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COMMENTS ON THE PAPERS " ARCH. MATH. (BRNO), 42(2006), 51-58" " THAI J. MATH., 3(2005), 63-70" AND "MATH. COMMUNICATIONS 13(2008), 85-96"

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Abstract. Using Dotsonś convexity structure, the authors in [16, 17, 18] established some deterministic and random common fixed point results. In this note, we comment that the proofs of the results in [16, 17, 18] are incomplete and incorrect.

1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real-valued function $(0 < p \leq 1)$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$

(ii) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$

(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

for all x, $y \in X$ and all scalars α. The pair $(X, \|.\|_p)$ is called a p-normed space. It is a metric linear space with a translation invariant metric d_p defined by $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of a normed space. It is well-known that the topology of every Hausdorff locally bounded

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topological linear space is given by some p-norm, $0 < p \leq 1$(see [7, 14, 19]).

Let X be a metric linear space and M a nonempty subset of X. Let $I : M \to X$ be a mapping. A mapping $T : M \to X$ is called I-Lipschitz if there exists $k \geq 0$ such that $d(Tx, Ty) \leq kd(Ix, Iy)$ for any $x, y \in M$. If $k < 1$ (respectively, $k = 1$), then T is called I-contraction (respectively, I-nonexpansive). The map $T : M \to X$ is said to be completely continuous if $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges strongly to Tx. The map $T : M \to X$ is demiclosed at 0 if for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ convergent strongly to 0, we have $Tx = 0$. The set of best approximations of $u \in X$ from M is defined by $P_M(u) = \{x \in M : d(x, u) = dist(u, M) = inf_{y \in M} d(u, y)\}$. The set of fixed points of T(resp. I) is denoted by $F(T)$(resp. $F(I)$). A point $x \in M$ is a common fixed (coincidence) point of I and T if $x = Ix = Tx$ $(Ix = Tx)$. The set of coincidence points of I and T is denoted by $C(I, T)$. Two selfmaps I and T of M are called:
(1) commuting if $ITx = TIx$ for all $x \in M$;
(2) R-weakly commuting if for all $x \in M$ there exists $R > 0$ such that $d(ITx, TIx) \leq Rd(Ix, Tx)$;
(3) compatible if $\lim_n d(TIx_n, ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M;
(4) weakly compatible if they commute at their coincidence points, i.e. $ITx = TIx$ whenever $Ix = Tx$.

The set M is called q-starshaped with $q \in M$ if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x, is contained in M for all $x \in M$. Suppose M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant in a p-normed space X. Then T and I are called:
(5) R-subcommuting on M if there exists a real number $R > 0$ such that $\|ITx - TIx\|_p \leq \frac{R}{k}\|(kTx + (1-k)q) - Ix\|_P$ for all $x \in M$, $k \in (0, 1]$. If $R = 1$, then the maps are called 1-subcommuting;
(6) R-subweakly commuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $\|ITx - TIx\|_p \leq$R $dist(Ix, [q, Tx])$;
(7) C_q-commuting if $ITx = TIx$ for all $x \in C_q(I, T)$, where $C_q(I, T) = \cup\{C(I, T_k) : 0 \leq k \leq 1\}$ and $T_k x = (1-k)q + kTx$.

Clearly, commuting maps are R-subweakly commuting, R-subweakly commuting maps are R-subcommuting and R-subcommuting maps are C_q-commuting but the converse, in each case, does not hold in general (see [8, 11] and references therein).

Following important extension of the concept of starshapedness was defined by Dotson [4] and has been studied by many authors (see [2]-[7],[9]-[18],[20]).

Definition 1.1. (Dotsonś convexity). Let M be subset of a p-normed space X and $\mathbb{F} = \{f_x\}_{x \in M}$ a family of functions from $[0, 1]$ into M such that $f_x(1) = x$ for each $x \in M$. The family $\mathbb{F}$ is said to be contractive [4, 5, 12, 14] if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $x, y \in M$ and all $t \in (0,1)$, we have $\|f_x(t) - f_y(t)\|_p \leq [\phi(t)]^p \|x - y\|_p$. The family $\mathbb{F}$ is said to be jointly (weakly) continuous if $t \to t_0$ in [0,1] and $x \to x_0$ ($x \to x_0$ weakly) in M, then $f_x(t) \to f_{x_0}(t_0)$ ($f_x(t) \to f_{x_0}(t_0)$ weakly) in M. We observe that if $M \subset X$ is q-starshaped and $f_x(t) = (1-t)q + tx$, ($x \in M; t \in [0,1]$), then $\mathbb{F} = \{f_x\}_{x \in M}$ is a contractive jointly continuous and jointly weakly continuous family with $\phi(t) = t$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets (see [3, 4, 6, 12, 14]).

2. Main Results

In the papers [16, 17] under consideration, the author defines the so called (S)-convex structure for a linear space X which is absurd as starshaped sets and hence linear spaces satisfy the so called (S)-convex structure. Therefore, we always define convex and starshaped structure on a nonempty subset M of X. Thus Definition 1 in [15], Definition 2.7 in [16] and Definition 2.3 in [17] should be modified in the context of a nonempty subset of a linear space X(see definition 1.1 above). Condition (iv) of the definition has no meanings and should be deleted and in Condition (v) the function ϕ should be from $(0,1) \to (0,1)$. Similarly, Definition 2.8 [16] should be modified as follows(see [4, 6, 12, 14]):

Let T be a selfmap of the set M having a family of functions $\mathbb{F} = \{f_x\}_{x \in M}$ as defined above. Then T is said to satisfy the property (A), if $T(f_x(t)) = f_{Tx}(t)$ for all $x \in M$ and $t \in [0,1]$.

Example 2.1. An affine map T defined on q-starshaped set with $Tq = q$ satisfies the property (A). For this note that each q-starshaped set M has a contractive jointly continuous family of functions $\mathbb{F} = \{f_x\}_{x \in M}$ defined by $f_x(t) = tx + (1-t)q$, for each $x \in M$ and $t \in [0,1]$. Thus $f_x(1) = x$ for all $x \in M$. Also, if the selfmap T of M is affine and $Tq = q$, we have $T(f_x(t)) = T(tx + (1-t)q) = tTx + (1-t)q = f_{Tx}(t)$ for all $x \in M$ and all $t \in [0,1]$. Thus T satisfies the property (A); a property considered first time in 2000, by Khan, the author and Thaheem (see [12], Theorems 3.7,3.10,3.12). This signifies that (S)-convex structure should be introduced on a nonempty subset M of a linear space X.

Here is the main result of Nashine [16].

Theorem 2.2. *Let X be a p-normed space with a (S)-convex structure. Let T, $I : X \to X$, C a subset of X such that $T(\partial C) \subset C$ and $u \in F(T) \cap F(I)$. Suppose that $D = P_M(u)$ and T is I-nonexpansive on $D \cup u$, I satisfies property (A), I is continuous, $TI = IT$ on D, $cl(T(D))$ is compact on D. Also assume, range of f_α is contained in $I(D)$. If D is nonempty, closed and if $I(D) \subset D$, then $D \cap F(I) \cap F(T) \neq \emptyset$.*

My comments to Theorem 2.2 are as follows:

(a) The condition "range of f_α is contained in $I(D)$" makes the result trivial. As a matter of fact take $f_\alpha(t) = t\alpha$ for each $\alpha \in X$ and $t \in [0,1]$; now X is a linear space with zero element so $\{f_\alpha\}$ is a (S)-convex structure with range of f_α equal to X. Thus $X \subseteq I(D) \subseteq D \subseteq X$.

(b) The (S)-convex structure is not a hereditary property so the set D here is without any convexity structure and hence the statement in the proof of this theorem "T_n is a well-defined map from D into D for each n" makes no sense; it is worth mentioning that the entire proof depends on this important fact. Same concerns the proof of Theorem 2 in [15].

(c) The statement in the proof of Theorem 2.2, "Since $cl(T(D))$ is compact, each $cl(T_n(D))$ is compact" needs to be verified which is crucial for the application of Theorem 2.9 stated in [16]. Actually, when D is q-starshaped, it has (S)-convex structure $f_x(t) = tx + (1-t)q$, for each $x \in D$ and $t \in [0,1]$. Further, if $T_n x = (1-k_n)q + k_n Tx$ for all $x \in D$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1, then $cl(T_n(M))$ is compact for each n provided $cl(T(D))$ is compact.

The second and last result in [16] is the following:

Theorem 2.3. *Let X be a complete p-normed space whose dual separates the points of X with a (S)-convex structure. Let T, $I : X \to X$, C a subset of X such that $T(\partial C) \subset C$ and $u \in F(T) \cap F(I)$. Suppose that T is I-nonexpansive on $D \cup u$, I satisfies property (A), I is weakly continuous, $TI = IT$ on D. Also assume that range of f_α is contained in $I(D)$. If D is nonempty, weakly compact and if $I(D) \subset D$, then $D \cap F(I) \cap F(T) \neq \emptyset$.*

The above comments (a) and (b) apply to Theorem 2.3 as well.

(d) The author has utilized Theorem 3.2 (stated in [16]) in the proof of Theorem 2.3 (see p.56, line 15) which holds for a compact metric space whereas the underlying set D here is assumed to be weakly compact and I is not continuous as well.

(e) The author seems to claim in equality (3.1) that $y_m \to 0$ which can not be true unless $Tx_m \to Ty$ which is impossible under the assumed hypotheses. If we assume that T is completely continuous to assure $Tx_m \to Ty$, then the condition "$I - T$ is demiclosed" becomes superfluous and we directly get the conclusion(see [5, 6, 10, 12, 14]). Thus the proof of Theorem 2.3 is incomplete and incorrect. Consequently, Remark 3.5–Remark 3.9 in [16] are invalid.

(f) For more general and comprehensive results for noncommuting maps namely, R-subweakly commuting, R-subcommuting and C_q-commuting maps defined on the set M satisfying the Dotsonś convexity condition (or the so called (S)-convex structure), we refer the reader to [5, 6, 10, 11].

Comments on the results in [17]

(g) The author defines in the proofs of Theorems 3.1 and 3.2 in [17]; $T_n : \Omega \times P_M(x_0) \to P_M(x_0)$ by $T_n(\omega, x) = f_{T(\omega,x)}(k_n)$ and claims that each T_n is a random operator without proving the measurability of T_n. The measurability of T_n is still an open problem(see [2, 13] and references therein). Thus all the results, Theorems 3.1-3.3 in [17], are deterministic in nature and hence are simple corollaries to more general results in [5, 6, 10, 11].

(h) The author has utilized Lemma 2.5 (stated in [17]) in the proof of his Theorem 3.2(see p.67, line 29) which holds for a compact metric space whereas the underlying set $P_M(x_0)$ here is assumed to be weakly compact and g is not continuous as well.

(i) The author seems to claim in lines 7 to 12 on page 68, that $y_m \to 0$ strongly which can not be true unless $T(\omega, \xi_m(\omega)) \to T(\omega, \xi(\omega))$. This is impossible as T is not assumed to have any type of continuity. Thus the proof of Theorem 3.2 is incomplete and incorrect. Consequently, Remark 3.5–Remark 3.7 in [17] are invalid.

Comments on the results in [18]

The proofs of all the results in [18] depends on the following statement:

If the maps I and T are compatible, then I and T_n are also compatible for each $n \geq 1$ where $T_n(x) = (1 - k_n)q + k_n Tx$ for fixed sequence of real numbers $k_n(0 < k_n < 1)$ converging to 1.

Here we give an example to show that the above statement is not correct.

Example 2.4. Let $X = \mathbb{R}$ with usual norm and $M = [1, \infty)$. Let $I(x) = 2x - 1$ and $T(x) = x^2$, for all $x \in M$. Let $q = 1$. Then M is q-starshaped with $Iq = q$. Note that I and T are compatible. Further $C(I, T_{\frac{2}{3}}) = \{1, 2\}$ and $IT_{\frac{2}{3}}(2) \neq T_{\frac{2}{3}}I(2)$, which implies that I and $T_{\frac{2}{3}}$ are not weakly compatible. Thus I and $T_{\frac{2}{3}}$ are not compatible maps. Consequently, all the results proved in [18] are incorrect.

The results in [18] can be corrected if the compatibility of I and T is replaced by the condition of subcompatibility (see [1]).

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