma 2.24. *Assume that* $\delta \in \mathbb{C}\backslash\mathbb{Z}_{-\infty}^{\infty}$. *One has*

$$\mathfrak{L}_{0}\left\{\frac{(\cdot-jr)_{+}^{\overline{m\delta+\alpha-1}}}{\Gamma(m\delta+\alpha)}\right\}(s) = \frac{(1-s)^{jr}}{s^{m\delta+\alpha}}, \quad |1-s|<1,$$

where $j, m \in \mathbb{Z}_0$, $\alpha \in \mathbb{R}$ *and* $(\sigma)_{+} = \max\{\sigma, 0\}$.

Proof. Indeed,

$$\begin{aligned}\mathfrak{L}_{0}\left\{\frac{(\cdot-jr)_{+}^{\overline{m\delta+\alpha-1}}}{\Gamma(m\delta+\alpha)}\right\}(s) &= \sum_{\sigma=1}^{\infty}(1-s)^{\sigma-1}\frac{(\sigma-jr)_{+}^{\overline{m\delta+\alpha-1}}}{\Gamma(m\delta+\alpha)} \quad \sigma-jr=l\\ &= (1-s)^{jr}\sum_{l=-pr}^{\infty}(1-s)^{l-1}\frac{(l)_{+}^{\overline{m\delta+\alpha-1}}}{\Gamma(m\delta+\alpha)}\\ &= (1-s)^{jr}\sum_{l=1}^{\infty}(1-s)^{l-1}\frac{(l)_{+}^{\overline{m\delta+\alpha-1}}}{\Gamma(m\delta+\alpha)} = \frac{(1-s)^{jr}}{s^{m\delta+\alpha}}.\end{aligned}$$

□

Lemma 2.25. *Assume* $\|\Delta_1\|<1$, $\delta>0$, *and* $\mu \in \mathbb{R}$. *Then*

$$\mathfrak{L}_{0}\left\{E_{\delta,\mu}^{\Delta_1}(\cdot,a)\right\}(s) = s^{\delta-\mu-1}\left(s^{\delta}I-\Delta_1\right)^{-1}$$

for $|1-s|<1$, $\left|s^{\delta}\right|>\|\Delta_1\|$.

Lemma 2.26. *Assume* $\|\Delta_1\|<1$, $\delta>0$, *and* $\mu \in \mathbb{R}$. *Then*

$$\begin{aligned}\mathfrak{L}_{0}\left\{\mathfrak{S}_{\delta,\delta-\mu,r}^{\Delta_1,\Delta_2}(\cdot-m)\right\}(s) &= (1-s)^{m}s^{\mu}\left(s^{2\delta}I-\Delta_1-(1-s)^{r}\Delta_2\right)^{-1},\\ \mathfrak{L}_{0}\left\{\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\cdot-m)\right\}(s) &= (1-s)^{m}\left(s^{2\delta}I-\Delta_1-(1-s)^{r}\Delta_2\right)^{-1},\\ (1-s)^{m}s^{2\delta-1}\left(s^{2\delta}I-\Delta_1-(1-s)^{r}\Delta_2\right)^{-1} &= \mathfrak{L}_{0}\left\{\mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\cdot-m)\right\}(s),\\ (1-s)^{m}s^{2\delta-2}\left(s^{2\delta}I-\Delta_1-(1-s)^{r}\Delta_2\right)^{-1} &= \mathfrak{L}_{0}\left\{\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\cdot-m)\right\}(s),\end{aligned}$$

for $|1-s|<1$, $\left|s^{2\delta}\right|>\|\Delta_1+(1-s)^{r}\Delta_2\|$, $m\in\mathbb{Z}_0$.

Proof. Under given assumptions, one easily calculates

$$(1-s)^{m}s^{\mu}\left(s^{2\delta}I-\Delta_1-(1-s)^{r}\Delta_2\right)^{-1} = \sum_{l=0}^{\infty}\sum_{j=0}^{l}R(l+1,j)(1-s)^{m+jr}\frac{s^{\mu}}{s^{2l\delta+2\delta}}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}\sum_{j=0}^{l} R(l+1,j)\, \mathfrak{L}_0 \left\{ \frac{(\cdot - m - jr)_+^{\overline{2l\delta+2\delta-\mu-1}}}{\Gamma(2l\delta+2\delta-\mu)} \right\}(s) \\
&= \mathfrak{L}_0 \left\{ \sum_{l=0}^{\infty}\sum_{j=0}^{l} R(l+1,j) \frac{(\cdot - m - jr)_+^{\overline{2l\delta+2\delta-\mu-1}}}{\Gamma(2l\delta+2\delta-\mu)} \right\}(s) \\
&= \mathfrak{L}_0 \left\{ \mathfrak{S}_{\delta,\delta-\mu,r}^{\Delta_1,\Delta_2}(\sigma - m) \right\}.
\end{aligned}
$$

The proofs of the remnant identities are omitted because they can be proved in the same manner as the first one. □

3. Delay discrete fractional matrix functions

Within this section, we present the discrete fractional cosine-type and sine-type delayed matrix functions for non-permutable real matrices Δ_1 and Δ_2. These functions are utilized to derive the precise solution of the Caputo fractional delayed difference system.

Definition 3.1 ([27]). Let $\alpha \in \mathbb{R}$, $\|\Delta_1\| < 1$, $\delta > 0$. The discrete fractional Mittag-Leffler matrix function is defined as

$$
E_{\delta,\alpha}^{\Delta_1}(\sigma, a) := \sum_{i=0}^{\infty} \Delta_1^i J_{i\delta+\alpha-1}(\sigma, a), \quad \sigma \in \mathbb{Z}_0.
$$

Definition 3.2. The discrete fractional cosine-type delayed matrix function $\mathfrak{C}_{\delta,\mu,r}^{\Delta_1,\Delta_2}$ generated by Δ_1, Δ_2 is defined as follows

$$
\mathfrak{C}_{\delta,\mu,r}^{\Delta_1,\Delta_2}(\sigma) := \begin{cases} \Theta, & \sigma \in \mathbb{Z}^{-1}, \\ \sum\limits_{i=0}^{\infty} \Delta_1^i \frac{(\sigma)_+^{\overline{2i\delta+\mu-1}}}{\Gamma(2i\delta+\mu)} + \sum\limits_{i=1}^{\infty} R(i+1,1) \frac{(\sigma - r)_+^{\overline{2i\delta+\mu-1}}}{\Gamma(2i\delta+\mu)} & \\ \quad + \cdots + \sum\limits_{i=p}^{\infty} R(i+1,p) \frac{(\sigma - pr)_+^{\overline{2i\delta+\mu-1}}}{\Gamma(2i\delta+\mu)}, & \sigma \in \mathbb{Z}_{pr}^{(p+1)r}, \end{cases}
$$

Definition 3.3. The discrete fractional sine-type delayed matrix function $\mathfrak{S}_{\delta,\mu,r}^{\Delta_1,\Delta_2}$ generated by Δ_1, Δ_2 is defined as follows

$$
\mathfrak{S}_{\delta,\mu,r}^{\Delta_1,\Delta_2}(\sigma) := \begin{cases} \Theta, & \sigma \in \mathbb{Z}^{-1}, \\ \sum\limits_{i=0}^{\infty} \Delta_1^i \frac{(\sigma)_+^{\overline{(2i+1)\delta+\mu-1}}}{\Gamma((2i+1)\delta+\mu)} + \sum\limits_{i=1}^{\infty} R(i+1,1) \frac{(\sigma - r)_+^{\overline{(2i+1)\delta+\mu-1}}}{\Gamma((2i+1)\delta+\mu)} & \\ \quad + \cdots + \sum\limits_{i=p}^{\infty} R(i+1,p) \frac{(\sigma - pr)_+^{\overline{(2i+1)\delta+\mu-1}}}{\Gamma((2i+1)\delta+\mu)}, & \sigma \in \mathbb{Z}_{pr}^{(p+1)r}, \end{cases}
$$

where Θ is the zero matrix.

Definition 3.4. The discrete cosine-type and sine-type delayed matrix function $\mathfrak{C}_{\delta,\mu,r}^{\Delta_2}$, generated by Δ_2 are defined as follows

$$
\mathfrak{C}_{\delta,\mu,r}^{\Delta_2}(\sigma) := \begin{cases} \Theta, & \sigma \in \mathbb{Z}^{-1}, \\ \frac{(\sigma)_+^{\overline{\mu-1}}}{\Gamma(\mu)} + \Delta_2 \frac{(\sigma - r)_+^{\overline{2\delta+\mu-1}}}{\Gamma(2\delta+\mu)} + \cdots + \Delta_2^p \frac{(\sigma - pr)_+^{\overline{2p\delta+\mu-1}}}{\Gamma(2p\delta+\mu)}, & \sigma \in \mathbb{Z}_{pr}^{(p+1)r}, \end{cases}
$$

$$
\mathfrak{S}_{\delta,\mu,r}^{\Delta_2}(\sigma) := \begin{cases} \Theta, & \sigma \in \mathbb{Z}^{-1}, \\ \frac{(\sigma)_+^{\overline{\delta+\mu-1}}}{\Gamma(\delta+\mu)} + \Delta_2 \frac{(\sigma - r)_+^{\overline{3\delta+\mu-1}}}{\Gamma(3\delta+\mu)} + \cdots + \Delta_2^p \frac{(\sigma - pr)_+^{\overline{(2p+1)\delta+\mu-1}}}{\Gamma((2p+1)\delta+\mu)}, & \sigma \in \mathbb{Z}_{pr}^{(p+1)r}. \end{cases}
$$

Remark 3.5. It is clear that $\mathfrak{C}^{\Delta_2}_{\delta,\mu,r}(\sigma) = \mathfrak{C}^{\Theta,\Delta_2}_{\delta,\mu,r}(\sigma)$ and $\mathfrak{S}^{\Delta_2}_{\delta,\mu,r}(\sigma) = \mathfrak{S}^{\Theta,\Delta_2}_{\delta,\mu,r}(\sigma)$.

We can reexpress the delayed discrete fractional sine and cosine-type matrix functions $\mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,\mu,r}$, $\mathfrak{C}^{\Delta_1,\Delta_2}_{\delta,\mu,r}$ in terms of the fractional Taylor monomial as follows $J_\delta(\sigma, a)$,

$$\mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,\mu,r}(\sigma) := \begin{cases} \Theta, & \sigma \in \mathbb{Z}^{-1}, \\ \sum_{i=0}^{\infty} \Delta_1^i J_{(2i+1)\delta+\mu-1}(\sigma, 0) + \sum_{i=1}^{\infty} R(i+1, 1) J_{(2i+1)\delta+\mu-1}(\sigma, r) & \\ \quad + \cdots + \sum_{i=p}^{\infty} R(i+1, p) J_{(2i+1)\delta+\mu-1}(\sigma, pr), & \sigma \in \mathbb{Z}^{(p+1)r}_{pr}. \end{cases}$$

$$\mathfrak{C}^{\Delta_1,\Delta_2}_{\delta,\mu,r}(\sigma) := \begin{cases} \Theta, & \sigma \in \mathbb{Z}^{-1}, \\ \sum_{i=0}^{\infty} \Delta_1^i J_{2i\delta+\mu-1}(\sigma, 0) + \sum_{i=1}^{\infty} R(i+1, 1) J_{2i\delta+\mu-1}(\sigma, r) & \\ \quad + \cdots + \sum_{i=p}^{\infty} R(i+1, p) J_{2i\delta+\mu-1}(\sigma, pr), & \sigma \in \mathbb{Z}^{(p+1)r}_{pr}. \end{cases}$$

Lemma 3.6. *We have the following inequalities*

$$\left\| \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,\mu,r}(\sigma) \right\| \leqslant \mathfrak{S}^{\|\Delta_1\|,\|\Delta_2\|}_{\delta,\mu,r}(\sigma), \quad \left\| \mathfrak{C}^{\Delta_1,\Delta_2}_{\delta,\mu,r}(\sigma) \right\| \leqslant \mathfrak{C}^{\|\Delta_1\|,\|\Delta_2\|}_{\delta,\mu,r}(\sigma).$$

Proof. Based on the properties of norms, one can easily obtain

$$\begin{aligned} \left\| \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,\mu,r}(\sigma) \right\| &\leqslant \sum_{i=0}^{\infty} \|\Delta_1\|^i J_{(2i+1)\delta+\mu-1}(\sigma, 0) + \sum_{i=1}^{\infty} \binom{i}{1} \|\Delta_1\|^{i-1} \|\Delta_2\| J_{(2i+1)\delta+\mu-1}(\sigma, r) \\ &\quad + \cdots + \sum_{i=p}^{\infty} \binom{i}{p} \|\Delta_1\|^{i-p} \|\Delta_2\|^p J_{(2i+1)\delta+\mu-1}(\sigma, pr) = \mathfrak{S}^{\|\Delta_1\|,\|\Delta_2\|}_{\delta,\mu,r}(\sigma). \end{aligned}$$

The proof of the latter is so easy to prove observing that of the former. □

4. The explicit solution

The main goal of this section is to explore a representation of an exact solution to the linear Caputo fractional delayed difference system.

The initial focus is on the homogeneous Caputo fractional delayed difference system outlined:

$$\begin{cases} {}^C\nabla_0^{2\delta} Z(\sigma) = \Delta_1 Z(\sigma) + \Delta_2 Z(\sigma - r), & \sigma \in \mathbb{Z}_1, \\ Z(\sigma) = \Theta, & \sigma \in \mathbb{Z}^0_{1-r}, \quad r \in \mathbb{Z}_2, \end{cases}$$

where ${}^C\nabla_0^{2\delta}$ is the Caputo fractional difference of order $1 < 2\delta \leqslant 2$, $Z : \mathbb{Z}_1 \to \mathbb{R}^n$, $r \in \mathbb{Z}_2$ is a delay parameter, Δ_1, Δ_2 are $n \times n$ real matrices.

Theorem 4.1.

(a) *The delayed discrete fractional sine-type matrix function* $\mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,-\delta+2,r}$ *satisfies homogeneous system:*

$$\begin{cases} {}^C\nabla_0^{2\delta} Z(\sigma) = \Delta_1 Z(\sigma) + \Delta_2 Z(\sigma - r), \quad \sigma \in \mathbb{Z}_1, \\ Z(\sigma) = \Theta, \quad \sigma \in \mathbb{Z}^0_{1-r}, \quad r \in \mathbb{Z}_2. \end{cases}$$

(b) *The delayed discrete fractional cosine-type matrix function* $\mathfrak{C}^{\Delta_1,\Delta_2}_{\delta,1,r}$ *satisfies homogeneous system:*

$$\begin{cases} {}^C\nabla_0^{2\delta} Z(\sigma) = \Delta_1 Z(\sigma) + \Delta_2 Z(\sigma - r), \quad \sigma \in \mathbb{Z}_1, \\ Z(0) = I, \quad Z(\sigma) = \Theta, \quad \sigma \in \mathbb{Z}^{-1}_{1-r}, \quad r \in \mathbb{Z}_2. \end{cases}$$

Proof. Now, for $\sigma \in \mathbb{Z}_1$ we show that

$$^{C}\nabla_0^{2\delta}\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) = \Delta_1\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) + \Delta_2\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma - r). \tag{4.1}$$

For $\sigma \in \mathbb{Z}_1$, there exists $p \in \mathbb{Z}_1$ such that $\sigma \in \mathbb{Z}_{pr}^{(p+1)r}$. We apply the mathematical induction on $p \in \mathbb{Z}_0$ to demonstrate its trueness. It is easy to check the trueness of the identity for $p = 0$. For $p = 1$, we consider

$$\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} R(i+1,j)\, J_{2i\delta+1}(\sigma, jr) = \sum_{i=0}^{\infty}\sum_{j=0}^{1} R(i+1,j)\, J_{2i\delta+1}(\sigma, jr), \quad \sigma \in \mathbb{Z}_0^{r}. \tag{4.2}$$

Taking the Caputo fractional difference $^{C}\nabla_0^{2\delta}$ of (4.2), we acquire

$$^{C}\nabla_0^{2\delta}\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) = \sum_{i=0}^{\infty}\sum_{j=0}^{1} R(i+1,j)\ {}^{C}\nabla_0^{2\delta} J_{2i\delta+1}(\sigma, jr).$$

After taking into account the subintervals and Lemma 2.13, we get

$$\begin{aligned}
^{C}\nabla_0^{2\delta}\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) &= {}^{C}\nabla_0^{2\delta}\sum_{i=0}^{\infty}\left(\Delta_1^{i} J_{2i\delta+1}(\sigma,0) + R(i+1,1)\, J_{2i\delta+1}(\sigma,r)\right)\\
&= {}^{C}\nabla_0^{2\delta}\sum_{i=0}^{\infty}\left(\Delta_1^{i} J_{2i\delta+1}(\sigma,0) + R(i+1,1)\, J_{2i\delta+1}(\sigma-r,0)\right)\\
&= \sum_{i=1}^{\infty}\left(\Delta_1^{i} J_{2i\delta-2\delta+1}(\sigma,0) + R(i+1,1)\, J_{2i\delta-2\delta+1}(\sigma-r,0)\right)\\
&= \sum_{i=0}^{\infty}\left(\Delta_1^{i+1} J_{2i\delta+1}(\sigma,0) + R(i+2,1)\, J_{2i\delta+1}(\sigma-r,0)\right)\\
&= \Delta_1\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) + \Delta_2\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma-r), \quad \sigma \in \mathbb{Z}_0^{r},
\end{aligned}$$

where the information $^{C}\nabla_0^{2\delta} J_1(\sigma,0) = {}^{C}\nabla_0^{2\delta} J_1(\sigma-r,0) = 0$ has been used. Now, let us assume its validity for $p = n$, that is

$$\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) = \sum_{i=0}^{\infty}\sum_{j=0}^{n} R(i+1,j)\, J_{2i\delta+1}(\sigma, jr), \quad \sigma \in \mathbb{Z}_{nr}^{(n+1)r},$$

satisfies (4.1). In the case where $p = n+1$, we can examine the same calculations as in the initial scenario for $\sigma \in \mathbb{Z}_{(n+1)r}^{(n+2)r}$,

$$\begin{aligned}
^{C}\nabla_0^{2\delta}\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) &= {}^{C}\nabla_0^{2\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{n+1} R(i+1,j)\, J_{2i\delta+1}(\sigma, jr)\\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{n+1} R(i+1,j)\, J_{2i\delta-2\delta+1}(\sigma, jr)\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{n+1} R(i+2,j)\, J_{2i\delta+1}(\sigma, jr)\\
&= \Delta_1\sum_{i=0}^{\infty}\sum_{j=0}^{n+1} R(i+1,j)\, J_{2i\delta+1}(\sigma, jr) + \Delta_2\sum_{i=0}^{\infty}\sum_{j=0}^{n+1} R(i+1,j-1)\, J_{2i\delta+1}(\sigma, jr)
\end{aligned}$$

$$
\begin{aligned}
&= \Delta_1 \sum_{i=0}^{\infty}\sum_{j=0}^{n+1} R(i+1,j)\, J_{2i\delta+1}(\sigma, jr) + \Delta_2 \sum_{i=0}^{\infty}\sum_{j=0}^{n} R(i+1,j)\, J_{2i\delta+1}(\sigma - r, jr) \\
&= \Delta_1 \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,-\delta+2,r}(\sigma) + \Delta_2 \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,-\delta+2,r}(\sigma - r),
\end{aligned}
$$

which is what we look for the craved result. It is easily obtained from the definition of $\mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,-\delta+2,r}(\sigma)$ that it satisfies the initial condition. The proof of the latter is omitted because it can be proved in the same manner as the former. □

We examine the Caputo fractional delayed difference system

$$
\begin{cases} \nabla_0^{2\delta} \nu(\sigma) = \Delta_1 \nu(\sigma) + \Delta_2 \nu(\sigma - r) + f(\sigma), & \sigma \in \mathbb{Z}_1, \\ \nu(\sigma) = \psi(\sigma),\ \nabla\nu(\sigma) = \nabla\psi(\sigma), & \sigma \in \mathbb{Z}^0_{1-r}, \quad r > 0, \end{cases} \tag{4.3}
$$

$\psi : \mathbb{Z}^0_{1-r} \to \mathbb{R}^n$ is an initial function. Let $f : \mathbb{Z}_1 \to \mathbb{R}$ be a function of exponential order, we perform the Laplace transform to acquire an analytical form of a solution of (4.3).

Theorem 4.2. *Then the representation of the unique solution of the fractional IVP (4.3) is offered as follows*

$$
\begin{aligned}
\nu(\sigma) &= \mathfrak{C}^{\Delta_1,\Delta_2}_{\delta,1,r}(\sigma)\nu(0) + \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,-\delta+2,r}(\sigma)\nabla\nu(0) \\
&\quad + \int_{-r}^{0} \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,\delta,r}(\sigma - \rho(k) - r)\Delta_2\psi(k)\,\nabla k + \int_0^{\sigma} \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,\delta,r}(\sigma - \rho(k))\, f(k)\nabla k, \quad \sigma \in \mathbb{Z}_1.
\end{aligned}
$$

Under the assumptions $1 < 2\delta < 2$, $\|\Delta_1\| < 1$, $f : \mathbb{Z}_1 \to \mathbb{R}$ *is exponentially ordered.*

Proof. Implementing $\mathfrak{L}_0$ to both sides of (4.3),

$$
\mathfrak{L}_0\left\{{}^C\nabla_0^{2\delta}\nu\right\}(s) = \Delta_1 \mathfrak{L}_0\{\nu\}(s) + \Delta_2 \mathfrak{L}_0\{\nu(\cdot - r)\} + \mathfrak{L}_0\{f\}(s),
$$

and using Lemma 2.23, which is expressed with the special choices as follows, we get

$$
\begin{aligned}
\mathfrak{L}_0\left\{{}^C\nabla_0^{2\delta}\nu\right\}(s) &= s^{2\delta}\mathfrak{L}_0\{\nu\}(s) - s^{2\delta-1}\nu(0) - s^{2\delta-2}\nabla\nu(0), \\
s^{2\delta}\mathfrak{L}_0\{\nu\}(s) - s^{2\delta-1}\nu(0) - s^{2\delta-2}\nabla\nu(0) &= \Delta_1\mathfrak{L}_0\{\nu\}(s) + \Delta_2(1-s)^r\mathfrak{L}_0\{\nu\}(s) \\
&\quad + \sum_{l=1-r}^{0}(1-s)^{l+r-1}\Delta_2\psi(l) + \mathfrak{L}_0\{f\}(s).
\end{aligned}
$$

Rearranges the terms,

$$
\left(s^{2\delta}I - \Delta_1 - (1-s)^r\Delta_2\right)\mathfrak{L}_0\{\nu\}(s) = s^{2\delta-1}\nu(0) + s^{2\delta-2}\nabla\nu(0) + \sum_{l=1-r}^{0}(1-s)^{l+r-1}\Delta_2\psi(l) + \mathfrak{L}_0\{f\}(s).
$$

By settling out the equation for $\mathfrak{L}_0\{\nu\}(s)$ one gets

$$
\begin{aligned}
\mathfrak{L}_0\{\nu\}(s) &= s^{2\delta-1}\left(s^{2\delta} - \Delta_1 - (1-s)^r\Delta_2\right)^{-1}\nu(0) + s^{2\delta-2}\left(s^{2\delta} - \Delta_1 - (1-s)^r\Delta_2\right)^{-1}\nabla\nu(0) \\
&\quad + \sum_{m=1-r}^{0}(1-s)^{m+r-1}\left(s^{2\delta} - \Delta_1 - (1-s)^r\Delta_2\right)^{-1}\Delta_2\psi(m) \\
&\quad + \left(s^{2\delta} - (1-s)\Delta_1 - (1-s)^r\Delta_2\right)^{-1}\mathfrak{L}_0\{f\}(s).
\end{aligned}
$$

By applying the inverse Laplace transform to nabla, and utilizing Lemma 2.26 along with the convolution property (Theorem 2.18), we obtain

$$
\nu(\sigma) = \mathfrak{C}^{\Delta_1,\Delta_2}_{\delta,1,r}(\sigma)\nu(0) + \mathfrak{S}^{\Delta_1,\Delta_2}_{\delta,-\delta+2,r}(\sigma)\nabla\nu(0)
$$

$$+\sum_{m=1-r}^{0}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k)-r)\Delta_2\psi(k)+\int_0^\sigma\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k))f(k)\nabla k,$$

which finishes the proof. □

The subsequent theorem demonstrates that it is possible to eliminate the requirement that $f:\mathbb{Z}_1\to\mathbb{R}$ possesses an exponential order.

Theorem 4.3. *Let $1<2\delta<2$ and $f:\mathbb{Z}_1\to\mathbb{R}$, $\|\Delta_1\|<1$. Then a representation of the unique solution of the difference system (1.1) is offered as follows*

$$\begin{aligned}\nu(\sigma)&=\mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma)\nu(0)+\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0)\\&\quad+\int_{-r}^{0}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k)-r)\Delta_2\psi(k)\nabla k+\int_0^\sigma\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k))f(k)\nabla k,\quad\sigma\in\mathbb{Z}_1.\end{aligned}\tag{4.4}$$

Proof. One has to implement the operator ${}^C\nabla_0^{2\delta}$ to (4.4). It is so easy to observe the expression

$$\nu(\sigma)=\mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma)\nu(0)+\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0)+\int_{-r}^{0}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k)-r)\Delta_2\psi(k)\nabla k$$

fulfills the equation ${}^C\nabla_0^{2\delta}\nu(\sigma)=\Delta_1\nu(\sigma)+\Delta_2\nu(\sigma-r)$. So, it is enough to show that f fulfills (1.1),

$${}^C\nabla_0^{2\delta}\nu(\sigma)={}^C\nabla_0^{2\delta}\left[\int_0^\sigma\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k))f(k)\nabla k\right].$$

One makes the following simple calculations

$$\begin{aligned}{}^C\nabla_0^{2\delta}\nu(\sigma)&={}^C\nabla_0^{2\delta}\sum_{k=1}^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k))f(k)\\&={}^C\nabla_0^{2\delta}\sum_{k=1}^{\sigma}\sum_{i=0}^{\infty}\Delta_1{}^iJ_{(2i+1)\delta+\delta-1}(\sigma-\rho(k),0)f(k)\\&\quad+{}^C\nabla_0^{2\delta}\sum_{k=1}^{\sigma}\sum_{i=1}^{\infty}R(i+1,1)J_{(2i+1)\delta+\delta-1}(\sigma-\rho(k),r)f(k)\\&\quad+\cdots+{}^C\nabla_0^{2\delta}\sum_{k=1}^{\sigma}\sum_{i=p}^{\infty}R(i+1,p)J_{(2i+1)\delta+\delta-1}(\sigma-\rho(k),pr)f(k)\\&=\sum_{k=1}^{\sigma}\sum_{i=0}^{\infty}\Delta_1{}^iJ_{2i\delta-1}(\sigma-\rho(k),0)f(k)+\sum_{k=1}^{\sigma}\sum_{i=1}^{\infty}R(i+1,1)J_{2i\delta-1}(\sigma-\rho(k),r)f(k)\\&\quad+\cdots+\sum_{k=1}^{\sigma}\sum_{i=p}^{\infty}R(i+1,p)J_{2i\delta-1}(\sigma-\rho(k),pr)f(k).\end{aligned}$$

Due to $\sum_{s=0}^{\sigma}J_{-1}(\sigma-\rho(k),0)f(k)=f(\sigma)$, we have

$$\begin{aligned}{}^C\nabla_0^{2\delta}\sum_{s=0}^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(k))f(k)&=f(\sigma)+\Delta_1\sum_{k=1}^{\sigma}\sum_{i=0}^{\infty}\Delta_1{}^iJ_{2i\delta+\delta-1}(\sigma-\rho(k),0)f(k)\\&\quad+\sum_{k=1}^{\sigma}\sum_{i=0}^{\infty}R(i+2,1)J_{2i\delta+\delta-1}(\sigma-\rho(k),r)f(k)\end{aligned}$$

$$+\cdots+\sum_{k=1}^{\sigma}\sum_{i=p-1}^{\infty}R\left(i+2,p\right)J_{2i\delta+\delta-1}\left(\sigma-\rho(k),pr\right)f(k).$$

Based on the recurrence equation of the determining equation (2.1), one gets

$$\begin{aligned}{}^{C}\nabla_{0}^{2\delta}\nu(\sigma)&=f(\sigma)+\Delta_1\int_0^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}\left(\sigma-\rho(t)\right)f(t)\nabla t+f(\sigma)+\Delta_2\int_0^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}\left(\sigma-r-\rho(t)\right)f(t)\nabla t\\&=f(\sigma)+\Delta_1\int_0^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}\left(\sigma-\rho(t)\right)f(t)\nabla t+f(\sigma)+\Delta_2\int_0^{\sigma-r}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}\left(\sigma-r-\rho(t)\right)f(t)\nabla t\\&=\Delta_1\nu\left(\sigma\right)+\Delta_2\nu\left(\sigma-r\right)+f\left(\sigma\right),\end{aligned}$$

which is what we want to prove. □

Remark 4.4. In the case $\Delta_1=\Theta$, $\mathfrak{C}_{\delta,\mu,r}^{\Delta_2}\left(\sigma\right)=\mathfrak{C}_{\delta,\mu,r}^{\Theta,\Delta_2}\left(\sigma\right)$ and $\mathfrak{S}_{\delta,\mu,r}^{\Delta_2}\left(\sigma\right)=\mathfrak{S}_{\delta,\mu,r}^{\Theta,\Delta_2}\left(\sigma\right)$ are finite sums given by

$$\mathfrak{C}_{\delta,\mu,r}^{\Delta_2}\left(\sigma\right):=\begin{cases}\Theta, & \sigma\in\mathbb{Z}^{-1},\\ \sum_{j=0}^{p}{\Delta_2}^{j}\dfrac{(\sigma-jr)_{+}^{\overline{2j\delta+\mu-1}}}{\Gamma\left(2j\delta+\mu\right)}, & \sigma\in\mathbb{Z}_{pr}^{(p+1)r},\end{cases}$$

and

$$\mathfrak{S}_{\delta,\mu,r}^{\Delta_2}\left(\sigma\right):=\begin{cases}\Theta, & \sigma\in\mathbb{Z}^{-1},\\ \sum_{j=0}^{p}{\Delta_2}^{j}\dfrac{(\sigma-jr)_{+}^{\overline{(2j+1)\delta+\mu-1}}}{\Gamma\left((2j+1)\,\delta+\mu\right)}, & \sigma\in\mathbb{Z}_{pr}^{(p+1)r},\end{cases}$$

so the explicit solution transforms to the following formula

$$\begin{aligned}\nu\left(\sigma\right)&=\mathfrak{C}_{\delta,1,r}^{\Delta_2}(\sigma)\nu\left(0\right)+\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_2}(\sigma)\nabla\nu\left(0\right)+\int_{-r}^{0}\mathfrak{S}_{\delta,\delta,r}^{\Delta_2}(\sigma-\rho(k)-r)\Delta_2\psi\left(k\right)\nabla k\\&+\int_0^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_2}\left(\sigma-\rho(k)\right)f(k)\nabla k,\quad\sigma\in\mathbb{Z}_1.\end{aligned}$$

Remark 4.5. Δ_1 and Δ_2 are permutable, i.e., $\Delta_1\Delta_2=\Delta_2\Delta_1$. Then, the representation of the explicit solution is of the same structure, but the cosine-type and sine-type functions make into the following shapes

$$\mathfrak{C}_{\delta,\mu,r}^{\Delta_1,\Delta_2}\left(\sigma\right):=\begin{cases}\Theta, & \sigma\in\mathbb{Z}^{-1},\\ \sum\limits_{i=0}^{\infty}\Delta_1^{i}\dfrac{(\sigma)_{+}^{\overline{2i\delta+\mu-1}}}{\Gamma\left(2i\delta+\mu\right)}+\sum\limits_{i=1}^{\infty}\begin{pmatrix}i+1\\1\end{pmatrix}{\Delta_1}^{i-1}\Delta_2\dfrac{(\sigma-r)_{+}^{\overline{2i\delta+\mu-1}}}{\Gamma\left(2i\delta+\mu\right)} & \\ \quad+\cdots+\sum\limits_{i=p}^{\infty}\begin{pmatrix}i+1\\p\end{pmatrix}\Delta_1^{i-p}{\Delta_2}^{p}\dfrac{(\sigma-pr)_{+}^{\overline{2i\delta+\mu-1}}}{\Gamma\left(2i\delta+\mu\right)}, & \sigma\in\mathbb{Z}_{pr}^{(p+1)r},\end{cases}$$

and

$$\mathfrak{S}_{\delta,\mu,r}^{\Delta_1,\Delta_2}\left(\sigma\right):=\begin{cases}\Theta, & \sigma\in\mathbb{Z}^{-1},\\ \sum\limits_{i=0}^{\infty}{\Delta_1}^{i}\dfrac{(\sigma)_{+}^{\overline{(2i+1)\delta+\mu-1}}}{\Gamma\left((2i+1)\,\delta+\mu\right)}+\sum\limits_{i=1}^{\infty}\begin{pmatrix}i+1\\1\end{pmatrix}{\Delta_1}^{i-1}\Delta_2\dfrac{(\sigma-r)_{+}^{\overline{(2i+1)\delta+\mu-1}}}{\Gamma\left((2i+1)\,\delta+\mu\right)} & \\ \quad+\cdots+\sum\limits_{i=p}^{\infty}\begin{pmatrix}i+1\\p\end{pmatrix}{\Delta_1}^{i-p}{\Delta_2}^{p}\dfrac{(\sigma-pr)_{+}^{\overline{(2i+1)\delta+\mu-1}}}{\Gamma\left((2i+1)\,\delta+\mu\right)}, & \sigma\in\mathbb{Z}_{pr}^{(p+1)r}.\end{cases}$$

5. Ulam-Hyers stability: linear systems

One of the most important concept in the theory of differential equations is the data dependence. There are some special data dependence in the theory of functional equations such as Ulam-Hyers, Ulam-Hyers-Rassias, finite-time, Lyapunov, etc. Especially, the Ulam-Hyers-type stabilities were taken up by a number of mathematicians and the study of this area has become one of the central subjects in the mathematical analysis area. So, we discuss distinct types of Ulam-Hyers stabilities for the linear system.

Definition 5.1. System (4.3) is of Ulam-Hyers stability when there is a real number $K > 0$, for each $\epsilon > 0$ and each $y : \mathbb{Z}_0^T \to \mathbb{R}^n$ fulfilling

$$\begin{cases} \left\| {}^C\nabla_0^{2\delta} y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma - r) - f(\sigma) \right\| \leqslant \epsilon, \ \sigma \in \mathbb{Z}_1^T, \ r \in \mathbb{Z}_2, \\ \| y(\sigma) - \psi(\sigma) \| \leqslant \epsilon, \ \sigma \in \mathbb{Z}_{-r+1}^0, \end{cases} \tag{5.1}$$

there exits an explicit solution $v : \mathbb{Z}_0^T \to \mathbb{R}^n$ of (4.3) such that $\| v(\sigma) - y(\sigma) \| \leqslant K\epsilon$, $\sigma \in \mathbb{Z}_1^T$.

Definition 5.2. System (4.3) is of Ulam-Hyers-Rassias stability w.r.t. Ψ when there is a real number $K_\Psi > 0$, for each $\epsilon > 0$, $\psi > 0$, a nonincreasing $\Psi : \mathbb{Z}_0^T \to \mathbb{R}$, and each $y : \mathbb{Z}_0^T \to \mathbb{R}^n$ fulfilling

$$\begin{cases} \left\| {}^C\nabla_0^{2\delta} y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma - r) - f(\sigma) \right\| \leqslant \epsilon \Psi(\sigma), \ \sigma \in \mathbb{Z}_1^T, \ r \in \mathbb{Z}_2, \\ \| y(\sigma) - \psi(\sigma) \| \leqslant \epsilon \psi, \ \sigma \in \mathbb{Z}_{-r+1}^0, \end{cases} \tag{5.2}$$

there exists an explicit solution $v : \mathbb{Z}_0^T \to \mathbb{R}^n$ of (4.3) such that $\| v(\sigma) - y(\sigma) \| \leqslant K_\Psi \Psi(\sigma)\epsilon$, $\sigma \in \mathbb{Z}_1^T$, where K_Ψ is a constant of Hyers-Ulam-Rassias.

Remark 5.3. From (5.1), there is $g : \mathbb{Z}_0^T \to \mathbb{R}^n$ fulfilling

$$\begin{cases} {}^C\nabla_0^{2\delta} y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma - r) - f(\sigma) = g(\sigma), \ \ \sigma \in \mathbb{Z}_1^T, \ r \in \mathbb{Z}_2, \\ y(\sigma) = \psi(\sigma), \ \nabla v(\sigma) = \nabla \psi(\sigma), \ \sigma \in \mathbb{Z}_{-r+1}^0, \end{cases}$$

where $\| g(\sigma) \| \leqslant \epsilon$, $\sigma \in \mathbb{Z}_1^T$.

Remark 5.4. From (5.2), there is $G : \mathbb{Z}_0^T \to \mathbb{R}^n$ fulfilling

$$\begin{cases} {}^C\nabla_0^{2\delta} y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma - r) - f(\sigma) = G(\sigma), \ \sigma \in \mathbb{Z}_1^T, \ r \in \mathbb{Z}_2, \\ y(\sigma) = \psi(\sigma), \ \nabla v(\sigma) = \nabla \psi(\sigma), \ \sigma \in \mathbb{Z}_{-r+1}^0, \end{cases}$$

where $\| G(\sigma) \| \leqslant \Psi(\sigma)\epsilon$, $\sigma \in \mathbb{Z}_1^T$.

Theorem 5.5. (4.3) *is stable in the sense of Ulam-Hyers on* $\mathbb{Z}_1^T$.

Proof. For an arbitrary real number $\epsilon > 0$, $y : \mathbb{Z}_1^T \to \mathbb{R}^n$ fulfills

$$\begin{cases} \left\| {}^C\nabla_0^{2\delta} y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma - r) - f(\sigma) \right\| \leqslant \epsilon, \ \sigma \in \mathbb{Z}_1^T, \ r \in \mathbb{Z}_2, \\ \| y(\sigma) - \psi(\sigma) \| \leqslant \epsilon, \ \| \nabla y(\sigma) - \nabla \psi(\sigma) \| \leqslant \epsilon, \ \sigma \in \mathbb{Z}_{-r+1}^0. \end{cases}$$

With the aid of Remark 5.3, there is $g : \mathbb{Z}_1^T \to \mathbb{R}^n$ fulfilling

$$\begin{cases} {}^C\nabla_0^{2\delta} y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma - r) - f(\sigma) = g(\sigma), \ \sigma \in \mathbb{Z}_1^T, \ r \in \mathbb{Z}_2, \\ y(\sigma) = \psi(\sigma), \ \nabla v(\sigma) = \nabla \psi(\sigma), \ \sigma \in \mathbb{Z}_{-r+1}^0, \end{cases}$$

where $\| g(\sigma) \| \leqslant \epsilon$, $\sigma \in \mathbb{Z}_1^T$. Base on Theorem 4.3 we have

$$y(\sigma) = \mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma) v(0) + \mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma) \nabla v(0) + \int_{-r}^{0} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(m) - r) \Delta_2 \psi(m) \nabla m$$

$$+\int_0^{\sigma} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\,[f(s)+g(s)]\,\nabla s,\ \sigma\in\mathbb{Z}_1.$$

therefore, we can get

$$\begin{aligned}\|\nu(\sigma)-y(\sigma)\| &= \left\|\int_0^{\sigma} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\,g(s)\,\nabla s\right\| \\ &\leqslant \sum_{s=1}^{\sigma}\left\|\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\right\|\|g(s)\| \\ &\leqslant \left(\sum_{s=1}^{\sigma}\left\|\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\right\|\right)\epsilon \\ &\leqslant \left(\sum_{s=1}^{T}\left\|\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\right\|\right)\epsilon \leqslant \left(\sum_{s=1}^{T}\mathfrak{S}_{\delta,\mu,r}^{\|\Delta_1\|,\|\Delta_2\|}(\sigma-\rho(s))\right)\epsilon = K\epsilon,\end{aligned}$$

where $K=\sum_{s=1}^{T}\mathfrak{S}_{\delta,\mu,r}^{\|\Delta_1\|,\|\Delta_2\|}(\sigma-\rho(s))$. (1.1) is stable in the setting of Ulam-Hyers on $\mathbb{Z}_1^T$ based on Definition 5.1. □

Theorem 5.6. (1.1) *is stable in the sense of Ulam-Hyers-Rassias on* $\mathbb{Z}_1^T$.

Proof. For an arbitrary real number $\epsilon>0$, $y:\mathbb{Z}_{-r+1}^T\to\mathbb{R}^n$ fulfills

$$\begin{cases}\left\|{}^C\nabla_0^{2\delta}y(\sigma)-\Delta_1 y(\sigma)-\Delta_2 y(\sigma-r)-f(\sigma)\right\|\leqslant\epsilon, & \sigma\in\mathbb{Z}_1^T,\ r\in\mathbb{Z}_2,\\ \|y(\sigma)-\psi(\sigma)\|\leqslant\epsilon\psi,\ \|\nabla y(\sigma)-\nabla\psi(\sigma)\|\leqslant\epsilon\psi, & \sigma\in\mathbb{Z}_{-r+1}^0.\end{cases}$$

With the help of Remark 5.4, there is $G:\mathbb{Z}_1^T\to\mathbb{R}^n$ holding

$$\begin{cases}{}^C\nabla_0^{2\delta}y(\sigma)-\Delta_1 y(\sigma)-\Delta_2 y(\sigma-r)-f(\sigma)=G(\sigma), & \sigma\in\mathbb{Z}_1^T,\ r\in\mathbb{Z}_2,\\ y(\sigma)=\psi(\sigma),\ \nabla y(\sigma)=\nabla\psi(\sigma), & \sigma\in\mathbb{Z}_{-r+1}^0,\end{cases}$$

where $\|G(\sigma)\|\leqslant\epsilon\Psi(\sigma)$, $\sigma\in\mathbb{Z}_1^T$. Base on Theorem 4.3 we have

$$\begin{aligned}y(\sigma) &= \mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma)\nu(0)+\mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0)+\int_{-r}^{0}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(m)-r)\Delta_2\psi(m)\,\nabla m\\ &\quad+\int_0^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\,[f(s)+G(s)]\,\nabla s,\ \sigma\in\mathbb{Z}_1.\end{aligned}$$

Therefore, we can get

$$\begin{aligned}\|\nu(\sigma)-y(\sigma)\| &= \left\|\int_0^{\sigma}\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\,G(s)\,\nabla s\right\| \\ &\leqslant \sum_{s=1}^{\sigma}\left\|\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\right\|\|G(s)\| \\ &\leqslant \left(\sum_{s=1}^{\sigma}\left\|\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\right\|\right)\epsilon\Psi(s) \\ &\leqslant \left(\sum_{s=1}^{T}\left\|\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\right\|\right)\epsilon\Psi(\sigma) \leqslant \left(\sum_{s=1}^{T}\mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(\sigma-\rho(s))\right)\epsilon\Psi(\sigma) = K_\Psi\Psi(\sigma)\,\epsilon,\end{aligned}$$

where $K_\Psi=\sum_{s=1}^{T}\mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(\sigma-\rho(s))$. The (1.1) is stable in the sense of Ulam-Hyers-Rassias on $\mathbb{Z}_1^T$ based on the Definition 5.2. □

6. Semilinear fractional difference system

The focus of this section is on analyzing the existence and uniqueness of solutions to the nabla fractional nonlinear difference equation (6.1),

$$\begin{cases} {}^{C}\nabla_{0}^{2\delta}\nu(\sigma) = \Delta_1\nu(\sigma) + \Delta_2\nu(\sigma - r) + f(\sigma, \nu(\sigma)), \ \sigma \in \mathbb{Z}_1, \\ \nu(\sigma) = \psi(\sigma), \ \sigma \in \mathbb{Z}_{1-r}^{0}, \ r \in \mathbb{Z}_2, \end{cases} \tag{6.1}$$

where the fractional order 2δ is between 1 and 2. Regarding the system (6.1), it is our consistent assumption that the function $f(\sigma,\nu)$ satisfies the Lipschitz condition with respect to the second variable:

$$\|f(\sigma,\nu) - f(\sigma,y)\| \leqslant \mathfrak{L}_f \|z - y\|, \ \sigma \in \mathbb{Z}_1^{T},$$

with $0 < \mathfrak{L}_f < 1$. By Theorem 4.3 the equation (6.1) is equivalent to

$$\begin{aligned} \nu(\sigma) &= \mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma)\nu(0) + \mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0) \\ &\quad + \int_{-r}^{0} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(m) - r)\Delta_2\psi(m)\,\nabla m + \int_{0}^{\sigma} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(s))\, f(s,\nu(s))\nabla s, \ \sigma \in \mathbb{Z}_1. \end{aligned} \tag{6.2}$$

Theorem 6.1. *Under the assumption that* $f(\sigma,\nu)$ *fulfills the Lipschitz condition and* $\mathfrak{S}_{\delta,\mu,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f T < 1$, *the system* (6.1) *possesses a unique solution* $\nu(\sigma)$.

Proof. It is enough to show that (6.1) is of a unique solution. To do this, one defines an operator $\Pi : \mathbb{Z}_1^{T} \to \mathbb{Z}_1^{T}$ by

$$\Pi\nu(\sigma) = z_0(\sigma) + \int_{0}^{\sigma} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(s))\, f(s,\nu(s))\nabla s, \ n \in \mathbb{Z}_1,$$

where

$$z_0(\sigma) = \mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma)\nu(0) + \mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0) + \int_{-r}^{0} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(m) - r)\Delta_2\psi(m)\,\nabla m.$$

The following estimation can be easily made:

$$\begin{aligned} \max_{\sigma\in\mathbb{Z}_1^{T}} \|\Pi\nu(\sigma) - \Pi y(\sigma)\| &\leqslant \max_{\sigma\in\mathbb{Z}_1^{T}} \int_{0}^{\sigma} \left\|\mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(s))\right\| \|f(s,\nu(s) - f(s,y(s)\|\,\nabla s \\ &\leqslant \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f \max_{\sigma\in\mathbb{Z}_1^{T}} \sum_{s=0}^{\sigma} \|\nu(s) - y(s)\| \\ &\leqslant \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f T \max_{s\in\mathbb{Z}_1^{T}} \|\nu(s) - y(s)\|. \end{aligned}$$

According to the Banach fixed point theorem, the equation (6.2) possesses a unique solution within $\mathbb{Z}_1^{T}$. □

Through the utilization of our theoretical results, we will now investigate Ulam-Hyers stability for the nabla fractional nonlinear difference equation within a finite time interval in this particular section. Our attention now shifts to:

$$\begin{cases} {}^{C}\nabla_{0}^{2\delta}\nu(\sigma) = \Delta_1\nu(\sigma) + \Delta_2\nu(\sigma - r) + f(\sigma, \nu(\sigma)), \ \sigma \in \mathbb{Z}_1, \\ \nu(\sigma) = \psi(\sigma), \ \nabla y(\sigma) = \nabla\psi(\sigma), \ \sigma \in \mathbb{Z}_{1-r}^{0}, \ r \in \mathbb{Z}_2, \end{cases} \tag{6.3}$$

where $\|\Delta_1\| < 1$, $1 < 2\delta < 2$.

Definition 6.2. If any $\varepsilon > 0$ and $\nu(\sigma)$ satisfies the inequality

$$\left\| {}^{C}\nabla_0^{2\delta}\nu(\sigma) - \Delta_1\nu(\sigma) - \Delta_2\nu(\sigma - r) - f(\sigma, \nu(\sigma)) \right\| \leqslant \varepsilon, \ \sigma \in \mathbb{Z}_1^T, \tag{6.4}$$

and there is such a solution formula $y(\sigma)$ to (6.3) that

$$\|\nu(\sigma) - y(\sigma)\| \leqslant C\varepsilon, \ \|\nabla\nu(\sigma) - \nabla y(\sigma)\| \leqslant C\varepsilon, \ \sigma \in \mathbb{Z}_1^T,$$

when C is a positive constant that does not depend on $y(\sigma)$ and $\nu(\sigma)$, the system (6.3) is classified as Ulam-Hyers stable.

Prior to exploring the Ulam-Hyers stability criterion for the nabla fractional difference equation within a finite time interval, it is crucial to lay the foundation with the following Remark, which proves to be highly useful.

Remark 6.3. Assume that $1 < 2\delta < 2$ and $\varepsilon > 0$. From (6.4), there exists $\chi : \mathbb{Z}_0^T \to \mathbb{R}^n$ satisfying

$$\begin{cases} {}^{C}\nabla_0^{2\delta}y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma - r) - f(\sigma) = \chi(\sigma), \ \sigma \in \mathbb{Z}_1^T, \ r \in \mathbb{Z}_2, \\ y(\sigma) = \psi(\sigma), \ \nabla y(\sigma) = \nabla\psi(\sigma), \ \sigma \in \mathbb{Z}_{-r+1}^0, \end{cases}$$

where $\|\chi(\sigma)\| \leqslant \varepsilon$, $\sigma \in \mathbb{Z}_1^T$.

Theorem 6.4. *Suppose that* $f(\sigma, \nu)$ *is the Lipschitzan* $\|f(\sigma, \nu) - f(\sigma, y)\| \leqslant \mathfrak{L}_f \|z - y\|$ *and the Lipschitz constant* $\mathfrak{S}_{\delta,\mu,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f T < 1$. *Then system* (6.3) *on* $\sigma \in \mathbb{Z}_1^T$ *is Ulam-Hyers stable with*

$$C = \frac{\mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,T}{1 - \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f T}.$$

Proof. Suppose that $\chi(\sigma)$, $\sigma \in \mathbb{Z}_1^T$ fulfilling

$$\begin{cases} \chi(\sigma) = {}^{C}\nabla_0^{2\delta}\nu(\sigma) - \Delta_1\nu(\sigma) - \Delta_2\nu(\sigma - r) - f(\sigma, \nu(\sigma)), \\ \nu(\sigma) = \psi(\sigma), \ \nabla y(\sigma) = \nabla\psi(\sigma), \ \sigma \in \mathbb{Z}_{1-r}^0, \ r \in \mathbb{Z}_2. \end{cases} \tag{6.5}$$

Then, $\|\chi(\sigma)\| \leqslant \varepsilon$, the solutions $\nu(\sigma)$ and $y(\sigma)$ to systems (6.3) and (6.5) are

$$\begin{aligned} \nu(\sigma) = {} & \mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma)\nu(0) + \mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0) \\ & + \int_{-r}^{0} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(k) - r)\Delta_2\psi(k)\,\nabla k + \int_0^{\sigma} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(k))\, f(k, \nu(k))\nabla k, \end{aligned}$$

and

$$\begin{aligned} y(\sigma) = {} & \mathfrak{C}_{\delta,1,r}^{\Delta_1,\Delta_2}(\sigma)\nu(0) + \mathfrak{S}_{\delta,-\delta+2,r}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0) \\ & + \int_{-r}^{0} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(k) - r)\Delta_2\psi(k)\,\nabla k + \int_0^{\sigma} \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(s))\,[f(k, y(k)) + \chi(k)]\,\nabla k, \end{aligned}$$

respectively. Therefore, one admits

$$\begin{aligned} \|\nu(\sigma) - y(\sigma)\| & \leqslant \int_{-1}^{\sigma} \left\| \mathfrak{S}_{\delta,\delta,r}^{\Delta_1,\Delta_2}(\sigma - \rho(k)) \right\| \|f(k, \nu(k) - f(k, y(k)\| + \|\chi(s)\|\,\nabla k \\ & \leqslant \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f \int_0^{\sigma} \|\nu(k) - y(k)\|\,\nabla k + \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T) \int_0^{\sigma} \|\chi(k)\|\,\nabla k \\ & = \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f \sum_{s=0}^{\sigma} \|\nu(k) - y(k)\| + \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T) \sum_{s=0}^{\sigma} \|\chi(k)\|, \end{aligned}$$

So one gets

$$\max_{1\leqslant\sigma\leqslant T}\|\nu(\sigma)-y(\sigma)\| \leqslant \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f T \max_{1\leqslant\sigma\leqslant T}\|\nu(\sigma)-y(\sigma)\| + \mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\sum_{s=0}^{\sigma}\|\chi(s)\|,$$

$$\max_{1\leqslant\sigma\leqslant T}\|\nu(\sigma)-y(\sigma)\| \leqslant \frac{\mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)}{1-\mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f T}\sum_{s=0}^{\sigma}\|\chi(s)\| < \frac{\mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,T}{1-\mathfrak{S}_{\delta,\delta,r}^{\|\Delta_1\|,\|\Delta_2\|}(T)\,\mathfrak{L}_f T}\varepsilon.$$

□

7. Illustrative examples

In this section, we would like to concretize our theoretical findings with the help of an illustrative examples.

Example 7.1. Consider the following delayed Caputo fractional linear discrete systems:

$$\begin{cases} {}^{C}\nabla_0^{1.6}\nu(\sigma) = \Delta_1\nu(\sigma) + \Delta_2\nu(\sigma-2) + f(\sigma), \ \sigma\in\mathbb{Z}_1^6, \\ \nu(\sigma) = \psi(\sigma), \ \nabla\nu(\sigma) = \nabla\psi(\sigma), \ \sigma\in\mathbb{Z}_{-1}^0, \end{cases} \tag{7.1}$$

with

$$\begin{cases} \left\|{}^{C}\nabla_0^{2\delta}y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma-2) - f(\sigma)\right\| \leqslant 0.8, \ \sigma\in\mathbb{Z}_1^6, \\ \|y(\sigma)-\psi(\sigma)\| \leqslant 1, \ \|\nabla y(\sigma)-\nabla\psi(\sigma)\| \leqslant 1, \ \sigma\in\mathbb{Z}_{-1}^0, \end{cases}$$

and

$$\begin{cases} \left\|{}^{C}\nabla_0^{2\delta}y(\sigma) - \Delta_1 y(\sigma) - \Delta_2 y(\sigma-2) - f(\sigma)\right\| \leqslant 0.8\Psi(\sigma), \ \sigma\in\mathbb{Z}_1^6, \\ \|y(\sigma)-\psi(\sigma)\| \leqslant 1, \ \|\nabla y(\sigma)-\nabla\psi(\sigma)\| \leqslant 1, \ \sigma\in\mathbb{Z}_{-1}^0, \end{cases}$$

where

$$\mathfrak{A} = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix}, \ \mathfrak{B} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \ \psi = 0.5, \ \psi(\sigma) = \left[\cos\left(\sigma+\frac{\pi}{2}\right), \ \cos\left(\sigma+\frac{\pi}{2}\right)\right]^T, \ \Psi(\sigma) = e^{\sigma}.$$

If $y(\cdot):\mathbb{Z}_1\to\mathbb{R}^2$ satisfies (7.1), the there exists $g(\cdot):\mathbb{Z}_1\to\mathbb{R}^2$ such that $\|g(\sigma)\|\leqslant\varepsilon$, moreover

$$\begin{cases} {}^{C}\nabla_0^{2\delta}y(\sigma) = \Delta_1 y(\sigma) + \Delta_2 y(\sigma-2) + f(\sigma) + g(\sigma), \ \sigma\in\mathbb{Z}_1^6, \\ y(\sigma) = \psi(\sigma), \ \nabla\nu(\sigma) = \nabla\psi(\sigma), \ \sigma\in\mathbb{Z}_{-1}^0, \end{cases} \tag{7.2}$$

also the solution of (7.1) is

$$\begin{aligned} y(\sigma) &= \mathfrak{C}_{\delta,1,2}^{\Delta_1,\Delta_2}(\sigma)\nu(0) + \mathfrak{S}_{\delta,-\delta+2,2}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0) \\ &\quad + \int_{-2}^{0}\mathfrak{S}_{\delta,\delta,2}^{\Delta_1,\Delta_2}(\sigma-\rho(m)-2)\Delta_2\psi(m)\,\nabla m + \int_0^{\sigma}\mathfrak{S}_{\delta,\delta,2}^{\Delta_1,\Delta_2}(\sigma-\rho(s))\,[f(s)+g(s)]\,\nabla s, \ \sigma\in\mathbb{Z}_1. \end{aligned}$$

Let $\varepsilon = 0.8$, and $g(\cdot):\mathbb{Z}_1\to\mathbb{R}^2$, be as given below

$$g(\sigma) = \left[\frac{3}{5}\cos\left(\sigma+\frac{\pi}{2}\right), \ \frac{3}{5}\sin\left(\sigma+\frac{\pi}{2}\right)\right]^T,$$

then clearly

$$\|g(\sigma)\| = \left(\sqrt{\left(\frac{3}{5}\cos\left(\sigma+\frac{\pi}{2}\right)\right)^2 + \left(\frac{3}{5}\sin\left(\sigma+\frac{\pi}{2}\right)\right)^2}\right) = \frac{3}{5} < \varepsilon = 0.8.$$

The Caputo fractional delayed difference system (7.2) is of the following unique solution

$$y(\sigma) = \mathfrak{C}_{\delta,1,2}^{\Delta_1,\Delta_2}(\sigma)\nu(0) + \mathfrak{S}_{\delta,-\delta+2,2}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu(0)$$

$$+\int_{-2}^{0}\mathfrak{S}_{\delta,\delta,2}^{\Delta_1,\Delta_2}(\sigma-\rho(m)-2)\Delta_2\psi\left(m\right)\nabla m+\int_{0}^{\sigma}\mathfrak{S}_{\delta,\delta,2}^{\Delta_1,\Delta_2}\left(\sigma-\rho(s)\right)\left[f(s)+g\left(s\right)\right]\nabla s,\ \sigma\in\mathbb{Z}_1.$$

Then we have

$$\begin{aligned}\left\|\nu\left(\sigma\right)-y\left(\sigma\right)\right\|&=\left\|\sum_{s=1}^{\sigma}\mathfrak{S}_{0.8,0.8,2}^{\Delta_2}\left(\sigma-s+1\right)g\left(s\right)\nabla s\right\|\\&=\left\|\mathfrak{S}_{0.8,0.8,2}^{\Delta_2}\left(\sigma\right)g\left(1\right)+\mathfrak{S}_{0.8,0.8,2}^{\Delta_2}\left(\sigma-1\right)g\left(2\right)+\cdots+\mathfrak{S}_{0.8,0.8,2}^{\Delta_2}\left(1\right)g\left(\sigma\right)\right\|,\end{aligned}$$

when $\sigma\in\mathbb{Z}_0^6$.

For some $K>0$, according to Theorem 5.5, we have $\left\|\nu\left(\sigma\right)-y\left(\sigma\right)\right\|\leqslant K\varepsilon$.

Example 7.2. It follows from Theorem 4.2 that a representation of an explicit solution to the following delayed Caputo fractional linear discrete systems

$$\begin{cases}{}^{C}\nabla_0^{1.2}\nu\left(\sigma\right)=1\nu\left(\sigma\right)+2\nu\left(\sigma-2\right)+\sigma,\ \sigma\in\mathbb{Z}_1^6,\\ \nu\left(\sigma\right)=1,\ \nabla\nu\left(\sigma\right)=0,\ \sigma\in\mathbb{Z}_{-1}^0,\end{cases}$$

is given by

$$\begin{aligned}\nu\left(\sigma\right)&=\mathfrak{C}_{0.6,1,2}^{1,2}(\sigma)\nu\left(0\right)+\mathfrak{S}_{0.6,1.4,2}^{1,2}(\sigma)\nabla\nu\left(0\right)\\&\quad+\int_{-2}^{0}\mathfrak{S}_{0.6,0.6,2}^{1,2}(\sigma-k-1)2\psi\left(k\right)\nabla k+\int_{0}^{\sigma}\mathfrak{S}_{0.6,0.6,2}^{1,2}\left(\sigma-k+1\right)f(k)\nabla k,\ \sigma\in\mathbb{Z}_1,\end{aligned}\tag{7.3}$$

whose graph is presented in Figure 1.

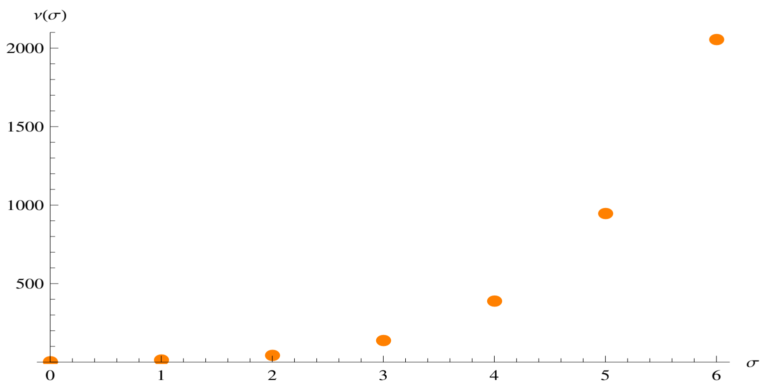

Figure 1: The simulation of the explicit solution (7.3).

Example 7.3. We take the following delayed Caputo fractional linear discrete systems into consideration:

$$\begin{cases}{}^{C}\nabla_0^{1.2}\nu\left(\sigma\right)=\Delta_1\nu\left(\sigma\right)+\Delta_2\nu\left(\sigma-2\right)+f\left(\sigma\right),\ \sigma\in\mathbb{Z}_1^6,\\ \nu\left(\sigma\right)=\psi\left(\sigma\right),\ \nabla\nu\left(\sigma\right)=\nabla\psi\left(\sigma\right),\ \sigma\in\mathbb{Z}_{-1}^0,\end{cases}\tag{7.4}$$

where $\Delta_1=\begin{pmatrix}1&2\\3&4\end{pmatrix}$, $\Delta_2=\begin{pmatrix}5&6\\7&8\end{pmatrix}$, $\psi\left(\sigma\right)=\begin{pmatrix}\sigma\\0\end{pmatrix}$, $f\left(\sigma\right)=\begin{pmatrix}\sigma^2\\0\end{pmatrix}$. A representation of an explicit solution to the system (7.4), which is directly calculated from Theorem 4.2, is of the following structure:

$$\begin{aligned}\nu\left(\sigma\right)&=\mathfrak{C}_{0.6,1,2}^{\Delta_1,\Delta_2}(\sigma)\nu\left(0\right)+\mathfrak{S}_{0.6,1.4,2}^{\Delta_1,\Delta_2}(\sigma)\nabla\nu\left(0\right)\\&\quad+\int_{-2}^{0}\mathfrak{S}_{0.6,0.6,2}^{\Delta_1,\Delta_2}(\sigma-k-1)\Delta_2\psi\left(k\right)\nabla k+\int_{0}^{\sigma}\mathfrak{S}_{0.6,0.6,2}^{\Delta_1,\Delta_2}\left(\sigma-k+1\right)f(k)\nabla k,\ \sigma\in\mathbb{Z}_1.\end{aligned}$$

With $\nu\left(\sigma\right)=\left[\nu_1\left(\sigma\right)\ \ \nu_2\left(\sigma\right)\right]^{\intercal}$, each graph of $\nu_i\left(\sigma\right)$, $i=1,2$ is offered in Figure 2.

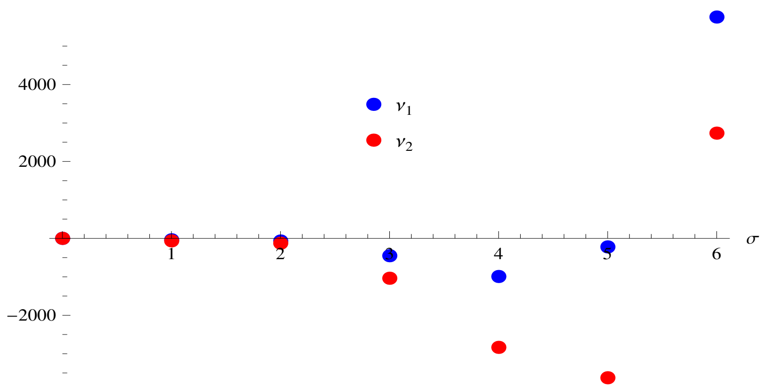

Figure 2: The graphs of components of the solution $\nu(\sigma)$ to system (7.4).

8. Conclusion

The significance of the Mittag-Leffler function is evident in its application to fractional calculus and discrete fractional calculus models, which make use of newly defined operators. In our research, fresh forms of the delayed nabla fractional cosine-type and sine-type matrix functions were introduced, specifically created for discrete applications, and labeled as the delayed discrete cosine-type and sine-type matrix functions. The investigation focused on analyzing the expression of the solution for the delayed Caputo fractional difference system, incorporating the noncommutative coefficients Δ_1 and Δ_2. In order to derive a clear and explicit representation of the solution, the study employed the delayed discrete cosine-type and sine-type matrix functions, along with the nabla Laplace transform. In addition, we have explored particular cases. The stabilities of a fractional delayed difference system with constant noncommutative coefficients, according to Ulam-Hyers and Ulam-Hyers-Rassias criteria, have been established through the analytical representation of the exact solution for the Caputo fractional delayed difference system.

By considering the system at hand, we are able to broaden the scope to include the creation of various discrete fractional operators that are fundamental in nature. These operators include, but are not confined to, the Atangana-Baleanu fractional difference operators, the Caputo-Fabrizio fractional difference operators. Furthermore, our findings can be augmented to encompass different types, such as multi-retarded types, types with variable coefficients, higher-order linear discrete types. It is imperative to thoroughly investigate each of these newly developed systems in order to analyze their respective properties.

References

[1] T. Abdeljawad, *Fractional difference operators with discrete generalized Mittag-Leffler kernels*, Chaos Solitons Fractals, **126** (2019), 315–324. 1

[2] T. Abdeljawad, D. Baleanu, *Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels*, Adv. Difference Equ., **2016** (2016), 18 pages.

[3] J. Alzabut, T. Abdeljawad, *A generalized discrete fractional Gronwall inequality and its application on the uniqueness of solutions for nonlinear delay fractional difference system*, Appl. Anal. Discrete Math., **12** (2018), 36–48. 1

[4] F. M. Atıcı, M. Atıcı, N. Nguyen, T. Zhoroev, G. Koch, *A study on discrete and discrete fractional pharmacokinetics-pharmacodynamics models for tumor growth and anti-cancer effects*, Comput. Math. Biophys., **7** (2019), 10–24. 1

[5] F. M. Atıcı, P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ., **2009** (2009), 12 pages. 1

[6] F. M. Atici, P. W. Eloe, *Linear systems of fractional nabla difference equations*, Rocky Mountain J. Math., **41** (2011), 353–370. 1

[7] F. M. Atıcı, P. W. Eloe, *Gronwall's inequality on discrete fractional calculus*, Comput. Math. Appl., **64** (2012), 3193–3200. 1

[8] M. Aydin, N. I. Mahmudov, *Iterative learning control for impulsive fractional order time-delay systems with nonpermutable constant coefficient matrices*, Int. J. Adapt. Control Signal Process., **36** (2022), 1419–1438. 1

[9] M. Aydin, N. I. Mahmudov, *On a study for the neutral Caputo fractional multi-delayed differential equations with noncommutative coefficient matrices*, Chaos Solitons Fractals, **161** (2022), 11 pages.

[10] M. Aydin, N. I. Mahmudov, *ψ-Caputo type time-delay Langevin equations with two general fractional orders*, Math. Methods Appl. Sci., **46** (2023), 9187–9204. 1

[11] D. Baleanu, G.-C. Wu, *Some further results of the Laplace transform for variable-order fractional difference equations*, Fract. Calc. Appl. Anal., **22** (2019), 1641–1654. 1

[12] B. Bonilla, M. Rivero, J. J. Trujillo, *On systems of linear fractional differential equations with constant coefficients*, Appl. Math. Comput., **187** (2007), 68–78. 1

[13] X. Cao, J. Wang, *Finite-time stability of a class of oscillating systems with two delays*, Math. Methods Appl. Sci., **41** (2018), 4943–4954. 1

[14] J. Čermák, T. Kisela, L. Nechvátal, *Discrete Mittag-Leffler functions in linear fractional difference equations*, Abstr. Appl. Anal., **2011** (2011), 21 pages. 1

[15] J. Cheng, *Fractional difference equation theory*, Xiamen University Press, Xiamen, (2011). 1

[16] C. F. M. Coimbra, *Mechanics with variable-order differential operators*, Ann. Phys., 12 (2003), 692–703. 1

[17] J. Diblík, M. Fečkan, M. Pospíšil, *Representation of a solution of the Cauchy problem for an oscillating system with two delays and permutable matrices*, Ukrainian Math. J., **65** (2013), 64–76. 1

[18] J. Diblík, D. Y. Khusainov, *Representation of solutions of linear discrete systems with constant coefficients and pure delay*, Adv. Difference Equ., **2006** (2006) 13 pages. 1

[19] J. Diblík, D. Y. Khusainov, *Representation of solutions of discrete delayed system* $x(k+1) = Ax(k) + Bx(k-m) + f(k)$ *with commutative matrices*, J. Math. Anal. Appl., **318** (2006), 63–76. 1

[20] J. Diblík, D. Y. Khusainov, J. Lukáčová, M. Ružicková, *Control of oscillating systems with a single delay*, Adv. Difference Equ., **2010** (2010), 15 pages. 1

[21] J. Diblík, B. Morávková, *Discrete matrix delayed exponential for two delays and its property*, Adv. Difference Equ., **2013** (2013), 18 pages. 1

[22] J. Diblík, B. Morávková, *Representation of the solutions of linear discrete systems with constant coefficients and two delays*, Abstr. Appl. Anal., **2014** (2014), 19 pages. 1

[23] K. Diethelm, *The analysis of fractional differential equations*, Springer-Verlag, Berlin, (2010). 1

[24] F. Du, B. Jia, *Finite-time stability of a class of nonlinear fractional delay difference systems*, Appl. Math. Lett., **98** (2019), 233–239. 1

[25] F. Du, J.-G. Lu, *Exploring a new discrete delayed Mittag-Leffler matrix function to investigate finite-time stability of Riemann-Liouville fractional-order delay difference systems*, Math. Methods Appl. Sci., **45** (2022), 9856–9878.

[26] P. Eloe, Z. Ouyang, *Multi-term linear fractional nabla difference equations with constant coefficients*, Int. J. Difference Equ., **10** (2015), 91–106. 1

[27] C. Goodrich, A. C. Peterson, *Discrete fractional calculus*, Springer, Cham, (2015). 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.22, 3.1

[28] N. Heymans, I. Podlubny, *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*, Rheol. Acta, **45** (2006), 765–771. 1

[29] L.-L. Huang, G.-C. Wu, D. Baleanu, H.-Y. Wang, *Discrete fractional calculus for interval–valued systems*, Fuzzy Sets and Systems, **404** (2021), 141–158. 1

[30] B. Jia, L. Erbe, A. Peterson, *Comparison theorems and asymptotic behavior of solutions of discrete fractional equations*, Electron. J. Qual. Theory Differ. Equ., **2015** (2015), 18 pages. 1

[31] B. Jia, L. Erbe, A. Peterson, *Comparison theorems and asymptotic behavior of solutions of Caputo fractional equations*, Int. J. Difference Equ., **11** (2016), 163–178. 1

[32] D. Y. Khusainov, J. Diblík, M. Ružicková, J. Lukácová, *Representation of a solution of the Cauchy problem for an oscillating system with pure delay*, Nonlinear Oscill., **11** (2008), 276–285. 1

[33] D. Y. Khusainov, G. V. Shuklin, *Linear autonomous time-delay system with permutation matrices solving*, Stud. Univ. Žilina Math. Ser., **17** (2003), 101–108. 1

[34] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, (2006). 1

[35] C. Li, D. Qian, Y. Chen, *On Riemann-Liouville and Caputo derivatives*, Discrete Dyn. Nat. Soc., **2011** (2011), 15 pages.

[36] M. Li, J. Wang, *Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations*, Appl. Math. Comput., **324** (2018), 254–265. 1

[37] C. Liang, J. Wang, *Analysis of iterative learning control for an oscillating control system with two delays*, Trans. Inst. Meas. Control, **40** (2018), 1757–1765. 1

[38] C. Liang, J. Wang, D. O'Regan, *Controllability of nonlinear delay oscillating systems*, Electron. J. Qual. Theory Differ. Equ., **2017** (2017), 18 pages.

[39] C. Liang, J. Wang, D. O'Regan, *Representation of a solution for a fractional linear system with pure delay*, Appl. Math. Lett., **77** (2018) 72–78.

[40] C. Liang, W. Wei, J. Wang, *Stability of delay differential equations via delayed matrix sine and cosine of polynomial degrees*, Adv. Difference Equ., **2017** (2017), 17 pages. 1

[41] N. I. Mahmudov, *Representation of solutions of discrete linear delay systems with non permutable matrices*, Appl. Math. Lett., **85** (2018), 8–14. 1

[42] N. I. Mahmudov, *Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations*, Math. Methods Appl. Sci., **42** (2019), 5489–5497. 1, 2.1, 2.2

[43] N. I. Mahmudov, *A novel fractional delayed matrix cosine and sine*, Appl. Math. Lett., **92** (2019), 41–48. 1

[44] N. I. Mahmudov, *Delayed linear difference equations: the method of Z-transform*, Electron. J. Qual. Theory Differ. Equ.,

2020 (2020), 12 pages. 1

[45] N. I. Mahmudov, M. Aydın, *Representation of solutions of nonhomogeneous conformable fractional delay differential equations*, Chaos Solitons Fractals, **150** (2021), 8 pages. 1

[46] A. D. Obembe, M. Enamul Hossain, S. A. Abu-Khamsin, *Variable-order derivative time fractional diffusion model for heterogeneous porous media*, J. Pet. Sci. Eng., **152** (2017), 391–405. 1

[47] M. Pospíšil, *Representation of solutions of delayed difference equations with linear parts given by pairwise permutable matrices via* Z*-transform*, Appl. Math. Comput., **294** (2017), 180–194. 1

[48] N. H. Sweilam, S. M. Al-Mekhlafi, *Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives*, J. Adv. Res., **7** (2016), 271–283. 1

[49] V. E. Tarasov, *Handbook of fractional calculus with applications*, De Gruyter, Berlin, (2019). 1

[50] G.-C. Wu, T. Abdeljawad, J. Liu, D. Baleanu, K.-T. Wu, *Mittag-Leffler stability analysis of fractional discrete-time neural networks via fixed point technique*, Nonlinear Anal. Model. Control, **24** (2019), 919–936. 1

[51] G.-C. Wu, D. Baleanu, W.-H. Luo, *Lyapunov functions for Riemann-Liouville-like fractional difference equations*, Appl. Math. Comput., **314** (2017), 228–236. 1

[52] G.-C. Wu, D. Baleanu, S.-D. Zeng, *Finite-time stability of discrete fractional delay systems: Gronwall inequality and stability criterion*, Commun. Nonlinear Sci. Numer. Simul., **57** (2018), 299–308. 1

[53] G.-C. Wu, Z.-G. Deng, D. Baleanu, D.-Q. Zeng, *New variable-order fractional chaotic systems for fast image encryption*, Chaos, **29** (2019), 11 pages. 1