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SEPARATION THEOREM WITH RESPECT TO SUB-TOPICAL FUNCTIONS AND ABSTRACT CONVEXITY

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ABSTRACT. This paper deals with topical and sub-topical functions in a class of ordered Banach spaces. The separation theorem for downward sets and sub-topical functions is given. It is established some best approximation problems by sub-topical functions and we will characterize sub-topical functions as superimum of elementary sub-topical functions.

1. INTRODUCTION AND PRELIMINARIES

Topical functions are intensively studied (see [2,3]), and they have many applications in various parts of applied mathematics in particular in the modeling of discrete event system (see [2,3]). Topical functions are also interesting from a different point of view, namely as a tool in the study of the so-called downward sets. Downward set arise in the study of some problems of mathematical economics and game theory (see [4]).
Moreover, topical functions have studied in much more general class of sub-topical (*increasing plus-sub-homogeneous*) functions (see [9]). In section 1, we recall some definitions and establish some results related to topical functions of $\varphi(x, y) := \sup\{\lambda \in \mathbb{R} : \lambda \cdot 1 \leq x + y\}$. In section 2, we will prove some basic properties of sub-topical functions, we prove separability theorem for downward sets and sub-topical functions. In section 3, it is given other form of separation theorem with respect to sub-topical function. We would characterize best approximation problem by sub-topical functions.It is given separation theorem for

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downward sets and sub-topical functions. Finally we establish best approximation problem by sub-topical functions and characterize sub-topical functions as superimum of elementary sub-topical functions.

Let $(X, \| \ . \ \|)$ be a Banach space and C be a closed convex cone in X such that $C \cap (-C) = \{0\}$ and int$C \neq \emptyset$. X is equipped with the relation $\geq$ i.e; generated by $C : y \leq x$ if and only if $x - y \in C$ $(x, y \in X)$ and $C : y < x$ if and only if $x - y \in intC$ $(x, y \in X)$. Assume that C is a normal cone. Recall that a cone C is called *normal* if there exits a constant $m > 0$ such that $\|x\| \leq m\|y\|$, whenever $0 \leq x \leq y$ and $x, y \in X$. Let $\mathbf{1} \in int\ C$ and

$$B = \{x \in X : \textbf{-1} \leq x \leq \mathbf{1}\}. \ (1)$$

It is well known and easy to check that B can be considered as the unit ball under a certain norm $\|.\|_1$, which is equivalent to the initial norm $\|.\|$. Without loss of generality one can assume that $\|.\| = \|.\|_1$ (see [6]).
For any subset W of X, denote by $intW$, clW and bdW the interior, closure and boundary of W respectively.
For a non-empty subset W of X and $x \in X$, define (see[6])

$$d(x, W) = \inf_{w \in W} \|x - w\|. \ (2)$$

Recall (see [6]), a point $w_o \in W$ is called *best approximation* for $x \in X$, if

$$\|x - w_o\| = d(x, W).$$

Let $W \subset X$. For $x \in X$ it is denoted by $P_W(x)$ the set of all best approximations of x in W:

$$P_W(x) = \{w \in W : \|x - w\| = d(x, W)\}. \ (3)$$

It is well-known that $P_W(x)$ is a closed and bounded subset of X. If $x \in X \diagdown W$ then $P_W(X)$ is located in the boundary of W (see [6]).
For $x \in X$ and $r > 0$, according to (1),

$$B(x, r) := \{y \in X : \|x - y\| \leq r\} = \{y \in X : x - r \cdot \mathbf{1} \leq y \leq x + r \cdot \mathbf{1}\}. \ (4)$$

Definition 1.1. [5, 6] A function $f : X \longrightarrow \overline{\mathbb{R}}$ to the set of extended real numbers is a *topical function* if
a) (*Plus-homogeneous*), i.e, $f(x + \lambda.\mathbf{1}) = f(x) + \lambda$ for $\forall x \in X$ and $\forall \lambda \in \mathbb{R}$,
b) (*Increasing function*), i.e; if $x \leq y$ then $f(x) \leq f(y)$.

Let $\varphi : X \times X \longrightarrow \mathbb{R}$ be a function which is defined by

$$\varphi(x, y) := \sup\{\lambda \in \mathbb{R} : \lambda \cdot \mathbf{1} \leq x + y\} \ (\forall x, y \in X). \ (5)$$

From (5) it is easy to see that the set $\{\lambda \in \mathbb{R} : \lambda \cdot \mathbf{1} \leq x + y\}$ is non-empty and bounded above by $\|x + y\|$. Clearly this set is closed. It follows from the definition of φ φ enjoys the following properties:

$$-\infty < \varphi(x, y) \leq \|x + y\| \text{ for all } x, y \in X. \ (6)$$

$$\varphi(x,y)\cdot \mathbf{1} \leq x+y \text{ for all } x,y\in X. \ (7)$$

$$\varphi(x,y)=\varphi(y,x) \text{ for all } x,y\in X. \ (8)$$

$$\varphi(x,-x)=\sup\{\lambda\in\mathbb{R}:\lambda\cdot\mathbf{1}\leq x-x=0\}=0 \text{ for all } x\in X. \ (9)$$

For each $y\in X$, define a function $\varphi_y : X \longrightarrow \mathbb{R}$ by

$$\varphi_y(x) := \varphi(x,y) \ \forall x\in X. \ (10)$$

Let $f : X \longrightarrow \mathbb{R}$. Recall that directional derivative $f'_+(x,u)$ of f at $x\in X$ in direction of $u\in X$ is defined by (see [7]),

$$f'_+(x,u) := \lim_{t\longrightarrow 0^+} \frac{f(x+tu)-f(x)}{t}. \ (11)$$

2. Basic properties of sub-topical functions

Definition 2.1. [9] A function $f : X \longrightarrow \overline{\mathbb{R}}$ is called *plus-sub-homogeneous* if

$$f(x+\lambda.\mathbf{1}) \leq f(x)+\lambda \ \forall x\in X \textit{ and } \forall\lambda\in\mathbb{R}_+. \ (12)$$

f is called *sub-topical* if it be increasing and plus-sub-homogeneous.In the following, we characterize plus-sub-homogenous which its proof is direct.

Lemma 2.2. *A function $f : X \longrightarrow \overline{\mathbb{R}}$ is plus-sub-homogeneous if and only if*

$$f(x+\lambda.\mathbf{1}) \geq f(x)+\lambda \ \forall x\in X \textit{ and } \forall\lambda\in\mathbb{R}_-. \ (13)$$

Let us present some examples of sub-topical functions:

Example 2.3. .
a) Every topical function is sub-topical.
b) Every sub-linear function such that $f(\mathbf{1}) \leq 1$ is sub-topical. Indeed

$$f(x+\lambda.\mathbf{1}) \leq f(x)+\lambda f(\mathbf{1}) \leq f(x)+\lambda \ \forall x\in X \textit{ and } \forall\lambda\in\mathbb{R}_+.$$

Following is a result for showing Lipschitz continuity of a sub-topical function. Its proof is direct.

Theorem 2.4. *Let $f : X \longrightarrow \mathbb{R}$ be a sub-topical function, then f is Lipschitz continuous.*

Theorem 2.5. *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a sub-topical function. Then the following assertions are true.*
a) If there exists $x\in X$ such that $f(x)=\infty$, then $f\equiv\infty$.
b) If there exists $x\in X$ such that $f(x)=-\infty$, then $f\equiv-\infty$.

Proof.
a) Suppose that there exists $x \in X$ such that $f(x) = \infty$. Let $y \in X$. $\lambda = \varphi(-x, y)$. There are two cases:

Case (1): If $\lambda < 0$, by (7) we have $\varphi(-x, y) \cdot \mathbf{1} \le y - x$, then $x \le y - \lambda \cdot \mathbf{1}$. Since $-\lambda > 0$ and f is sub-topical, so

$$f(x) \le f(y) - \lambda.$$

It implies that $f(y) = \infty$.

Case (2): If $\lambda \ge 0$, then $0 \le \varphi(-x, y) \cdot \mathbf{1} \le y - x$, so $x \le y$ and f is an increasing. Then $f(x) \le f(y)$, so $f(y) = \infty$.
b) Let $y \in X$ be an arbitrary, $\lambda = \varphi(x, -y)$. Then the remind of proof is similar to that one in (a). ♦

Theorem 2.6. *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be an increasing function. Then f is plus-sub-homogeneous if and only if,*
$f_x : \mathbb{R}_+ \longrightarrow \overline{\mathbb{R}}$ given by $f_x(\alpha) = f(x + \alpha \cdot \mathbf{1}) - \alpha$,
is decreasing.

Proof. If $f(x) = \infty$ (or, $-\infty$) for some $x \in X$, by theorem 2.5 $f \equiv \infty$ (or, $-\infty$), then $f_x \equiv \infty(-\infty)$ and so f_x is decreasing for all $x \in X$. Therefore, $f : X \longrightarrow \mathbb{R}$ is sub-topical and $0 \le \alpha \le \beta$.

$$\begin{aligned} f_x(\beta) = f(x + \beta \cdot \mathbf{1}) - \beta &= f(x + \alpha \cdot \mathbf{1} + (\beta - \alpha) \cdot \mathbf{1}) - \beta \\ &\le f(x + \alpha \cdot \mathbf{1}) + \beta - \alpha - \beta = f_x(\alpha). \end{aligned}$$

Conversely, if f_x is decreasing, $f_x(0) \ge f_x(\alpha)$ for all $\alpha \ge 0$. Hence, $f(x) \ge f(x + \alpha \cdot \mathbf{1}) - \alpha$ and f is plus-sub-homogeneous. ♦

Theorem 2.7. *Let $f : X \longrightarrow \mathbb{R}$ be an increasing function. Then f is plus-sub-homogeneous if and only if $f'_+(x, \mathbf{1}) \le 1$ $(\forall x \in X)$.*

Proof. $(\Longrightarrow)$. According to theorem 2.6, f_x is decreasing. Then $(f_x)'_+(\lambda) \le 0$ $(\forall x \in X)$.

$$(f_x)'_+(\lambda) = \lim_{t \longrightarrow 0^+} \frac{f_x(\lambda + t) - f_x(\lambda)}{t}$$

$$\lim_{t \longrightarrow 0^+} \frac{f(x + \lambda \cdot \mathbf{1} + t \cdot \mathbf{1}) - \lambda - t - f(x + \lambda \cdot \mathbf{1}) + \lambda}{t}.$$

If $z := x + \lambda \cdot \mathbf{1}$

$$(f_x)'_+(\lambda) = \lim_{t \longrightarrow 0^+} \frac{f(z + t \cdot \mathbf{1}) - t - f(z)}{t} = (f_z)'_+(0) = f'_+(z, \mathbf{1}) - 1.$$

Therefore, $f'_+(x, \mathbf{1}) \le 1$, $\forall x \in X$.
$(\Longleftarrow)$ Conversely, if $f'_+(x, 1) \le 1$ for all $x \in X$, then $(f_x)'_+(\lambda) \le 0$. Indeed $(f_x)'_+(\lambda) = f'_+(x + \lambda \cdot \mathbf{1}, \mathbf{1}) - 1$. Therefore, f_x is decreasing and f is plus-sub-homogeneous. ♦

Let $\{f_\alpha : \alpha \in I\}$ be a family of sub-topical functions. Then $\overline{f}(x) = \sup\limits_{\alpha \in I} f_\alpha(x)$ and $\underline{f}(x) = \inf\limits_{\alpha} f_\alpha(x)$ are sub-topical functions.

Next result is an example for a sub-topical function which is not topical.

Example 2.8. If $\alpha > 1$, we define

$$\varphi_\alpha(x, y) := \sup\{\lambda \in \mathbb{R} : \lambda\alpha \cdot \mathbf{1} \leq x + y\} \ (\forall x, y \in X). \ (14)$$

It follows from (14) that the set $\{\lambda \in \mathbb{R} : \lambda\alpha \cdot \mathbf{1} \leq x + y\}$ is non-empty and bounded from above (*by* $\alpha^{-1}\|x + y\|$). Clearly this set is closed. It follows from the definition of φ_α that φ_α enjoys the following properties:

$$-\infty < \varphi_\alpha(x, y) \leq \alpha^{-1}\|x + y\| \text{ for all } x, y \in X. \ (15)$$

$$\varphi_\alpha(x, y) \cdot \mathbf{1} \leq \alpha^{-1}(x + y) \text{ for all } x, y \in X. \ (16)$$

$$\varphi_\alpha(x, y) = \varphi_\alpha(y, x) \text{ for all } x, y \in X. \ (17)$$

$$\varphi_\alpha(x, -x) = \sup\{\lambda \in \mathbb{R} : \lambda\alpha \cdot \mathbf{1} \leq x - x = 0\} = 0 \text{ for all } x \in X. \ (18)$$

For each $y \in X$ define the function $\varphi_{\alpha,y} : X \longrightarrow \mathbb{R}$ by

$$\varphi_{\alpha,y}(x) := \varphi_\alpha(x, y) \ \forall x \in X. \ (19)$$

Then,

$$\varphi_{\alpha,y}(x) = \varphi_\alpha(x, y) = \alpha^{-1}\varphi(x, y) = \alpha^{-1}\varphi_y(x).$$

Lemma 2.9. *Let φ_α be the function defined by (14). Then*

a) For $1 \leq \alpha \leq \beta$, then $\varphi_\beta \leq \varphi_\alpha \leq \varphi$.

b) $\lim\limits_{\alpha \longrightarrow 1^+} \varphi_\alpha(x, y) = \sup\limits_{\alpha > 1} \varphi_\alpha(x, y) = \varphi(x, y)$

Proof. (a). $\varphi_\beta = \beta^{-1}\varphi \leq \alpha^{-1}\varphi = \varphi_\alpha$
(b). $\lim\limits_{\alpha \longrightarrow 1^+} \varphi_\alpha(x, y) = \lim\limits_{\alpha \longrightarrow 1^+} \alpha^{-1}\varphi(x, y) = \varphi(x, y)$. ♦

Consider $X_{\varphi_\alpha} = \{\varphi_{\alpha,y} : \alpha > 1, \ y \in X\}$. Lemma 2.2 shows that, elements of X_{φ_α} can be elementary function for φ, (i.e; $\varphi_y(x) = \sup\{\varphi_{\alpha,y}(x) : \varphi_{\alpha,y} \in X_{\varphi_\alpha}\}$).

Remark.1. The function $\varphi_{\alpha,y}$ defined by (19) is sub-topical, so by theorem 2.4 is Lipschitz continuous.

Now it is given a characterization of downward sets in terms of separation from outside points by sub-topical functions instead of topical functions.

Theorem 2.10. *Let φ_α be the function defined by (14). Then for a nonempty subset W of X the following assertions are equivalent:*
i) W is a downward subset of X.
ii) For each $x \in X \setminus W$,

$$\varphi_\alpha(w, -x) < 0, \ (\forall w \in W). \ (20)$$

iii) For each $x \in X \setminus W$, there exists $l \in X$ such that

$$\varphi_\alpha(w, l) < 0 \leq \varphi_\alpha(x, l), \ (\forall w \in W). \ (21)$$

Proof. $(i) \Longrightarrow (ii)$. Suppose that (i) holds and there exists $x \in X \setminus W$. It is known (in [6]) that, $\varphi(w, -x) < 0$. Therefore, $\varphi_\alpha(w, -x) = \alpha^{-1}\varphi(w, -x) < 0$ $(\forall w \in W)$.
$(ii) \Longrightarrow (iii)$. Assume that (ii) holds and $x \in X \setminus W$ is arbitrary. Then by hypothesis, $\varphi_\alpha(w, -x) < 0$ $(\forall w \in W)$. Let $l = -x \in X$,

$$\varphi_\alpha(w, -x) = \varphi_\alpha(w, l) < 0 = \varphi_\alpha(x, -x) = \varphi_\alpha(x, l).$$

$(iii) \Longrightarrow (i)$. Suppose that (iii) holds and W is not downward set. There is $x \leq w$ such that $w \in W$ and $x \in X \setminus W$. There is $l \in X$ such that $\forall \alpha > 1, \ \varphi_\alpha(w, l) < 0 \leq \varphi_\alpha(x, l)$. But $\varphi_{\alpha,l}(\cdot)$ is increasing. Therefore,

$$\varphi_\alpha(x, l) \leq \varphi_\alpha(w, l) < 0,$$

which is a contradiction. ♦

Theorem 2.11. [6] *For a function $f : X \longrightarrow \overline{\mathbb{R}}$, the following assertions are equivalent:*
i) f is topical.
ii) For each $y \in X$, there exists $l_y \in X$ such that

$$\varphi_{l_y} : X \longrightarrow \mathbb{R}$$

satisfies in

$$\varphi_{l_y} \leq f \ \text{and} \ f(y) = \varphi_{l_y}(y) \ (y \in dom f).$$

iii) f in $X_\varphi -$ convex. ♦

Example 2.12. Let the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x$ is topical and for $\alpha = 2$, $\varphi_2 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\varphi_2(x, y) = 2^{-1}(x + y) \ (\forall x, y \in X).$$

For arbitrary but fixed $y \in \mathbb{R}$, if there exist $l_y \in \mathbb{R}$ such that

$$\varphi_{2,l_y}(x) \leq f(x) \ \forall x \in \mathbb{R} \ \text{and} \ \varphi_{2,l_y}(y) = f(y),$$

then $l_y = y$ and $y \leq x \ \forall x \in \mathbb{R}$ which is a contradiction.

3. Separation theorem and abstract convexity by sub-topical functions

Remark.2. Define, $S_{y,d}(\cdot) := min\{\varphi_y(\cdot), d\}$ if $y \in X$ and $d \in \mathbb{R}$, then $S_{y,d}$ is sub-topical. Indeed $S_{y,d}$ is increasing since φ_y is increasing. Also

$$(*)\ y_1 \leq y_2 \Longleftrightarrow S_{y_1,d} \leq S_{y_2,d}$$

$$(**)\ d_1 \leq d_2 \Longleftrightarrow S_{y,d_1 \leq} S_{y,d_2}.$$

and if $x \in X$, $\lambda > 0$

$$S_{y,d}(x + \lambda \cdot \mathbf{1}) = min\{\varphi_y(x + \lambda \cdot \mathbf{1}), d\} = min\{\varphi_y(x) + \lambda, d = d_1 + \lambda\}$$

$$= min\{\varphi_y(x), d_1\} + \lambda = S_{y,d_1}(x) + \lambda \leq S_{y,d}(x) + \lambda.$$

Theorem 3.1. *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a function. The following assertions are equivalent:*
i) f is topical.
ii) $f(x) \geq S_{y,d}(x) + f(-y)$ for all $x, y \in X$, $d \in \mathbb{R}_+$.

Proof. Suppose that (i) holds and since $S_{y,d}(x) = min\{\varphi(x,y), d\} \leq \varphi_y(x)$. Then by (see [1]),

$$S_{y,d}(x) \leq \varphi_y(x) \leq f(x) - f(-y).$$

It implies that

$$f(x) \geq S_{y,d}(x) + f(-y).$$

Conversely, assume that (ii) holds, if $f(x) = \infty(or - \infty)$ for some $x \in X$, by hypothesis $f \equiv \infty(or - \infty)$. Then f is topical. We assume that $f : X \longrightarrow \mathbb{R}$.

$$f(x + \lambda \cdot \mathbf{1}) \geq S_{-x,|\lambda|}(x + \lambda \cdot \mathbf{1}) + f(x).$$

By definition of $S_{y,d} \in S$,

$$f(x + \lambda \cdot \mathbf{1}) \geq f(x) + \lambda\ (I)$$

and

$$f(x) + \lambda \geq S_{-x-\lambda \cdot \mathbf{1},|\lambda|}(x) + f(x + \lambda \cdot 1) + \lambda = f(x + \lambda \cdot \mathbf{1})\ (II)$$

Therefore, (I) and (II) imply that $f(x + \lambda \cdot \mathbf{1}) = f(x) + \lambda$. We show that f is increasing. Let $x \leq y$ for $x, y \in X$. According to (9):

$$0 = \varphi(x, -x) \leq \varphi(y, -x)$$

and

$$0 = S_{-x,0}(y) = min\{\varphi_{-x}(y), 0\} \leq \varphi_{-x}(y) \leq f(y) - f(x).$$

Therefore, $f(x) \leq f(y)$. ♦

Theorem 3.2. *The map $\xi : X \times \mathbb{R} \longrightarrow S = \{S_{y,d} : y \in X, d \in \mathbb{R}\}$ which $(y, d) \mapsto S_{y,d}$ is bijection.*

Proof. ξ is obviously onto. We show f is one-to-one. If $S_{y_1,d_1} = S_{y_2,d_2}$, $S_{y_1,d_1}(-y_1 + d_1 \cdot \mathbf{1}) = d_1 = S_{y_2,d_2}(-y_1 + d_1 \cdot \mathbf{1}) = min\{\varphi(y_2, -y_1) + d_1, d_2\}$. Then $d_1 \leq d_2$ and $d_1 \leq \varphi(y_2, -y_1) + d_1$, so $0 \leq \varphi(y_2, -y_1)$ and implies that $0 \leq \varphi(y_2, -y_1) \cdot \mathbf{1} \leq y_2 - y_1$, so $y_1 \leq y_2$. Also since, $S_{y_2,d_2}(-y_2 + d_2 \cdot \mathbf{1}) = d_2 = S_{y_1,d_1}(-y_2 + d_2 \cdot \mathbf{1}) = min\{\varphi(y_1, -y_2) + d_2, d_1\}$, then $d_2 \leq d_1$ and $d_2 \leq \varphi(y_1, -y_2) + d_2$, so $0 \leq \varphi(y_1, -y_2)$ and implies that $0 \leq \varphi(y_1, -y_2) \cdot \mathbf{1} \leq y_1 - y_2$. Therefore, $y_2 \leq y_1$. It follows that $y_1 = y_2$ and $d_1 = d_2$. ♦

Theorem 3.3. *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a function. If f is a topical function then there exists a set $M = Y \times \mathbb{R}_+ \subseteq X \times \mathbb{R}$ such that*

$$f(x) = \sup_{(y,d)\in M} S_{y,d}(x). \quad (22)$$

In this case, one can take $Y = \{y \in X : f(-y) \geq 0\}$.

Proof. Let f be a topical. If $f(x) = \infty$ for some $x \in X$, by theorem (3.6), $f \equiv \infty$. Then $Y = X$, so $f(x) = \sup_{y\in Y} \varphi(x, y) = \infty$. If $f(x) = -\infty$ for some $x \in X$, by theorem 2.5, $f \equiv -\infty$. Then $Y = \emptyset$ and $f(x) = \sup_{y\in Y} \varphi(x, y) = -\infty$. Suppose that $f : X \longrightarrow \mathbb{R}$, be a topical function. According to theorem 2.11, $\forall x \in X$ there exists $y \in Y$ such that,

$$\varphi_y \leq f, \ \varphi_y(x) = f(x).$$

Choose $d = |f(x)|$, $S_{y,d} = \min\{\varphi_y, d\} \leq f$ and $S_{y,d}(x) = f(x)$. Therefore, $f(x) = \sup_{(y,d)\in M} S_{y,d}(x)$. ♦

Definition 3.4. The lower polar-function of $f : X \longrightarrow \overline{\mathbb{R}}$ is the function $f^\star : S \longrightarrow \overline{\mathbb{R}}$

$$f^\star(S_{y,d}) := \sup_{x\in X}\{S_{y,d}(x) - f(x)\}, \ (\forall S_{y,d} \in S). \quad (23)$$

Theorem 3.5. *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a function, then*

$$f^\star(S_{y,d}) \geq d - f(-y + d \cdot \mathbf{1}) \ (\forall S_{y,d} \in S). \quad (24)$$

f is topical if and only if

$$f^\star(S_{y,d}) = -f(-y) \ (\forall S_{y,d} \in S). \quad (25)$$

Proof. $f^\star(S_{y,d}) = \sup_{x\in X}\{S_{y,d}(x) - f(x)\}$
$\geq S_{y,d}(-y + d \cdot \mathbf{1}) - f(-y + d \cdot \mathbf{1}) = d - f(-y + d \cdot \mathbf{1})$.
Indeed $S_{y,d}(-y + d \cdot \mathbf{1}) = d$. Then $f^\star(S_{y,d}) \geq d - f(-y + d \cdot \mathbf{1})$.
If f is a topical function. Let $x, y \in X$ be arbitrary. It follows from (7) that $S_{y,d}(x) \cdot \mathbf{1} \leq x + y$ and hence $S_{y,d}(x) \cdot \mathbf{1} - y \leq x$. Since f is topical function,

$$S_{y,d}(x) - f(x) \leq -f(-y) \ (x, y \in X).$$

Then

$$f^\star(S_{y,d}) = \sup_{x\in X}\{S_{y,d}(x) - f(x)\} \leq -f(-y), \ (y \in Y).$$

From (24), $d-f(-y+d\cdot\mathbf{1}) = d-f(-y)-d = -f(-y) \leq f^*(S_{y,d}) \leq -f(-y)$. Therefore, $f^\star(S_{y,d}) = -f(-y)$.
Conversely, we assume that (25) holds. Let $x, y \in X$ be arbitrary. $f^\star(S_{y,d}) \geq S_{y,d}(x) - f(x)$. By (25), $-f(-y) \geq S_{y,d}(x) - f(x)$, so $f(x) \geq S_{y,d}(x) + f(-y)$. By theorem 3.3, f is a topical function which it completes the proof.

Definition 3.6. Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a topical function and $S_{l,d} \in S$. Define the X_s- subdifferential $\partial_{X_s} f(y)$ of at a point $y \in X$ by,

$$\partial_{X_s} f(y) = \{(l,d) \in X \times \mathbb{R} : S_{l,d}(x) \leq f(x)\ \forall x \in X,\ and\ S_{l,d}(y) = f(y)\}, \quad (26)$$

where $X_s := \{(l,d) \in X \times \mathbb{R} : S_{l,d} \in S\}$.

Theorem 3.7. *Let $f : X \longrightarrow \mathbb{R}$ be a topical function and $y \in X$. Then*

$$\partial_{X_s} f(y) = \{(l,d) \in X \times \mathbb{R} : S_{l,d}(y) \geq f(y)\ and\ f(-l) = 0\}.$$

In particular, $(f(y)\cdot\boldsymbol{1} - y, f(y)) \in \partial_{X_s} f(y)$ and $(f(y)\cdot\boldsymbol{1} - y, |f(y)|) \in \partial_{X_s} f(y)$

Proof. Let

$$Q := \{(l,d) \in X \times \mathbb{R} : S_{l,d}(y) \geq f(y)\ and\ f(-l) = 0\}.$$

Let $(l,d) \in \partial_{X_s} f(y)$. Then $f(y) \leq S_{l,d}(y)$. It follows that $f(y)\cdot\mathbf{1} \leq S_{l,d}(y)\cdot\mathbf{1} \leq \varphi_l(y)\cdot\mathbf{1} \leq y + l$. Therefore, $y \geq f(y)\cdot\mathbf{1} - l$ and $f(y) \geq f(y) + f(-l)$. Then $f(-l) \leq 0$ (I). Since $f(x) \geq S_{l,d}(x)$, $\forall x \in X$ so $f(-l+d\cdot\mathbf{1}) \geq S_{l,d}(-l+d\cdot\mathbf{1})$. Therefore, $f(-l)+d \geq \min\{\varphi_l(-l+d\cdot\mathbf{1}), d\} = d$. This implies that $f(-l) \geq 0$ (II) by using $(I), (II)$, $f(-l) = 0$ and $(l,d) \in D$.
Conversely, if $(l,d) \in D$, there exists $x \in X$ such that $S_{l,d}(x) > f(x)$ which implies that there exists $r > 0$ such that $S_{l,d}(x) > f(x) + r$, and so $x > (f(x)+r)\cdot\mathbf{1} - l$. Since f is topical and $f(-l) = 0$. It shows that

$$f(x) > f(x) + r + f(-l),$$

which is a contradiction by choosing of r. Therefore, $S_{l,d}(x) \leq f(x)$, $\forall x \in X$. Also $S_{l,d}(y) \leq f(y)$. Since $(l,d) \in D$, then $f(y) \leq S_{l,d}(y)$. It implies $f(y) = S_{l,d}(y)$. Hence, $(l,d) \in \partial_{X_s} f(y)$. If $f(y)\cdot\mathbf{1} - y$, $d = |f(y)|$ or $d = f(y)$, then $(l,d) \in \partial_{X_s} f(y)$

$$S_{l,d}(y) = \min\{\varphi_l(y), d\} = f(y)\ and\ f(y - f(y)\cdot\mathbf{1}) = 0.$$

Then $(l,d) \in \partial_{X_s} f(y)$. ♦

It is worth noting that the function $S_{l,d}$ defined by remark (2) is sub-topical and by theorem (3.5) Lipschitz continuous. We now give characterizations of downward sets in terms of separation from outside points.

Theorem 3.8. *Let $W \subseteq X$ and $S_{l,d}$ be a function defined by remark (2). Then the following assertions are equivalent:*
i) W is a downward set.
ii) For each $x \in X \setminus W$, there exists $(l,d) \in X \times \mathbb{R}_+$ such that $S_{l,d}(w) < 0 \leq S_{l,d}(x)$.

Proof.
$(i) \Longrightarrow (ii)$. Suppose that (i) holds and $x \notin W$. Let $l = -x$, $d \in \mathbb{R}_+$, then by [6], $\varphi_l(w) < 0 \leq \varphi_l(x)$ $\forall w \in W$. From the definition of $S_{l,d}(w) = min\{\varphi_l(w), d\}$,

$$S_{l,d}(w) < 0 \leq S_{l,d}(x) \ (\forall w \in W).$$

$(ii) \Longrightarrow (i)$. Suppose that (ii) holds and W is not a downward set. There is $w_0 \in W$ and $x_0 \in X \setminus W$ with $x_0 \leq w_0$. By hypothesis, there exists $l \in X, d \in \mathbb{R}^+$ such that

$$S_{l,d}(w) < 0 \leq S_{l,d}(x_0) \ (\forall w \in W).$$

Since $S_{l,d}$ is increasing, then $S_{l,d}(x_0) \leq S_{l,d}(w_0)$.
Therefore,

$$S_{l,d}(w_0) < 0 \leq S_{l,d}(x_0) \leq S_{l,d}(w_0).$$

This is a contradiction. ♦

Theorem 3.9. *Let $W \subseteq X$, and $S_{l,d}$ be the function defined by remark (2). Then the following assertions are equivalent:*
i) W is a closed downward subset of X.
ii) W is downward, and for each $x \in X$ the set

$$H = \{\lambda \in \mathbb{R} : x + \lambda \cdot \mathbf{1} \in W\}, \ (27)$$

is closed in $\mathbb{R}$.
iii) For each $x \in X \setminus W$. There is $(l, d) \in X \times \mathbb{R}_{++}$ such that

$$S_{l,d}(w) < 0 < S_{l,d}(x), \ (w \in W). \ (28)$$

vi) For each $x \in X \setminus W$ there exists $(l, d) \in X \times \mathbb{R}_{++}$ such that

$$\sup_{w \in W} S_{l,d}(w) < S_{l,d}(x). \ (29)$$

Proof.
$(i) \Longrightarrow (ii)$. The proof is the same as in [6]
$(ii) \Longrightarrow (iii)$. Suppose that (ii) holds and $x \in X \setminus W$ is arbitrary. There is $l \in X$ such that

$$\varphi(w, l) < 0 < \varphi(x, l) \ (\forall w \in W).$$

Let $d = \varphi(x, l) \in \mathbb{R}_{++}$, then

$$S_{l,d}(w) = min\{\varphi(w, l), d\} = \varphi(w, l) < 0 \ (\forall w \in W).$$

and

$$S_{l,d}(x) = min\{\varphi(x, l), d\} = \varphi(x, l) > 0.$$

Therefore,

$$S_{l,d}(w) < 0 < S_{l,d}(x) \ (\forall w \in W).$$

$(iii) \Longrightarrow (vi)$, is obvious.
$(vi) \Longrightarrow (i)$. Suppose that (vi) holds and W is not downward. There is $w_0 \in W$ and $x_0 \in X \setminus W$ with $x_0 \leq w_0$. By hypothesis, there exists $l \in X$, $d \in \mathbb{R}_{++}$ such that

$$\sup_{w \in W} S_{l,d}(w) < S_{l,d}(x_0).$$

Since $S_{l,d}(\cdot)$ is increasing, it follows that;

$$S_{l,d}(x_0) \leq S_{l,d}(w_0) \leq \sup_{w \in W} S_{l,d}(w) < S_{l,d}(x_0).$$

This is a contraction. Hence, W is a downward set. Finally, assume that W is not closed. There is a sequence $\{w_n\}_{n\geq 1} \subseteq W$ and $x_0 \in X \setminus W$ such that $\|w_n - x_0\| \longrightarrow 0$ as $n \longrightarrow \infty$. Since $x_0 \in X \setminus W$, there exists $l \in X, d \in \mathbb{R}_{++}$ such that;

$$\sup_{w \in W} S_{l,d}(w) < S_{l,d}(x_0).$$

Thus,

$$S_{l,d}(w_n) \leq \sup_{w \in W} S_{l,d}(w) \ (\forall n \geq 1).$$

From continuity of $S_{l,d}(\cdot)$, $S_{l,d}(x_0) \leq \sup_{w \in W} S_{l,d}(w)$. This is a contradiction. ♦

Lemma 3.10. *Let W be a closed downward subset of X, $w_0 \in bdW$ and let $S_{l,d}$ be the function defined by remark (2). Then $S_{-w_0,d}(w) \leq 0$ $(\forall w \in W,)$ and $d \in \mathbb{R}_+$.*

Proof. Suppose that this condition holds, by (see [6]), $\varphi(w, -w_0) \leq 0$ $(\forall w \in W)$. Therefore, if $d \in \mathbb{R}_+$,

$$S_{-w_0,d}(w) = min\{\varphi(w, -w_0), d\} = \varphi(w, -w_0) \leq 0 \ (\forall w \in W).$$

♦

Lemma 3.11. *Let W be a closed downward subset of X,$w_0 \in bdW, l = -w_0$ and $d \in \mathbb{R}_+$. Let $S_{l,d}$ be the function defined by remark (2). Then*

$$S_{l,d}(w) \leq 0 = S_{l,d}(w_0), \ (\forall w \in W).$$

Proof. By hypothesis and [6]

$$\varphi(w, l) \leq 0 = \varphi(w_0, l) \ (\forall w \in W).$$

Let $d = 0$

$$S_{l,d}(w) = min\{\varphi(w, l), d\} = \varphi(w, l) \leq 0,$$

and

$$S_{l,d}(w_0) = min\{\varphi(w_0, l), d\} = 0.$$

Therefore,

$$S_{l,d}(w) \leq 0 = S_{l,d}(w_0) \ (\forall w \in W).$$

♦

The following theorem gives a necessary and sufficient condition for the best approximation in terms of separation from outside points.

Theorem 3.12. *Let W be a closed downward subset of X and $x_0 \in X$. Let $y_0 \in W$ and $r_0 = \|x_0 - y_0\|$. Assume that $S_{l,d}$ is the function defined by remark (2). Then the following assertions are equivalent:*
i) $y_0 \in P_W(x_0)$.
ii) There exists $l \in X$ and $d \in \mathbb{R}_+$ such that;

$$S_{l,d}(w) \leq 0 \leq S_{l,d}(y) \ (\forall w \in W, y \in B(x_0, r_0)) \ (30)$$

Moreover, if (30) holds with $l = -y_0$, then $y_0 = minP_W(x_0)$.

Proof. $(i) \Longrightarrow (ii)$. Suppose that $y_0 \in P_W(x_0)$, then $r_0 = \|x_0 - y_0\| = d(W, x_0)$. Since W is closed downward subset of X, then by (see [6]), that the least element $w_0 = x_0 - r_0 \cdot 1$ of the set $P_W(x_0)$ exists. Let $l = -w_0 \in X$. Then,

$$\varphi(w, l) \leq 0 \leq \varphi(y, l) \ (\forall w \in W, y \in B(x_0, r_0)). \ (31)$$

Let $\varphi_l(x_0 + r_0 \cdot \mathbf{1}) = d$, then $d \in \mathbb{R}_+$. Indeed $\forall y \in B(x_0, r_0)$. By (4), $0 \leq y \leq x_0 + r_0 \cdot \mathbf{1}$, then by (31), $0 \leq \varphi(y, l) \leq \varphi(x_0 + r_0 \cdot \mathbf{1}, l)$. Therefore,

$$S_{l,d}(w) = \min\{\varphi(w, l), d\} = \varphi(w, l) \leq 0, \ (\forall w \in W)$$

and

$$S_{l,d}(y) = \min\{\varphi(y, l), d\} = \varphi(y, l) \geq 0. \ (\forall y \in B(x_0, r_0))$$

$(ii) \Longrightarrow (i)$. Assume that there exists $l \in X$ and $d \in \mathbb{R}_+$ such that

$$S_{l,d}(w) \leq 0 \leq S_{l,d}(y). \ (\forall w \in W, y \in B(x_0, r_0))$$

From (4), $x_0 - r_0 \cdot \mathbf{1} \in B(x_0, r_0)$. From the hypothesis $S_{l,d}(x_0 - r_0 \cdot \mathbf{1}) \geq 0$. According to the definition of $S_{l,d}$, $\varphi(x_0 - r_0 \cdot \mathbf{1}, l) \geq 0$. Indeed $S_{l,d}(x_0 - r_0 \cdot \mathbf{1}) = \min\{\varphi(x_0 - r_0 \cdot \mathbf{1}, l)\} \geq 0$. Since $\varphi(., l)$ is topical, $\varphi(x_0, l) \geq r_0$. Due to (7),

$$r_0 \cdot \mathbf{1} \leq \varphi(x_0, l) \cdot \mathbf{1} \leq x_0 + l. \ (32)$$

Let $w \in W$ be an arbitrary and $p_w = \varphi(w, -x_0) \cdot \mathbf{1} + x_0 \in X$. Then $\varphi(w, -x_0) \cdot \mathbf{1} \leq w - x_0$ and $p_w \leq w$. Since W is downward set and $w \in W$, it follows that $p_w \in W$. By hypothesis $S_{l,d}(p_w) \leq 0$ and since $d \in \mathbb{R}_+$, $\varphi(p_w, l) \leq 0$. Since $\varphi(p_w, .)$ is topical and (32) holds,

$$S_{-x_0,d}(p_w) \leq \varphi(p_w, -x_0) \leq \varphi(p_w, l) - r_0 \leq -r_0.$$

Since $\varphi(., -x_0)$ is topical

$$-r_0 \geq \varphi(p_w, -x_0) = \varphi(\varphi(w, -x_0) \cdot \mathbf{1} + x_0, -x_0) = \varphi(w, -x_0).$$

From Lipschitz continuity of φ_{-x_0},

$$r_0 \leq |\varphi(w, -x_0)| = |\varphi(x_0, -x_0) - \varphi(w, -x_0)| \leq \|x_0 - w\|.$$

Thus $r_0 \leq \|x_0 - w\|$ for all $(w \in W)$ and $\|x_0 - y_0\| = d(x_0, W)$. Consequently, $y_0 \in P_W(x_0)$. Finally, suppose that (30) holds with $l = -y_0$. From implication $(ii) \Longrightarrow (i)$, $y_0 \in P_W(x_0)$. Let $w \in P_W(x_0)$ be an arbitrary. Thus, $\|x_0 - w\| = d(x_0, W) = \|x_0 - y_0\| = r_0$, that is $w \in B(x_0, r_0)$. It follows from the hypothesis $S_{-y_0,d}(w) \geq 0$ and so $0 \leq S_{-y_0,d}(w) \cdot \mathbf{1} \leq \varphi(w, -y_0) \cdot \mathbf{1} \leq w - y_0$. This implies that $y_0 \leq w$ for all $w \in P_W(x_0)$. Hence, $y_0 = \min P_W(x_0)$, which it completes the proof. ⧫

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