J. Nonlinear Sci. Appl., 18 (2025), 64–74
ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.isr-publications.com/jnsa

Hyers-Ulam-Rassias stability criteria for second order nonlinear differential equations with nonlinear damping

I. Fakunle[a,*], P. O. Arawomo[b], S. O. Olagunju[a], A. A. Ganiyu[a]

[a]Department of Mathematics, Adeyemi Federal University of Education, Ondo, Nigeria.
[b]Department of Mathematics, University of Ibadan, Ibadan, Nigeria.

Abstract

This paper is concerned with the Hyers-Ulam-Rassias stability second order nonlinear differential equations with nonlinear damping. New criteria for Hyers-Ulam-Rassias stability are established which improve and extend the known results in the literature.

Keywords: Hyers-Ulam-Rassias constant, Hyers-Ulam-Rassias stability, nonlinear damped differential equation, integral inequality.

2020 MSC: 26A46, 34C10, 11R33, 35Q31.

1. Introduction

Over the last forty years, the stability theory of functional equations has been strongly developed very important contributions to this subject were brought by Ulam [42] who posed the following problem concerning the stability of functional equations:give conditions in order for a linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [18] when $\mathbf{G}_1$ and $\mathbf{G}_2$ are Banach spaces and the result of Hyers was generalized by Rassias [37]. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians ([2, 4, 7–10, 19, 31, 32, 36]). In 1998, the Hyers-Ulam stability of differential equation $u' = u$ was first investigated by Alsina and Ger [3]. In 2002, this result has been generalized by Takahasi et al. [40] for the Banach space-valued differential equation $u' = \lambda u$. In 2005, Jung [23] proved the generalized Hyers-Ulam stability of a linear equations of the first order. For more results on stability of differential equations see [1, 17, 20, 22, 24–30, 33, 43] and for more details on the Hyers-Ulam stability and related topics, the readers refer to [12–16, 22, 33, 34, 38, 39, 41, 43].

In this paper, we prove the Hyers-Ulam-Rassias stability criteria for the following second order nonlinear damped differential equations:

$$(r(t)K_1(t,u(t),u'(t)))' + p(t)K_2(t,u(t),u'(t))u'(t) + q(t)f(u(t)) = P(t,u(t),u'(t)), \tag{1.1}$$

*Corresponding author
Email addresses: fakunlesanmi@gmail.com (I. Fakunle), womopeter@gmail.com (P. O. Arawomo), lagsam2016@gmail.com (S. O. Olagunju), ganiyuiwajowa@gmail.com (A. A. Ganiyu)

doi: 10.22436/jnsa.018.01.06

Received: 2023-09-29 *Revised:* 2023-12-10 *Accepted:* 2024-01-31

$$(r(t)(u'(t)^{\alpha}))' + p(t)K_2(t, u(t), u'(t))u'(t) + q(t)f(u(t)) = P(t, u(t), u, (t)), \tag{1.2}$$

for all $t \geqslant t_0$ with initial conditions

$$u(t_0) = u'(t_0) = 0, \quad t \in \mathbf{I}, \tag{1.3}$$

where $r(t), p(t), q(t) \in C(\mathbf{R}_+)$, $f \in C(R_+)$, $K_1, K_2, P \in C(R_+ \times \mathbf{R^2})$, $R_+ = [0, \infty)$, $K_1(t_0, 0, 0,) = 0$, $K_2(t_0, 0, 0) = 0$, $P(t_0, 0, 0) = 0$, $\mathbf{I} = (t_0, \infty)$.

2. Preliminary

We begin our considerations by giving the following definitions, lemmas and theorems to obtain our results.

Definition 2.1. We say that equation (1.1) is Hyers-Ulam-Rassias stable with respect to φ if there exists $C_{\varphi} > 0$ such that for each solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|(r(t)K_1(t, u(t), u'(t)))' + p(t)K_2(t, u(t), u'(t))u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t))| \leqslant \varphi(t), \tag{2.1}$$

there exists a solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ to equation (1.1) with $|u(t) - u_0(t)| \leqslant C_{\varphi}\varphi(t)$, $t \in \mathbf{I}$, where $\varphi(t)$ is a non-decreasing and positive function defined as $\varphi : \mathbf{I} \to \mathbf{R}_+$.

Definition 2.2. Equation (1.2) is stable in the sense of Hyers-Ulam-Rassias with respect to φ if there exists $C_{\varphi} > 0$ such that for each solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|(r(t)u'(t)^{\alpha})' + p(t)K_2(t, u(t), u'(t))u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t))| \leqslant \varphi(t), \tag{2.2}$$

there exists a solution a solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ to equation (1.2) with $|u(t) - u_0(t)| \leqslant C_{\varphi}\varphi(t)$, $\forall t \in \mathbf{I}$, where $\varphi(t)$ is a non-decreasing and positive function defined as $\varphi : \mathbf{I} \to \mathbf{R}_+$.

Definition 2.3. A function $\omega : [0, \infty) \to [0, \infty)$ is said to belong to a class Λ if

(i) $\omega(u)$ is non-decreasing and continuous for $u \geqslant 0$;
(ii) $(\frac{1}{v})\omega(u) \leqslant \omega(\frac{u}{v})$ for all u and $v \geqslant 1$;
(iii) there exist a function ϕ, continuous on $[0, \infty)$ with $\omega(\alpha u) \leqslant \phi(\alpha)\omega(u)$ for $\alpha \geqslant 0$.

Lemma 2.4 ([5, 6])**.** *Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \leqslant t \leqslant b, (\leqslant \infty)$ and $K > 0$, $M \geqslant 0$. Further, let $\omega(u)$ be a nonnegative non-decreasing continuous function for $u \geqslant 0$, then the inequality*

$$u(t) \leqslant K + M\int_{t_0}^{t} f(s)\omega(u(s))ds, \ \ t_0 \leqslant t < b,$$

implies the inequality

$$u(t) \leqslant \Omega^{-1}\left(\Omega(k) + M\int_{t_0}^{t} f(s)ds\right), \ \ t_0 \leqslant t \leqslant b' \leqslant b,$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u. \tag{2.3}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(k) + M\int_{t_0}^{t} f(s)ds \in \mathrm{Dom}(\Omega^{-1}).$$

Lemma 2.5 ([21]). *Let* $r(t)$ *be an integrable function, then the* n*-successive integration of* r *over the interval* $[t_0, t]$ *is given by*

$$\int_{t_0}^{t} \cdots \int_{t_0}^{t} r(s)ds^n = \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} r(s)ds.$$

Theorem 2.6 ([11]). *Let*

(i) $u(t), r(t) : (0, \infty) \to (0, \infty)$ *and continuous on* $(0, \infty)$;
(ii) $\varpi \in \Lambda$;
(iii) $n > 0$ *be monotonic, non-decreasing and continuous on* $(0, \infty)$,

if

$$u(t) \leqslant n(t) + \int_{0}^{t} f(s)\varpi(u(s))ds, \quad 0 < t < \infty,$$

then

$$u(t) \leqslant n(t)\Omega^{-1}\left(\Omega(1) + \int_{0}^{t} f(s)ds\right), \; 0 < t \leqslant b,$$

where $(0, b) \subset (0, \infty)$, $\Omega(u)$ *is defined in* (2.3) *and* Ω^{-1} *is the inverse of* Ω *and* t *is in the sub-interval* $(0, b)$ *and is chosen such that*

$$\Omega(1) + \int_{0}^{t} f(s)ds \in \mathrm{Dom}(\Omega^{-1}).$$

Theorem 2.7 ([35]). *If* $f(t)$ *and* $g(t)$ *are continuous in* $[t_0, t] \subseteq \mathbf{I}$ *and* $f(t)$ *does not change sign in the interval, then there is a point* $\xi \in [t_0, t]$ *such that* $\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds$.

Theorem 2.8 ([15]). *Suppose* $u(t), r(t), h(t) \in C(\mathbf{I}, \mathbf{R}_+)$ *and* $\varpi(u), \beta(u) \in \Psi$ *are nonnegative, monotonic, non-decreasing, and continuous and* $\omega(u)$ *be sub-multiplicative for* $u > 0$. *Let*

$$u(t) \leqslant K + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds$$

for positive constants K, T, *and* L, *then*

$$u(t) \leqslant \Omega^{-1}\left(\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{s} r(\alpha)d\alpha\right)\right)ds\right)F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right),$$

where $\beta(u) \neq \varpi(u)$, Ω *is defined in equation* (2.3) *and* $F(u)$ *is defined as*

$$F(u) = \int_{u_0}^{u} \frac{ds}{\beta(s)}, \quad 0 < u_0 \leqslant u, \tag{2.4}$$

F^{-1}, Ω^{-1} *are the inverses of* F, Ω, *respectively, and* t *is in the sub-interval* $(0, b) \in \mathbf{I}$ *chosen such that*

$$F(1) + T\int_{t_0}^{t} r(s)ds \in \mathrm{Dom}(F^{-1})$$

and

$$\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds \in \mathrm{Dom}(\Omega^{-1}).$$

Corollary 2.9 ([15]). *Suppose* $\rho(t)$ *is a nonnegative, monotonic, non-decreasing continuous function on* $\mathbf{R}_+$. *Let*

$$u(t) \leqslant \rho(t) + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds,$$

for T *and* L *be positive constants, then*

$$u(t) \leqslant \rho(t)\Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds\right)F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right), \ t \in \mathbf{I},$$

where $\Omega(u)$ *and* $F(u)$ *are defined as in* (2.3) *and* (2.4), *respectively.*

Theorem 2.10 ([15]). *If* $u(t), r(t), h(t), \rho(t), g(t) \in C(\mathbf{R}_+)$ *and* $\omega, f, \gamma \in \Psi$ *be nonnegative, monotonic, and non-decreasing continuous functions, let* γ *be sub-multiplicative. If*

$$u(t) \leqslant \rho(t) + A\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds$$

for $K, A, B, L > 0$, *then*

$$\begin{aligned} u(t) \leqslant \rho(t)\Upsilon^{-1}&\left[\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \\ &\times \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t), \end{aligned}$$

where $T(t)$ *is given as*

$$T(t) = F^{-1}\left(F(1) + A\int_{t_0}^{t} r(s)ds\right)$$

and

$$\Upsilon(r) = \int_{t_0}^{t} \frac{ds}{\gamma(s)}, \quad 0 < r_0 \leqslant r,$$

and F^{-1}, Ω^{-1}, *and* Υ^{-1} *are the inverses of* F, Ω, Υ, *respectively, and* $t \in (0, b) \subset (I)$. *Consequently*

$$\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds \in \mathrm{Dom}(\Upsilon^{-1}).$$

3. Main results

In the section, we established the Hyers-Ulam-Rassias stability of the nonlinear damped differential equations (1.1), (1.2), and the case $P(t, u(t), u'(t)) = 0$. We shall begin to prove the Hyers-Ulam-Rassias stability of the nonlinear differential equation (1.2) with initial condition (1.2).

Theorem 3.1. *Suppose that*

(i) $|P(t, u(t), u'(t))| \leqslant \phi(t)g(|u(t)|)h(|u'(t)|)$;
(ii) $|K_1(t, u(t), u'(t)| \leqslant \alpha(t)\gamma(|u(t)|)b(|u'(t)|)(|u'(t)|)^n$, *where* $n \in \mathbf{N}$;
(iii) $|K_2(t, u(t), u'(t))| \leqslant \psi(t)\varpi(|u(t)|)(|u'(t)|)$;
(iv) *there exists constant* $\rho > 0$ *such that* $\int_{t_0}^{t} \varphi(s)ds \leqslant \rho\varphi(t)$;
(v) $\lim\limits_{t_0\to\infty} \int_{t_0}^{t} \phi(s)ds \leqslant n_1 < \infty$, *where* $n_1 > 0$;
(vi) $\lim\limits_{t_0\to\infty} \int_{t_0}^{t} p(s)\psi(s)ds \leqslant n_2 < \infty$, *where* $n_2 > 0$;
(vii) $\lim\limits_{t_0\to\infty} \int_{t_0}^{t} r(s)\alpha(s)ds \leqslant n_3 < \infty$, *where* $n_3 > 0$,

where $\phi, \alpha, \psi, \gamma, \varpi, g, h, b \in C(\mathbf{R}_+)$. In addition, let $\varpi(u(t)), g(u(t)), \gamma(t) \in \Lambda$ be continuous, non-decreasing, and monotonic, then equation (1.1) has the Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_1} = \frac{\lambda\rho}{\eta}\Upsilon^{-1}\left[\Upsilon(1) + \frac{n_1 h(\lambda)\lambda}{\eta} g\left[\Omega^{-1}\left(\Omega(1) + \frac{n_2\lambda}{\eta}\varpi\left(T^*\right)\right)T^*\right]\right]\Omega^{-1}\left(\Omega(1) + \frac{(n_2\lambda}{\eta}\varpi\left(T^*\right)\right)T^*,$$

where

$$T^* = F^{-1}\left(F(1) + \frac{b(\lambda)\lambda^{n+1}n_3}{\eta}\right).$$

Proof. Using inequality (1.3) and multiplying both sides of equation by $u'(t)$ we have

$$\begin{aligned}(r(t)K_1(t, u(t), u'(t))'u'(t) + p(t)K_2(t, u(t), u'(t))(u'(t))^2 \\ + q(t)f(u(t))u'(t) - P(t, u(t), u'(t))u'(t) \leqslant u'(t)\varphi(t).\end{aligned} \tag{3.1}$$

Integrating both sides of (3.1) twice from t_0 to t, and applying Lemma 2.5 we obtain

$$\begin{aligned}\int_{t_0}^{t} r(s)K_1(s, u(s), u'(s))u'(s)ds + t\int_{t_0}^{t} p(s)K_2(s, u(s), u'(s))(u'(s))^2 ds \\ + tq(t)F(u(t)) - t\int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leqslant t\epsilon\int_{t_0}^{t} u'(s)ds,\end{aligned}$$

where

$$F(u(t)) = \int_{u_0}^{u(t)} f(s)ds. \tag{3.2}$$

Using conditions (i)-(iii) and application of Theorem 2.7 implies there exists $\xi, \rho, \delta \in [t_0, t]$ such that

$$\begin{aligned}b(u'(\xi))u'(\xi)^{n+1}\int_{t_0}^{t} r(s)\alpha(s)\gamma(u(s))ds + tu'(\rho)\int_{t_0}^{t} p(s)\psi(s)\varpi(u(s))ds \\ + tq(t)F(u(t)) - th(u'(\delta))u'(\delta)\int_{t_0}^{t}\phi(s)g(u(s))ds \leqslant tu'(t)\int_{t_0}^{t}\varphi(s)ds.\end{aligned}$$

Set $|F(u(t)| \geqslant |u(t)|$ and $|u'(t)| \leqslant \lambda$, since $q(t)$ is an increasing function, there exists $\eta > 0$ such that $q(t) \geqslant \eta$, then

$$\begin{aligned}|u(t)| \leqslant \frac{\lambda}{\eta}\int_{t_0}^{t}\varphi(s)ds + \frac{b(\lambda)\lambda^{n+1}}{\eta}\int_{t_0}^{t} r(s)\alpha(s)\gamma(|u(s)|)ds \\ + \frac{\lambda}{\eta}\int_{t_0}^{t} p(s)\psi(s)\varpi(|u(s)|)ds + \frac{h(\lambda)\lambda}{\eta}\int_{t_0}^{t}\phi(s)g(|u(s)|)ds, \ \forall t \geqslant 1.\end{aligned}$$

The application of Theorem 2.10 gives

$$\begin{aligned}|u(t)| \leqslant \frac{\lambda}{\eta}\int_{t_0}^{t}\varphi(s)ds\Upsilon^{-1}\left[\Upsilon(1) + \frac{h(\lambda)\lambda}{\eta}\int_{t_0}^{t}\phi(s)g\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\eta}\int_{t_0}^{s} p(s)\psi(s)\varpi\left(T(\tau)\right)d\tau\right)T(s)\right]ds\right] \\ \times\Omega^{-1}\left(\Omega(1) + \frac{(\lambda)}{\eta}\int_{t_0}^{t} p(s)\psi s\varpi\left(T(s)\right)ds\right)T(t),\end{aligned}$$

where $T(t)$ is given as

$$T(t) = F^{-1}\left(F(1) + \frac{b(\lambda)\lambda^{n+1}}{\eta}\int_{t_0}^{t} r(s)\alpha(s)ds\right).$$

Applying the conditions (iv)-(vii), we arrive at

$$|u(t)| \leqslant \frac{\lambda\rho}{\eta}\Upsilon^{-1}\left[\Upsilon(1)+\frac{n_1h(\lambda)\lambda}{\eta}g\left[\Omega^{-1}\left(\Omega(1)+\frac{n_2\lambda}{\eta}\varpi\left(T^*\right)\right)T^*\right]\right]\Omega^{-1}\left(\Omega(1)+\frac{(n_2\lambda)}{\eta}\varpi\left(T^*\right)\right)T^*\varphi(t),$$

where T^* is defined as

$$T^* = F^{-1}\left(F(1)+\frac{b(\lambda)\lambda^{n+1}n_3}{\eta}\right).$$

Hence, $|u(t)-u(t_0)| \leqslant |u(t)| \leqslant K_1\epsilon$. Therefore,

$$C_{\varphi_1} = \frac{\lambda\rho}{\eta}\Upsilon^{-1}\left[\Upsilon(1)+\frac{n_1h(|u'(\delta)|)|u'(\delta)|}{\eta}g\left[\Omega^{-1}\left(\Omega(1)+\frac{n_2|u'(\rho)|}{\eta}\varpi\left(T^*\right)\right)T^*\right]\right]$$
$$\times\,\Omega^{-1}\left(\Omega(1)+\frac{(n_2|u'(\rho)|)}{\eta}\varpi\left(T^*\right)\right)T^*.$$

□

Theorem 3.2. *Assume that all the conditions of Theorem 3.1 are satisfied. In addition, let* $\lim_{t_0\to\infty}\int_{t_0}^{t} r(s)ds \leqslant n_4 < \infty$, *where* $n_4 > 0$, *then, equation (1.2) has Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant given as*

$$C_{\varphi_2} = \rho\left(\frac{\lambda}{\delta}+\frac{n_4(\lambda)^{\alpha+1}}{\delta}\right)\Omega^{-1}\left[\Omega(1)\frac{n_1h(\lambda)\lambda}{\delta}g\left(F^{-1}\left(F(1)+\frac{n_2(\lambda)^2}{\delta}\right)\right)\right)F^{-1}\left(F(1)+\frac{n_2(\lambda)^2}{\delta}\right).$$

Proof. It is easily seen by evaluation of inequality (2.2) with application of Lemma 2.5 that

$$\begin{aligned}\int_{t_0}^{t}\int_{t_0}^{t}(r(s)(u'(s))^{\alpha})'u'(s)dsds + t\int_{t_0}^{t} p(s)K_2(s,u(s),u'(s))u'(s)ds \\ + t\int_{t_0}^{t} q(s)f(u(s))u'(s)ds - t\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leqslant t\int_{t_0}^{t} u'(s)\varphi(s)ds,\end{aligned} \tag{3.3}$$

and it is clear from equation (3.2) that equation (3.3) turns to

$$\begin{aligned}\int_{t_0}^{t}\int_{t_0}^{t}(r(s)(u'(s))^{\alpha})'u'(s)dsds + t\int_{t_0}^{t} p(s)K_2(s,u(s),u'(s))u'(s)ds \\ + t\int_{t_0}^{t} q(s)\frac{d}{ds}F(u(s))ds - t\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leqslant t\int_{t_0}^{t} u'(s)\varphi(s)ds.\end{aligned} \tag{3.4}$$

Since $q(t)$ and $u'(t)$ are non-decreasing, then $q'(t) \geqslant 0$, $u''(t) \geqslant 0$, and $q(t) \geqslant \delta$, where $\delta > 0$, then inequality (3.4) becomes

$$\begin{aligned}\int_{t_0}^{t} r(s)(u'(s))^{\alpha+1}ds + t\int_{t_0}^{t} p(s)K_2(s,u(s),u'(s))u'(s)ds \\ + t\delta F(u(t)) - t\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leqslant t\int_{t_0}^{t} u'(s)\varphi(s)ds.\end{aligned}$$

Using conditions (iii), (iv), and the application of Theorem 2.7 implies there exists $\rho,\xi,\vartheta,\nu \in [t_0,t]$ such that

$$\begin{aligned}|F(u(t))| \leqslant \left(\frac{|u'(\vartheta)|}{\delta}+\frac{(|u'(\xi)|)^{\alpha+1}}{\delta}\int_{t_0}^{t} r(s)ds\right)\int_{t_0}^{t}\varphi(s)ds + \frac{(|u'(\rho)|)^2}{\delta}\int_{t_0}^{t} p(s)\psi(s)\varpi(u(s))ds \\ + \frac{h(|u'(\nu)|)|u'(\nu)|}{\delta}\int_{t_0}^{t}\phi(s)g(|u(s)|)ds,\ \forall t \geqslant 1.\end{aligned} \tag{3.5}$$

By application of Corollary 2.9, inequality (3.5) turns to

$$|u(t)| \leqslant \left(\frac{\lambda}{\delta} + \frac{(\lambda)^{\alpha+1}}{\delta}\int_{t_0}^{t} r(s)ds\right)\int_{t_0}^{t} \varphi(s)ds\Omega^{-1}\left[\Omega(1)\frac{h(\lambda)\lambda}{\delta}\int_{t_0}^{t} \phi(s)g\left(F^{-1}(F(1) + \frac{(|u'(\lambda)^2}{\delta}\int_{t_0}^{t} p(\mu)\psi(\mu)d\mu\right)\right)ds\right)F^{-1}\left(F(1) + \frac{(\lambda)^2}{\delta}\int_{t_0}^{t} p(s)\psi(s)ds\right),$$

where $|F(u(t))| \geqslant |u(t)|$ and $|u'(t)| \leqslant \lambda$ for $\lambda > 0$. Using conditions (iv), (v), and (vi) of Theorem 3.1, and hypothesis of Theorem 3.2, we obtain

$$|u(t)| \leqslant \rho\left(\frac{\lambda}{\delta} + \frac{n_4\lambda^{\alpha+1}}{\delta}\right)\Omega^{-1}\left[\Omega(1)\frac{n_1 h(\lambda)\lambda}{\delta}g\left(F^{-1}\left(F(1) + \frac{n_2(\lambda)^2}{\delta}\right)\right)\right)F^{-1}\left(F(1) + \frac{n_2(\lambda)^2}{\delta}\right)\varphi(t),$$

hence, $|u(t) - u(t_0)| \leqslant C_\varphi \varphi(t)$, with

$$C_{\varphi_2} = \rho\left(\frac{\lambda}{\delta} + \frac{n_4\lambda^{\alpha+1}}{\delta}\right)\Omega^{-1}\left[\Omega(1)\frac{n_1 h(\lambda)\lambda}{\delta}g\left(F^{-1}\left(F(1) + \frac{n_2(\lambda)^2}{\delta}\right)\right)\right)F^{-1}\left(F(1) + \frac{n_2(\lambda)^2}{\delta}\right).$$

□

For $P(t, u(t), u'(t)) = 0$ in equations (1.1) and (1.2) the results are given in the following theorems.

Theorem 3.3. *Let all the conditions of Theorem 3.1 remained satisfied. Then equation*

$$(r(t)K_1(t, u(t), u'(t)))' + p(t)K_2(t, u(t), u'(t))u'(t) + q(t)f(u(t)) = 0,$$

has the Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_3} = \rho\frac{\lambda}{\eta}\Omega^{-1}\left(\Omega(1) + \frac{n_2|u'(\lambda}{\eta}\varpi\left(F^{-1}\left(F(1) + \frac{b(\lambda)\lambda^{n+1}}{\eta}n_3\right)\right)\right)F^{-1}\left(F(1) + \frac{b(\lambda)\lambda^{n+1}}{\eta}n_3\right).$$

Proof. Since $P(t, u(t), u'(t)) = 0$, using inequality (2.1) and multiplying both sides of equation by $u'(t)$ we have

$$(r(t)K_1(t, u(t), u'(t))'u'(t) + p(t)K_2(t, u(t), u'(t))(u'(t))^2 + q(t)f(u(t))u'(t) \leqslant u'(t)\varphi(t). \tag{3.6}$$

Integrating both sides of (3.6) twice from t_0 to t, and applying Lemma 2.5 we obtain

$$\int_{t_0}^{t} r(s)K_1(s, u(s), u'(s))u'(s)ds + t\int_{t_0}^{t} p(s)K_2(s, u(s), u'(s))(u'(s))^2 ds + tq(t)F(u(t)) \leqslant tu(t)\int_{t_0}^{t} \varphi(s)ds,$$

where equation (3.2) is used. By conditions (ii)-(iii) of Theorem 3.1 and application of Theorem 2.7 implies there exists $\xi, \rho \in [t_0, t]$ such that

$$b(u'(\xi))u'(\xi)^{n+1}\int_{t_0}^{t} r(s)\alpha(s)\gamma(u(s))ds + tu'(\rho)\int_{t_0}^{t} p(s)\psi(s)\varpi(u(s))ds + tq(t)F(u(t)) \leqslant tu'(t)\int_{t_0}^{t} \varphi(s)ds.$$

Set $|F(u(t)| \geqslant |u(t)|$ and $|u'(t)| \leqslant \lambda$ for $\lambda > 0$ since $q(t)$ is an increasing function, there exists $\eta > 0$ such that $q(t) \geqslant \eta$, then

$$|u(t)| \leqslant \frac{\lambda}{\eta}\int_{t_0}^{t} \varphi(s)ds + \frac{b(\lambda)\lambda^{n+1}}{\eta}\int_{t_0}^{t} r(s)\alpha(s)\gamma(|u(s)|)ds + \frac{\lambda}{\eta}\int_{t_0}^{t} p(s)\psi(s)\varpi(|u(s)|)ds, \ \forall t \geqslant 1.$$

The application of Corollary 2.9 gives

$$|u(t)| \leqslant \frac{\lambda}{\eta}\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1)+\frac{\lambda}{\eta}\int_{t_0}^{t}p(s)\psi(s)\varpi\left(F^{-1}\left(F(1)+\frac{b(\lambda)\lambda^{n+1}}{\eta}\int_{t_0}^{s}r(\eta)\alpha(\eta)d\eta\right)\right)ds\right) \times F^{-1}\left(F(1)+\frac{b(\lambda)\lambda^{n+1}}{\eta}\int_{t_0}^{t}r(s)\alpha(s)ds\right),$$

with the application of the conditions (iv), (vi), and (vii) we have

$$|u(t)| \leqslant \rho\frac{\lambda}{\eta}\Omega^{-1}\left(\Omega(1)+\frac{n_2\lambda}{\eta}\varpi\left(F^{-1}\left(F(1)+\frac{b(\lambda)\lambda^{n+1}}{\eta}n_3\right)\right)\right)F^{-1}\left(F(1)+\frac{b(\lambda)\lambda^{n+1}}{\eta}n_3\right)\varphi(t).$$

Hence, $|u(t)-u(t_0)| \leqslant |u(t)| \leqslant C_\varphi\varphi(t)$. Therefore,

$$C_{\varphi_3} = \rho\frac{\lambda}{\eta}\Omega^{-1}\left(\Omega(1)+\frac{n_2\lambda}{\eta}\varpi\left(F^{-1}\left(F(1)+\frac{b(\lambda)\lambda^{n+1}}{\eta}n_3\right)\right)\right)F^{-1}\left(F(1)+\frac{b(\lambda)\lambda^{n+1}}{\eta}n_3\right).$$

□

Theorem 3.4. *Assume all the conditions of Theorem 3.1 are satisfied. In addition, let* $\lim_{t_0\to\infty}\int_{t_0}^{t}r(s)ds \leqslant n_4 < \infty$, *where* $n_4 > 0$, *then, equation*

$$(r(t)(u'(t)^\alpha))' + p(t)K_2(t,u(t),u'(t))u'(t) + q(t)f(u(t)) = P(t,u(t),u,(t)),$$

has Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_4} = \left(\frac{\lambda}{\delta}+\frac{\lambda^{\alpha+1}}{\delta}n_4\right)\Omega^{-1}\left(\Omega(1)+\frac{\lambda^2}{\delta}n_2\right).$$

Proof. If $P(t,u(t),u'(t)) = 0$, it is clear from inequality (2.2) with use of Lemma 2.5 that

$$\int_{t_0}^{t}\int_{t_0}^{t}(r(s)(u'(s))^\alpha)'u'(s)dsds + t\int_{t_0}^{t}p(s)K_2(s,u(s),u'(s))u'(s)ds + t\int_{t_0}^{t}q(s)f(u(s))u'(s)ds \leqslant t\int_{t_0}^{t}u'(s)\varphi(s)ds, \tag{3.7}$$

and we use equation (3.2) in inequality (3.7) to obtain

$$\int_{t_0}^{t}\int_{t_0}^{t}(r(s)(u'(s))^\alpha)'u'(s)dsds + t\int_{t_0}^{t}p(s)K_2(s,u(s),u'(s))u'(s)ds + t\int_{t_0}^{t}q(s)\frac{d}{ds}F(u(s))ds \leqslant t\int_{t_0}^{t}u'(s)\varphi(s)ds. \tag{3.8}$$

Let $q(t)$ and $u'(t)$ be non-decreasing, then $q'(t) \geqslant 0$, $u''(t) \geqslant 0$, and $q(t) \geqslant \delta$, where $\delta > 0$, then inequality (3.8) becomes

$$\int_{t_0}^{t}r(s)(u'(s))^{\alpha+1}ds + t\int_{t_0}^{t}p(s)K_2(s,u(s),u'(s))u'(s)ds + t\delta F(u(t)) \leqslant t\int_{t_0}^{t}u'(s)\varphi(s)ds.$$

Using condition (iv) and by the application of Theorem 2.7 implies there exists $\rho, \xi, \vartheta \in [t_0,t]$ such that

$$|F(u(t))| \leqslant \left(\frac{|u'(\vartheta)|}{\delta}+\frac{(|u'(\xi)|)^{\alpha+1}}{\delta}\int_{t_0}^{t}r(s)ds\right) \times \int_{t_0}^{t}\varphi(s)ds + \frac{(|u'(\rho)|)^2}{\delta}\int_{t_0}^{t}p(s)\psi(s)\varpi(u(s))ds,\ \forall t \geqslant 1. \tag{3.9}$$

Using Theorem 2.6 on inequality (3.9) we obtain

$$|u(t)| \leqslant \left(\frac{\lambda}{\delta} + \frac{(\lambda)^{\alpha+1}}{\delta}\int_{t_0}^{t} r(s)ds\right)\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{(\lambda)^2}{\delta}\int_0^t p(s)\psi(s)ds\right), \quad \forall t \geqslant 1.$$

where $|F(u(t))| \geqslant |u(t)|$, $u'(t) \leqslant \lambda$ for $\lambda > 0$. Using conditions (iv) and (vi) of Theorem 3.1, and hypothesis of Theorem 3.2, we obtain

$$|u(t)| \leqslant \left(\frac{\lambda}{\delta} + \frac{\lambda^{\alpha+1}}{\delta}n_4\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}n_2\right)\varphi(t),$$

hence, $|u(t) - u(t_0)| \leqslant C_\varphi \varphi(t)$, with

$$C_{\varphi_4} = \left(\frac{\lambda}{\delta} + \frac{\lambda^{\alpha+1}}{\delta}n_4\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}n_2\right).$$

□

Example 3.5. Consider the equation

$$\left(\frac{1}{t^5}u^2(t)(u'(t))^2\right)' + \frac{1}{t^3}u^2(t)(u'(t))^4 + t^2u^2(t) = \frac{1}{t^6}u^2(t)(u'(t))^8, \quad t \geqslant 1,$$

where $K_1(t, u(t), u'(t)) = \frac{1}{t^5}u^2(t)(u'(t))^2$, $K_2(t, u(t), u'(t)) = \frac{1}{t^3}u^2(t)(u'(t))^4$, $P(t, u(t), u'(t)) = \frac{1}{t^6}u^2(t) \times (u'(t))^8$, $r(t) = t^3$, $\alpha(t) = \frac{1}{t^8}$, $p(t) = \frac{1}{t^5}$, $\psi(t) = t^2$, $\phi(t) = \frac{1}{t^6}$. By Criteria of Theorem 3.1, we arrive at the result.

Example 3.6. Investigate Hyers-Ulam-Rassias stability of the equation

$$\left(\frac{1}{t^5}(u'(t))^5\right)' + \frac{1}{t^3}u^2(t)(u'(t))^4 + t^2u^2(t) = \frac{1}{t^6}u^2(t)(u'(t))^8, \quad t \geqslant 1,$$

where $K_2(t, u(t), u'(t)) = \frac{1}{t^3}u^2(t)(u'(t))^4$, $P(t, u(t), u'(t)) = \frac{1}{t^6}u^2(t)(u'(t))^8$, $r(t) = \frac{1}{t^5}$, $p(t) = \frac{1}{t^5}$, $\psi(t) = t^2$, $\phi(t) = \frac{1}{t^6}$. By Criteria of Theorem 3.2, we arrive at the result.

4. Conclusions

The results obtained in this paper on Hyers-Ulam-Rassias stability of second order nonlinear damped differential equations extended the aforementioned results. Also new criteria were obtained for Hyers-Ulam-Rassias stability.

We say at this point that there is no competing interest or conflicts that can hinder this article not to be published.

Acknowledgment

We would like to thank the reviewers for careful reading of our paper and lot of valuable suggestions on the first draft of the manuscript.

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