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CARTESIAN PRODUCTS OF $PQPM$-SPACES

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ABSTRACT. In this paper we define the concept of finite and countable Cartesian products of $PqpM$-spaces and give a number of its properties. We also study the properties of topologies of those products.

Introduction

Let (X, P_1) and (Y, P_2) are PM-spaces under triangle function $*$ and a pair $(X \times Y, P_1 \times P_2)$ is a finite product of PM-spaces (see Tardiff [11], Urazov [12]), when the function $P_1 \times P_2 : (X \times Y)^2 \to \Delta^+$ is given by formula:

$$P_1 \times P_2(u, v) = P_1(x_1, y_1) * P_2(x_2, y_2)$$

for any $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $X \times Y$. Convolution of Wald space [13], as well as several types of products of PM-spaces, where first defined by Istrăţescu and Vadura [4]. If T is a t-norm and $* = *_T$, then $*$-product is the T-product on defined independently by Egbert [1] and Xavier [14]. It is immediat that $*$-product of PM spaces is PM-space (see (Sherwood, Taylor [9]), (Höle [3]). In section 1 we extended this notion and results of T-product of $PqpM$-spaces. In section 2 we give definition and some results on countable products of $PqpM$-spaces of type $\{k_n\}$.

0. Preliminary notes and results

Definition 0.1 ([8]). A distance distribution function is a nondecreasing function $F : (-\infty, +\infty) \to [0, 1]$ which is left-continuous on $(-\infty, +\infty)$ and $F(0) = 0$

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and $\lim_{x\to\infty} F(x) = 1$. We denote by Δ^+ the set of all distribution functions and by ε_a specific distribution function by

$$\varepsilon_a(t) = \begin{cases} 1, & \text{for } t > a, \\ 0, & \text{for } t \le a, \ a \in R. \end{cases}$$

The element of Δ^+ are partially ordered by

$$F \le G \text{ if and only if } F(x) \le G(x), \text{ for } x \in R.$$

For eny $F, G \in \Delta^+$ and $h \in (0, 1]$, let (F, G, h) denote the condition

$$G(x) = F(x + h) + h \quad \text{for all} \ \ x \in (0, h^{-1})$$

and

$$d_L(F, G) = \inf\{h : \ \text{both } (F, G; h) \text{ and } (G, F; h) \text{ hold}\}.$$

As shown by Sibley [10] the function d_L is a metric in Δ^+ which is a modified form on the well-known Levy metric for distribution functions and the metric space (Δ^+, d_L) is compact and hence complete (see [8, pp. 45-49]).

Definition 0.2 ([8,15]). A binary operation $* : \Delta^+ \times \Delta^+ \to \Delta^+$ is a *triangle function* if $(\Delta^+, *)$ is an Abelian monoid with identity ε_0 in Δ^+ such that, for any $F, F', G, G' \in \Delta^+$,

$$F * G \le F' * G' \ \ \text{whenever} \ \ F \le F', \ G \le G'.$$

Note that a triangle function $*$ is continuous if it is continuous with respect to the metric topology induced by d_L.

Let $T(\Delta^+)$ denote the family of all triangle functions $*$ then the relation $\le$ defined by

$$*_1 \le *_2 \Leftrightarrow F *_1 G \le F *_2 G, \ \ \text{for all} \ \ F, G \in \Delta^+ \tag{0.2.1}$$

is a partial order in the family $T(\Delta^+)$.

The second relation in the set $T(\Delta^+)$ $\gg$ is defined by

$$*_1 \gg *_2 \Leftrightarrow ((E *_2 G) *_1 (F *_2 H)) \ge ((E *_1 F) *_2 (G *_1 H)), \tag{0.2.2}$$

for all $E, F, G, H \in \Delta^+$.

We can see the connection between the two relation: $*_1 \gg *_2$ implies $*_1 \ge *_2$ and following conditions: $\min \ge *$ and $\min \gg *$, for all $*$.

Definition 0.3 ([2]). A *probabilistic-quasi-pseudo-metric-space* (briefly, a *Pqp*-metric space) is a triple $(X, P, *)$, where X is a nonempty set, P is a function from $X \times X$ into $\Delta^+, *$ is a triangle function and the following conditions are satisfied (the value of P at (x, y) in X^2 will be denoted by P_{xy}):

$$P_{xx} = u_0, \ \text{for all} \ \ x \in X, \tag{0.3.1}$$

$$P_{xy} * P_{yz} \le P_{xz}, \ \ \text{for all} \ \ x, y, z \in X. \tag{0.3.2}$$

If P satisfies also the additional condition

$$P_{xy} \ne \varepsilon_0 \ \ \text{it} \ \ x \ne y, \tag{0.3.3}$$

then $(X, P, *)$ is called a probabilistic quasi-metric space.

Moreover, if P satisfies the condition of symmetry:

$$P_{xy} = P_{yz},$$

then $(X, P, *)$ is called a probabilistic metric spac (PM-space).

If the function $Q : X^2 \to \Delta^+$ be defined by

$$Q_{xy} = P_{yx}, \quad \text{for all} \quad x, y \in X, \tag{0.3.4}$$

then a triple $(X, Q, *)$ is also a probabilistic-quasi-pseudo-metric space. We say P and Q are conjugate each other.

Lemma 0.4. *Let $(X, P, Q, *)$ be a structure defined by Pqp-metric P and $*_1 \gg *$ (0.2.2). Then $(X, F^{*_1}, *)$ is a probabilistic pseudo-metric space whenever the function $F^{*_1} : X^2 \to \Delta^+$ is given by:*

$$F^{*_1}_{xy} = P_{xy} *_1 Q_{xy}, \quad \textit{for all} x, y \in X_0. \tag{0.4.1}$$

If additionally, P satisfies the condition

$$P_{xy} \neq u_0 \quad \textit{or} \quad Q_{xy} \neq u_0 \quad \textit{for} \quad x \neq y, \tag{0.4.2}$$

then $(X, F^{*_1}, *)$ is a PM-space.

Lemma 0.5 ([2, Example 9]). *If (X, p) is a quasi-pseudometric-space and the function $P_p : X^2 \to \Delta^+$ is defined by*

$$P_p(x, y) = \varepsilon_p(x, y), \quad \textit{for all} \quad x, y \in X$$

and $*$ is a triangle function such that

$$\varepsilon_a * \varepsilon_b \geq \varepsilon_{a+b} \quad \textit{for all} \quad a, b \in R^+,$$

then $(X, P_p, *)$ is a proper Pqp-metric space.

Theorem 0.6 ([2, Theorem 6]). *Let $(X, P, *)$ be a Pqp-metric space under a uniformly continuous t-function $*$ and, for any $x \in X$, and $t > 0$, the P-neighborhood of x be a set*

$$N_x^P(t) = \{y \in X : d_L(P_{xy}, u_0) < t\}.$$

Then the collection of all P-neighborhood form a base for the topology τ_P on X the Pqp-metric Q which is a conjugate of P generate a topology τ_Q on X. Thus natural structure associated with a Pqp-metric is a bitopological space (X, τ_P, τ_Q).

It is worthy of note that in the spaces $(X, P_p, Q_q, *)$, the τ_{P_p}-topology is equivalent to the q-quasi-pseudometric topology τ_{P_q} (see [2], [11]).

Lemma 0.7. *Let $(X, P, *)$ be a PqpM-space. Then the relation $\leq_P$ defined by*

$$x \leq_P y \ \textit{if and only if} \ \ P_{xy} = \varepsilon_\tau \tag{0.7.1}$$

is reflexive and transitive, i.e. it is a quasi-order on X.

Proof. Reflexivity follows immediately from (0.2.1) and transivites is a consequence of (0.3.2).

Corollary 0.8. *If Pqp-metric satisfies the assumption (0.4.2), then the relation $\leq_P$ is a partial ordering on X.*

Proof. Assume that $x \leq_P y$ and $y \leq_P x$. This means that

$$P_{xy} = \varepsilon_0 \text{ and } P_{yx} = u.$$

By (0.4.2), it follows that $P_{xy} = P_{yx} = \varepsilon_0$ if and only if $x = y$.

Corollary 0.9. *If $x \neq y$ imply $P_{xy} = \varepsilon_0$ and $P_{yx} \neq \varepsilon_0$ or $P_{xy} \neq \varepsilon_0$ and $P_{yx} = \varepsilon_0$, then $\leq_P$ is a linear ordering on X.*

Remark 0.10. If Q is a conjugate of a Pqp-metric P, then the relation $\leq_Q$ generated by Q is also a quasi-ordering on X and is the inverse relation of $\leq_P$.

1. Cartesian products of PqpM-spaces

In this section, we give some properties of Cartesian products of $PqpM$-spaces.

Definition 1.1. Let $(X, P_1, *)$ and $(Y, P_2, *)$ be $PqpM$-spaces. The $*$-product of (X, P_1) and (Y, P_2) is the pair $(X \times Y, P_1 \times P_2)$, where $P_1 \times P_2$ is the function from $(X \times Y)^2$ into Δ^+ given by

$$P_1 \times P_2(u, v) = P_1(x_1, y_1) * P_2(x_2, y_2) \tag{1.1}$$

for any $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $X \times Y$.

Theorem 1.2. *Let $(X, P_1, *)$ and $(X, P_2, *)$ be $PqpM$-spaces. Let a mapping $P_1 \times P_2 : (X \times Y)^2 \to \Delta^+$ be given by*

$$P_1 \times P_2(u, v) = (P_1(x_1, x_2) *_1 P_2(y_1, y_2)) \quad \textit{with} \quad *_1 \gg * \tag{1.2}$$

for any $u = (x_1, y_1)$, $v = (x_2, y_2) \in X \times Y$. Then $(X \times Y, P_1 \times P_2, *)$ is a $PqpM$-space.

Proof. If $u = v$, then $x_1 = x_2$ and $y_1 = y_2$. Thus, by (0.6.1), we have

$$P_1 \times P_2(u, v) = P_1(x_1, x_1) *_1 P_2(y_1, y_1) = u_0 *_1 u_0 = u_0.$$

Now, let $w = (x_3, y_3)\ X \times Y$. Then, by (0.3.2) and (0.2.2), we obtain

$$\begin{aligned} P_1 \times P_2(u, v) &= P_1(x_1, x_2) *_1 P_2(y_1, y_2) \\ &\geq (P_1(x_1, x_3) * P_1(x_3, x_2)) *_1 (P_2(y_1, y_3) * P_2(y_3, y_2)) \\ &\geq (P_1(x_1, x_3) *_1 P_2(y_1, y_3)) * ((P_1(x_3, x_2) *_1 P_2(y_3, y_2)) \\ &= P_1 \times P_2(u, v) * P_1 \times P_2(w, v). \end{aligned}$$

This completes the proof.

Definition 1.3. Let $(X, P_1, *)$ and $(X, P_2, *)$ be $PqpM$-spaces and let $*_1 \gg *$. Then $(X \times Y, P_1 \times P_2, *_1)$ is called a *Cartesian $*_1$-product* of $PqpM$-spaces provided that $P_1 \times P_2$ is given by the formula (1.1).

By Definition 0.2 and (0.2.2), it follows that Min$\gg *$ holds true for any t_{Δ^+}-norm $*$. Thus it follows that the function given by

$$\begin{aligned} P_1 \times P_2(u, v) =& \text{Min}\,(P_1(x_1, x_2), P_2(y_1, y_2)) \\ =& P_1(x_1, x_2) \times P_2(y_1, y_2) \end{aligned}$$

satisfies the conditions of Theorem 1.2.

Theorem 1.4. *Let $(X, G_{p_1}, *_M)$ and $(Y, G_{p_2}, *_M)$ be a PqpM-space defined by formula $G_p(x,y) = G\left(\frac{t}{p(x,y)}\right)$, where $G \in \Delta^+$ be distinct from ε_0 and ε_∞, which were generated by quasi-pseudo-metric p_1 and p_2, respectively. Let a function $p_1 \vee p_2\ (X \times Y)^2 \to \mathbb{R}^+$ be defined by*

$$p_1 \vee p_2(u,v) = \text{Max}\,(p_1(x_1,x_2), p_2(y_1,y_2)),$$

where $u = (x_1, x_2)$ and $v = (y_1, y_2)$ belong to $X \times Y$. Then the triple $(X \times Y, G_{p_1 \vee p_2}, *_M)$ is a $PqpM$-space generated by a quasi-pseudo-metric $p_1 \vee p_2$.

As a consequence of Theorem 1.2, we have the following:

Corollary 1.5. *Let $(X, P, *)$ be a PqpM-space. Let $*_1 =$Min. Then there are four Pqp-metrics on $X \times X$ generated by the function P, that is, $P \times P, P \times Q, Q \times P$ and $Q \times Q$, where Q is the Pqp-metric conjugate with P.*

Remark 1.6. Note that, by Definition 1.1, for all $u, v \in X \times X$, the following equalities hold:

$$P \times P(v,u) = Q \times Q(u,v), \quad P \times Q(v,u) = Q \times P(u,v).$$

Therefore, the pairs $P \times P$ and $Q \times Q$ as well as $P \times Q$ and $Q \times P$ are the mutually conjugate Pqp-metrics defined on $X \times X$. The function $M\ (X \times Y)^2 \to \Delta^+$ given by

$$M(u,v) = P \times Q(u,v) \wedge Q \times P(u,v) = P \times P(u,v) \wedge Q \times Q(u,v)$$

for all $(u,v) \in X \times X$ is a probabilistic pseudo-metric on $X \times X$.

Corollary 1.7. *Let $(X, P, *)$ be a PqpM-space. Let the t_{Δ^+}-norm $*$ be continuous at $(\varepsilon_0, \varepsilon_0)$ and $*_1 =$Min. Then the topology $T_{P\times P}$ generated by the function $P \times P$ is equivalent to the topology $T_P \times T_P$. Also, the topologies $T_P \times T_Q$ and $T_{P\times Q}, T_Q \times T_Q$ and $T_{P\times Q}$, and $T_Q \times T_Q$ and $T_{Q\times Q}$ are equivalent.*

Proof. For an illustration, we prove the first equivalence. Let $t_1, t_2 > 0$ and $x, y \in X$. Then we have

$$N_x^P(t_1) \times N_y^P(t_2) \in T_P \times T_P.$$

Let $t_3 = \max(t_1, t_2)$ and $u = (x, y) \in X \times X$. Then a $P \times P$-neighbourhood of a point $u \in X \times X$ is of the form:

$$\begin{aligned} N_u^{P\times P}(t_3) &= \{v = (x_1, x_2)\ P \times P(u,v)(t_3) > 1 - t_3\} \\ &= \{v = (x_1, x_2)\ P_{xx_1}(t_3) > 1 - t_3 \ \text{ and } \ P_{yy_1}(t_3) > 1 - t_3\}, \\ N_x^P(t_3) &\times N_y^P(t_3) \subset N_x^P(t_1) \times N_y^P(t_2). \end{aligned}$$

On the other hand, for each $t > 0$ and $u = (x, y) \in X \times X$, we have

$$N_u^{P\times P}(t) = N_x^P(t) \times N_y^P(t).$$

The remaining cases can be verified similarly. This completes the proof.

Theorem 1.8. *Let $(X, P, *)$ be a PqpM-space. Assume that the t_{Δ^+}-norm $*$ is continuous and let $\leq_P$ be the quasi-order generated by P (in the sense of Lemma 0.7). Then the set $G(\leq_P) = \{(x,y) \in X^2\ x \leq_P y\}$ is closed in the topology $T_{P\times Q}$.*

Proof. Assume that (x_1, y_1) belongs to the $P \times Q$-closure of $G(\leq_P)$ and does not belong to $G(\leq_P)$. Then, by Corollary 0.9, $P_{x_1y_1} \neq \varepsilon_0$ and there exists a sequence $\{(x_n, y_n)\}$ of $G(\leq_P)$ which is $P \times Q$-convergent to (x_1, y_1). This means that

$$P_{x_1x_n} \to \varepsilon_0, \quad Q_{y_1y_n} \to \varepsilon_0.$$

Thus, by (0.3.2), we have

$$\begin{aligned} \varepsilon_0 \neq P_{x_1y_1} &\geq P_{x_1x_n} * P_{x_ny_1} \geq p_{x_1x_n} * P_{x_ny_n} * P_{y_ny_1} \\ &= P_{x_1x_n} * P_{x_ny_n} * Q_{y_1y_n} \\ &= P_{x_1x_n} * \varepsilon_0 * Q_{y_1y_n} \\ &= P_{x_1x_n} * Q_{y_1y_n} \to \varepsilon_0, \end{aligned}$$

which is a contradiction. This completes the proof.

Lemma 1.9. *Let $(X, P, *)$ be a PqpM-space satisfying the condition* (0.3.4), *and let the t_{Δ^*}-norm $*$ be continuous. Then the set*

$$(\leftarrow, x] = \{y \in X \; y \leq_P x\},$$

where $\leq_P$ is the order generated by P, is a subset of $N_x^Q(t)$ for every $t > 0$.

Proof. If $y \in (\leftarrow, x]$, then $y \leq_P x$ and so, by (0.3.4), we have

$$P_{yx} = Q_{xy} = \varepsilon_0.$$

Therefore, we have $y \in N_x^Q(t)$ for every $t > 0$.

Corollary 1.10. *The set $(\leftarrow, x]$ is G_δ in the topology T_Q.*

Proof. For $t > 0$, there is a natural number n such that $\frac{1}{n} < t$. Then we have

$$Q_{xy}(t) \geq Q_{xy}(\frac{1}{n}) > 1 - \frac{1}{n} > 1 - t,$$

which means that

$$N_x^Q(\frac{1}{n}) \subset N_x^Q(t).$$

Therefore, we conclude that the family $\{N_x^Q(\frac{1}{n})\}_{n \in \mathbb{N}}$ satisfies the assertion. This completes the proof.

Lemma 1.11. *The set $(\leftarrow, x]$ is P-closed.*

Proof. Assume that y belongs to the P-closure of $(\leftarrow, x]$ and $y \notin (\leftarrow, x]$. Then $P_{yx} \neq \varepsilon_0$ and, for each $n \in \mathbb{N}$, there is $x_n \in (\leftarrow, x]$ such that

$$P_{yx_n} \to \varepsilon_0.$$

Finally, we have

$$\varepsilon_0 \neq P_{yx} \geq P_{yx_n} * P_{x_ny} = P_{yx_n} * \varepsilon_0 = P_{yx_n} \to \varepsilon_0,$$

which is a contradiction. This completes the proof.

Corollary 1.12. *The set $[x, \rightarrow) = \{y \in X \; x \leq_P y\}$ is a Q-closed and G_δ in the topology T_P.*

The following result is an immediate consequence of Lemma 1.2:

Theorem 1.13. *Let $(X, P, *)$ be a PqpM-space satisfying the condition of Corollary 0.9 and let the t_{Δ^+}-norm $*$ be continuous. Then the family $\{(\leftarrow, x]\}_{x\in X}$ forms a P-closed subbase of a topology, which is denoted by $T(\leftarrow]$. Similarly, the family $\{[x, \rightarrow)\}_{x\in X}$ forms a Q-closed subbase of $T[\rightarrow)$.*

We note that these families form, respectively, a P-closed and Q-closed base and that the function P generates such a partial order $\leq_P$ in X which is a lattice order.

Lemma 1.14. *Let $(X, P, *)$ be a PqpM-space satisfying the condition of Corollary 0.9 and let the t_{Δ^+}-norm $*$ be continuous. Then the set $(\leftarrow, x) = \{y \in X y <_P x\}$ is Q-open and the set $(x, \rightarrow) = \{y \in X\ x <_P y\}$ is P-open.*

Proof. By Corollary 0.9, it follows that $\leq_P$ orders X linearly. Hence we have $(\leftarrow, x) \subset N_x^Q(t)$ for all $t > 0$. On the other hand, for each $y \in (\leftarrow, x)$, we have $Q_{xy} \neq \varepsilon_0$. This means that there exists $t > 0$ such that

$$Q_{yx}(t) > 1 - t.$$

We thus have $N_y^Q(t) \subset (\leftarrow, x)$. This completes the proof.

Corollary 1.15. *Let $(X, P, *)$ be a PqpM-space satisfying the condition of Corollary 0.2 and let the t_{Δ^+}-norm $*$ be continuous. The family $\{(\leftarrow, x)\}_{x\in X}$ is a Q-open base for T_Q. Similarly, the family $\{(x, \rightarrow)\}_{x\in X}$ is a P-open base for the topology T_P.*

Theorem 1.16. *Let $(X, P, *)$ be a PqpM-space. Then the family $\{(\leftarrow, x]\}_{x\in X}$ is a complete neighbourhood system in the space X. It thus defines some topology on X. Similarly, $\{[x, \rightarrow)\}_{x\in X}$ forms a complete neighbourhood system in X.*

Proof. It suffices to observe that, for each $x \in X$, $x \in (\leftarrow, x]$ and, if $y \in (\leftarrow, x]$, then we have

$$(\leftarrow, y] \subset (\leftarrow, x].$$

2. Cartesian product in PqpM-spaces of the type $\{k_n\}$

The following result characterizes countable Cartesian products of PqpM-spaces.

Definition 2.1. Let $\{(X_n, P_n)\}$ be a sequence of PqpM-spaces and let the sequence $\{k_n\}$ of nonnegative numbers satisfy the condition $\sum_{n\in\mathbb{N}} k_n = 1$. Then the pair (X, P) is called a *Cartesian product* of PqpM-spaces of the type $\{k_n\}$ if $X = \prod_{n\in\mathbb{N}} X_n$ and $P : X^2 \to \Delta^+$ is given by

$$P_{xy} = \sum_{n\in\mathbb{N}} k_n P_n(x_n, y_n), \tag{2.1}$$

where $x = \{x_n\}$ and $y = \{y_n\}$.

Definition 2.2 ([5]). A function $T : I^2 \to I$ $(I = \langle 0, 1\rangle)$ is called a t-norm if it satisfies the following conditions

(T1) $T(a, b) = T(b, a)$
(T2) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $\le d$
(T3) $T(a, 1) = a$
(T4) $T(T(a, b), c) = T(a, T(b, c))$, for all $a, b, c, d \in I$.
(TA) The t-norm T is said to be Archimedean if for any $x, y \in (0, 1)$, there exists $n \in N$ such that

$$x^n < y, \text{ that is } x^n \le y \text{ and } x^n \ne y,$$

where $x^0 = 1, x^1 = x$ and $x^{n+1} = T(x^n, x)$, for all $n \ge 1$. We shall now establish the notation related to a few most important t-norm:

$$M(x, y) = \text{Min } (x, y), \tag{TM}$$

$$\Pi(x, y) = x \cdot y, \tag{TΠ}$$

$$W(x, y) = \text{Max } (x + y - 1, 0). \tag{TW}$$

The function W is continuous and Archimedean and we give the following relations among t-norms

$$M \ge \Pi \ge W. \tag{TR}$$

Definition 2.3. Let X be a nonempty set, $P : X^2 \to D$, and I in T_I-norm. The triple (X, P, T) is called a quasi-pseudo-Menger space if it satisfies the axioms:

(M1) $P_{xx} = \varepsilon_0,\ x \in X$,
(M2) $P_{xz}(t_1 + t_2) \ge T(P_{xy}(t_1), P_{yz}(t_2))$, for all $x, y, z \in X$ and $t_1, t_2 > 0$.

If P satisfies also the additional condition:

(M3) $P_{xy} \ne \varepsilon_0$ if $x \ne y$, then (X, P, T) is quasi-Menger space.

Moreover, if P satisfies the condition of symmetry $P_{xy} = P_{yx}$, then (X, P, T) is called a Menger-space (see [5]).

Definition 2.4. Let (X, P, T) be a probabilistic quasi-Menger space (PqM) and the function $Q : X^2 \to D$ be defined by

$$Q_{xy} = P_{yx}, \quad \text{for all} \quad x, y \in X.$$

Then the ordered triple (X, Q, T) is also PqM-space. The function Q is called a conjugate Pqp-metric of the P. By (X, P, Q, T) we denote the structure generated by the Pqp-metric P on X.

Lemma 2.5. *Let (X_n, P_n) be a sequence of proper $PqpM$-spaces (Lemma 0.5). Then the Cartesian product (X, P) of the type $\{k_n\}$ is also a proper $PqpM$-space. Also, if each (X_n, P_n) is a quasi-pseudo-Menger space with respect to the t_I-norm of type (TA), then so is the Cartesian product of type $\{k_n\}$. Moreover, the topology T_p of a Cartesian product of the type $\{k_n\}$ generated by P is equivalent to the product topology.*

Proof. For proper $PqpM$-spaces, the condition $F \ge u_a$ is equivalent to the statement that $F(a+) = 1$. It thus suffices to observe that, if, for some $a > 0$,

$$P_{x_n y_n}(a+) = 1$$

for all $x_n, y_n \in X_n$, then, by (2.1), we obtain

$$P_{xy}(a+) = \lim_{t \to a}(\Sigma k_n P_{x_n y_n}(t)) = \Sigma k_n = 1.$$

To prove the second part of the theorem, let us observe that, by the definition of the t_I-norm W and the Menger condition (M2), the following holds:

$$\begin{aligned} W(P_{xz}(t), P_{zy}(s)) =& \text{Max } (\Sigma k_n P_n(x_n, z_n)(t) + \Sigma k_n P_n(z_n, y_n)(s) - 1, 0) \\ =& \text{Max } ((\Sigma k_n (P_n(x_n, z_n)(t) + P_n(z_{n,n})(s) - 1, 0) \\ \leq& \text{Max } ((\Sigma k_n \text{ Max } (P_n(x_n, z_n)(t) + P_n(z_n, y_n)(s) - 1, 0), 0) \\ =& \Sigma k_n W(P_n(x_n, z_n)(t), P_n(z_n, y_n)(s) \\ \leq& \Sigma k_n P_n(x_n, y_n)(t + s) = P_{xy}(t + s). \end{aligned}$$

Therefore, we have proved that the Cartesian product of the type $\{k_n\}$ is a quasi-pseudo-Menger space.

In order to prove the third assertion, let us suppose that the sequence $\{x^n\}$ is P-convergent to $x = \{x_k\}$ in (X, P). Then, for each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$x^n \in N_x^P(t)$$

for all $n > n_0$. Suppose, further, that, for some $i_0 \in \mathbb{N}$, the sequence $\{x_{i_0}^n\}$ is not convergent to $x_{i_0} \in X_{i_0}$ which is the i_0-th coordinate of x. This means that, for some $t_0 > 0$, there exists $m_n > n$ for all n such that

$$P_i(x_{i_0}, x_{i_0}^{m_n})(t_0) > 1 - t_0.$$

Let $t = k_{i_0} t_0$. Then, for all $n > n_0$, we get

$$\begin{aligned} 1 - t &= 1 - k_{i_0} t_0 \\ &< P_{xx} m_n(t) \\ &= \sum_{i \in \mathbb{N}} k_i P_i(x_i, x_i^{m_n})(t) \\ &\leq \sum_{i = i_0} k_{i_0} + (1 - t_0) \\ &= 1 - k_{i_0} + k_{i_0} - k_{i_0} t_0 \\ &= 1 - t, \end{aligned}$$

which is a contradiction. This means that, if $\{x^n\}$ is P-converegent, then each sequence $\{x_i\}$ is P_i-convergent to x_i for all $i \in \mathbb{N}$. Thus the projections onto the i-th coordinate are continuous. Therefore, the topology of the Cartesian product of the type $\{k_n\}$ is stronger than the product topology.

Now, let U be a P-open set of T_P. Then, if $x \in U$, there exists a P-neighbourhood $N_x^P(t_0) \subset U$. Let $F \subset \mathbb{N}$ be a finite subset such that

$$\sum_{j \in F} k_j - (1 - t_0) > 0.$$

For every $j \in F$, we selsct $y_j \in N_{x_j}^{P_j}(t_0)$ and fix $t = 1-(1-t_0)(\sum_{j\in F} k_j)^{-1}$. Then, for each $y = \{y_j\}$ such that $y_i = y_j$ for $i = j$. where $j \in F$, we get

$$\begin{aligned} P_{xy}(t_0) &= \sum_{i\in\mathbb{N}} k_i P_i(x_j, y_j)(t_0) \\ &> \sum_{j\in F} k_j P_j(x_j, y_j)(t_0) \\ &> \sum_{j\in F} k_j(1-t) \\ &= 1 - t_0. \end{aligned}$$

Thus it follows that $y \in N_x^P(t_0)$. Let U_i be P_i-open with $U_i = X_i$ for $i \in \mathbb{N} - F$ and $U_i = N_{x_i}^{P_i}(t)$ for $i \in F$. Then we have

$$x \in \prod_{i\in\mathbb{N}} U_i \subset U,$$

which shows that U is open in the product topology. This completes the proof.

Corollary 2.6. *Each finite or countable Cartesian product of quasi-pseudo-metrizable spaces is quasi-pseudo-metrizable.*

Proof. By Lemma 2.5, it follows that each finite or countable cartesian product of quasi-pseudo-Menger spaces is a quasi-pseudo-Menger space with uspect to the t-norm W. Since that $\sup\{W(x,x) : x < 1\} = 1$ the topology of it is quasi-pseudo-metrizable (see [6], [7]). Indeed, let p be a quasi-pseudo-metric that generates the topology. Then (X, G_p) of Theorem 1.4 satisfies the required condition.

Remark 2.7. We note that the Cartesian products of PM-space were studied by Istratescu and Vadura [4], Egbert [1], Sherwood and Taylor [9] and Radu [6].

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