J. Nonlinear Sci. Appl., 18 (2025), 148–164
ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.isr-publications.com/jnsa

On the logarithmic Petrovsky equation with distributed delay: existence, decay, and blow up

Erhan Pişkin[a,*], Hazal Yüksekkaya[b], Jorge Ferreira[c], Mohammad Shahrouzi[d]

[a] *Department of Mathematics, Dicle University, Diyarbakir, Turkey.*
[b] *Department of Mathematics, Hakkari University, Hakkari, Turkey.*
[c] *Department of Exact Sciences, Federal Fluminense University, Brazil.*
[d] *Department of Mathematics, Jahrom University, Jahrom, P.O. Box 74137-66171, Iran.*

Abstract

In this article, we deal with a logarithmic Petrovsky equation with distributed internal delay. Firstly, we prove the local existence of solutions utilizing the semigroup theory. Later, we obtain the global existence of solutions by using the well-depth method. Moreover, under appropriate assumptions on the weight of the delay and that of frictional damping, we get the exponential decay. Finally, we establish the blow up of solutions.

Keywords: Logarithmic Petrovsky equation, existence, decay, blow up, distributed delay.

2020 MSC: 35B44, 35L05, 93D15, 93D20.

1. Introduction

In this work, we deal with the logarithmic Petrovsky equation with distributed delay as follows:

$$\begin{cases} u_{tt} + \Delta^2 u + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s)\, u_t(x,t-s)\, ds = u|u|^{p-2}\ln|u|^k, & x \in \Omega,\ t>0, \\ u(x,t) = \frac{\partial u(x,t)}{\partial \upsilon} = 0, & x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), & \text{in } (0,\tau_2), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n $(n \geqslant 1)$ with a smooth boundary $\partial\Omega$. Here, k and μ_1 are positive constants and the integral term denotes the distributed delay for $0 \leqslant \tau_1 < \tau_2$, such that $\mu_2 : [\tau_1, \tau_2] \to [0,\infty)$ is a bounded function. Also, u_0, u_1, f_0 are the initial data functions to be specified later and υ is the unit outward normal vector.

Generally, time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications, and medicine

*Corresponding author
Email addresses: episkin@dicle.edu.tr (Erhan Pişkin), hazally.kaya@gmail.com (Hazal Yüksekkaya), ferreirajorge2012@gmail.com (Jorge Ferreira), mshahrouzi@jahromu.ac.ir (Mohammad Shahrouzi)

doi: 10.22436/jnsa.018.03.01

Received: 2025-01-30 *Revised:* 2025-03-04 *Accepted:* 2025-03-06

[11]. Moreover, in 1986, Datko et al. [8], indicated that, in a boundary control, a small delay effect is a source of instability. From Quantum Field Theory, that such kind of ($u|u|^{p-2}\ln|u|^k$) logaritmic source term seems in inflation cosmology, nuclear physics, geophysics, and optics (see [3, 9]).

Firstly, for the literature review, we begin with the works of Birula and Mycielski [4, 5]. They looked into the following equation with logarithmic term:

$$u_{tt} - u_{xx} + u - \varepsilon u \ln|u|^2 = 0.$$

They are the pioneer of these kind of problems. They established that, in any number of dimensions, wave equations including the logarithmic term have localized, stable, soliton-like solutions. Cazenave and Haraux [6], in 1980, considered the following equation:

$$u_{tt} - \Delta u = u \ln|u|^k. \tag{1.2}$$

The authors established the existence and uniqueness of the equation (1.2). Gorka [9] obtained the global existence for one-dimensional of the equation (1.2). Bartkowski and Gorka [3], looked into the weak solutions and established the existence results. In [2], Al-Gharabli and Messaoudi studied the following plate equation with logarithmic term:

$$u_{tt} + \Delta^2 u + u + h(u_t) = ku \ln|u|. \tag{1.3}$$

The authors proved the existence results utilizing the Galerkin method and established an explicit and general decay rate results by using the multiplier method of the equation (1.3). Liu [15], considered the following logarithmic plate equation:

$$u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u \log|u|^k.$$

The author obtained the local existence utilizing the contraction mapping principle, and proved the global existence and decay of solutions. Also, the author established the blow up of solutions. In [16], Messaoudi considered the following equation:

$$u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u. \tag{1.4}$$

He proved the existence and blow up of solutions. Then, Chen and Zhou [7] extended these results. In the presence of the strong damping term ($-\Delta u_t$), Polat and Pişkin [20] proved the global existence and decay of solutions for the equation (1.4). In [17], Nicaise and Pignotti considered the following equation:

$$u_{tt} - \Delta u + a_0 u_t(x,t) + a u_t(x, t-\tau) = 0,$$

where a_0, $a > 0$. The authors established that, under the condition $0 \leqslant a \leqslant a_0$, the system is exponentially stable. They obtained a sequence of delays that shows the solution is instable, under the condition $a \geqslant a_0$. In [21], Yüksekkaya et al., studied the following delayed plate equation with the logarithmic source:

$$u_{tt} + \Delta^2 u + \alpha u_t(t) + \beta u_t(x, t-\tau) = u \ln|u|^\gamma. \tag{1.5}$$

They obtained the local and global existence of solutions and proved the stability and nonexistence results of the equation (1.5). In [12], Kafini and Messaoudi looked into the following equation:

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = |u|^{p-2} u \log|u|^k. \tag{1.6}$$

They obtained the local existence and blow up of solutions of the equation (1.6). In [10], Kafini studied the wave equation with logarithmic nonlinearity with distributed delay as follows:

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x, t-s) = u|u|^{p-2} \ln|u|^k, \tag{1.7}$$

he established the local and global existence results. Moreover, he proved the exponential decay of solutions for the equation (1.7).

In this paper, we consider the local existence, global existence, exponential decay, and blow up of solutions for the logarithmic Petrovsky equation (1.1) with distributed delay, motivated by above works. To our best knowledge, there is no research related to the logarithmic Petrovsky equation (1.1) with distributed delay ($\int_{\tau_1}^{\tau_2}\mu_2(s)u_t(x,t-s)\,ds$) term and logarithmic ($u|u|^{p-2}\ln|u|^k$) source term, therefore, our paper is the generalization of the above works.

This work consists of six sections in addition to the introduction. Firstly, in Section 2, we give some needed materials. Then, in Section 3, we get the local existence results by using the semigroup theory similar to the works of [10, 12]. Moreover, in Section 4, we establish the global existence results by the well-depth method. Furthermore, in Section 5, we prove the exponential decay results. Finally, in Section 6, we establish the blow up results for negative initial energy.

2. Preliminaries

In this section, we give some materials for our main results. As usual, the notation $\|.\|_p$ denotes L^p norm, and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$ (see [1, 19], for details).

Let $B_p > 0$ be the constant satisfying [1]

$$\|\nabla v\|_p \leqslant B_p\|\Delta v\|_p, \text{ for } v \in H_0^2(\Omega).$$

Similar to the [17], we introduce the new variable

$$z(x,\rho,s,t) = u_t(x,t-\rho s) \text{ in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Therefore, we have

$$sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \text{ in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Hence, the problem (1.1) takes the form:

$$\begin{cases} u_{tt} + \Delta^2 u + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)\,ds = u|u|^{p-2}\ln|u|^k, & \text{in } \Omega\times(\tau_1,\tau_2)\times(0,\infty),\\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, & \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty),\\ z(x,\rho,s,0) = f_0(x,-\rho s), & \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2),\\ u(x,t) = \frac{\partial u(x,t)}{\partial \upsilon} = 0, & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{2.1}$$

The energy functional related to the problem (2.1) is, for $\forall t \geqslant 0$,

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{k}{p^2}\|u\|_p^p \\ &\quad + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi+\mu_2(s))|z(x,\rho,s,t)|^2\,ds\,d\rho\,dx - \frac{1}{p}\int_\Omega |u|^p\ln|u|^k\,dx, \end{aligned}$$

where ξ is a positive constants satisfying

$$\mu_1 > \int_{\tau_1}^{\tau_2}\mu_2(s)\,ds + \frac{\xi}{2}(\tau_2-\tau_1), \tag{2.2}$$

under the condition $\mu_1 > \int_{\tau_1}^{\tau_2}\mu_2(s)\,ds$.

The following lemma shows that the related energy functional is nonincreasing.

Lemma 2.1. *Suppose that* (2.2) *holds. Then, along the solution of* (2.1) *and for some* $C_0 \geqslant 0$, *we get*

$$E'(t) \leqslant -C_0 \int_\Omega \left(|u_t|^2 + |z(x,1,s,t)|^2\right) dx \leqslant 0.$$

Proof. By multiplying the first equation in (2.1) by u_t and integrating over Ω and the second equation in (2.1) by $(\xi + \mu_2(s))\, z$ and integrating over $(\tau_1, \tau_2) \times (0,1) \times \Omega$ with respect to s, ρ, and x, summing up, we get

$$\begin{aligned} &\frac{d}{dt}\left(\begin{array}{c} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{k}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln |u|^k\, dx \\ +\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\,(\xi + \mu_2(s))\, |z(x,\rho,s,t)|^2\, ds\, d\rho\, dx \end{array}\right) \\ &\quad = -\mu_1 \int_\Omega |u_t|^2\, dx - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s))\, z z_\rho(x,\rho,s,t)\, ds\, d\rho\, dx \\ &\qquad - \int_\Omega u_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\, dx. \end{aligned} \tag{2.3}$$

Now, we handle the last two terms of the right-hand side of (2.3) as follows:

$$\begin{aligned} &-\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s))\, z z_\rho(x,\rho,s,t)\, ds\, d\rho\, dx \\ &\quad = -\frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 \frac{\partial}{\partial \rho}\left[(\xi + \mu_2(s))\, |z(x,\rho,s,t)|^2\right] d\rho\, ds\, dx \\ &\quad = \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds + \xi(\tau_2 - \tau_1)\right)\int_\Omega |u_t|^2\, dx - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s))\, |z(x,1,s,t)|^2\, ds\, dx \end{aligned}$$

and

$$-\int_\Omega u_t \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x,1,s,t)\, ds\, dx \leqslant \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \int_\Omega |u_t|^2\, dx + \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \int_\Omega |z(x,1,s,t)|^2\, dx\right).$$

Therefore, we get

$$\frac{dE(t)}{dt} \leqslant -\left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds - \frac{\xi}{2}(\tau_2 - \tau_1)\right)\int_\Omega |u_t|^2\, dx - \frac{\xi}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |z(x,1,s,t)|^2\, ds\, dx.$$

From (2.2), we obtain, for some $C_0 > 0$,

$$E'(t) \leqslant -C_0 \int_\Omega \left(|u_t|^2 + \int_{\tau_1}^{\tau_2} |z(x,1,s,t)|^2\, ds\right) dx \leqslant 0. \qquad \square$$

3. Local existence

In this section, we prove the local existence result by using the semigroup theory [13, 18]. Let $\upsilon = u_t$ and denote by

$$\Phi = (u, \upsilon, z)^T, \quad \Phi(0) = \Phi_0 = (u_0, u_1, f_0(\cdot, \rho s))^T, \quad \text{and} \quad J(\Phi) = \left(0, u|u|^{p-2}\ln |u|^k, 0\right)^T.$$

Hence, (2.1) can be written as an initial-value problem:

$$\begin{cases} \partial_t \Phi + \mathcal{A}\Phi = J(\Phi), \\ \Phi(0) = \Phi_0, \end{cases} \tag{3.1}$$

where the linear operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -\upsilon \\ \Delta^2 u + \mu_1 \upsilon + \int_{\tau_1}^{\tau_2} \mu_2(s) z(\cdot,1,s)\, ds \\ \frac{1}{s} z_\rho \end{pmatrix},$$

where $D(\mathcal{A})$ and $\mathcal{H}$ are introduced below. Now, we define

$$L_s^2(\tau_1,\tau_2) = \left\{ w : \int_{\tau_1}^{\tau_2} s\,|w(s)|^2\, ds < \infty \right\}.$$

The state space of Φ is the Hilbert space

$$\mathcal{H} = H_0^2(\Omega) \times L^2(\Omega) \times L^2\left(\Omega \times (0,1); L_s^2(\tau_1,\tau_2)\right),$$

equipped with the inner product

$$\left\langle \Phi, \tilde{\Phi} \right\rangle_{\mathcal{H}} = \int_\Omega \left(\Delta u \Delta \tilde{u} + \upsilon \tilde{\upsilon} \right) dx + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2(s)\, z\tilde{z}\, ds\, d\rho\, dx,$$

for all $\Phi = (u,\upsilon,z)^T$ and $\tilde{\Phi} = \left(\tilde{u},\tilde{\upsilon},\tilde{z}\right)^T$ in $\mathcal{H}$. The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{l} \Phi \in \mathcal{H} : u \in H^4(\Omega),\ \upsilon \in H_0^1(\Omega),\ z(\cdot,1,\cdot) \in L^2(\Omega), \\ z,\ z_\rho \in L^2\left(\Omega \times (0,1); L_s^2(\tau_1,\tau_2)\right),\ z(\cdot,0,\cdot) = \upsilon. \end{array} \right\}.$$

Now, we get the local existence of solutions as follows.

Theorem 3.1. *Suppose that* $\mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds$ *and*

$$\begin{cases} p \geqslant 2, & \text{if } n = 1,2,3,4, \\ 2 < p < \frac{2(n-2)}{n-4}, & \text{if } n \geqslant 5. \end{cases} \tag{3.2}$$

Then, for any $\Phi_0 \in \mathcal{H}$*, problem* (3.1) *has a unique weak solution* $\Phi \in C([0,T);\mathcal{H})$.

Proof. Firstly, for all $\Phi \in D(\mathcal{A})$, we get

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} &= -\int_\Omega \Delta\upsilon \Delta u\, dx + \int_\Omega \upsilon \left[\Delta^2 u + \mu_1 \upsilon + \int_{\tau_1}^{\tau_2} \mu_2(s) z(\cdot,1,\cdot)\, ds \right] dx \\ &\quad + \int_0^1 \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s)\, z z_\rho\, ds\, dx\, d\rho \\ &= \mu_1 \int_\Omega |\upsilon|^2\, dx + \int_\Omega \upsilon \int_{\tau_1}^{\tau_2} \mu_2(s) z(\cdot,1,\cdot)\, ds\, dx \\ &\quad + \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) |z(\cdot,1,\cdot)|^2\, ds\, dx - \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) |\upsilon|^2\, ds\, dx. \end{aligned} \tag{3.3}$$

Utilizing Young's inequality, the estimate (3.3) takes the form:

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} &\geqslant \mu_1 \int_\Omega |\upsilon|^2\, dx - \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) |\upsilon|^2\, ds\, dx - \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) |\upsilon|^2\, ds\, dx \\ &= \left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \right) \int_\Omega |\upsilon|^2\, dx \geqslant 0. \end{aligned}$$

Hence, $\mathcal{A}$ is a monotone operator. To show that $\mathcal{A}$ is maximal, we establish that for each $F = (f, g, h)^T \in \mathcal{H}$, there exists $V = (u, v, z)^T \in D(\mathcal{A})$ such that $(I + \mathcal{A}) V = F$. That is,

$$\begin{cases} u - v = f, \\ v + \Delta^2 u + \mu_1 v + \int_{\tau_1}^{\tau_2} \mu_2(s) z(\cdot, 1, s)\, ds = g, \\ sz + z_\rho = sh. \end{cases} \tag{3.4}$$

As $v = u - f$, by the third equation of (3.4), we infer that

$$z(\cdot, \rho, s) = (u - f) e^{-\rho s} + s e^{-\rho s} \int_0^\rho h(\cdot, r, s) e^{rs} dr. \tag{3.5}$$

Substituting (3.5) in the second equation of (3.4), we obtain $\sigma u + \Delta^2 u = G$, where

$$\begin{aligned} \sigma &= 1 + \mu_1 + \int_{\tau_1}^{\tau_2} \mu_2(s) e^{-s} ds > 1, \\ G &= g + \sigma f - \int_{\tau_1}^{\tau_2} \mu_2(s) s e^{-s} \int_0^1 h(\cdot, r, s) e^{rs} dr ds \in L^2(\Omega). \end{aligned} \tag{3.6}$$

We define, over $H_0^2(\Omega)$, the bilinear and linear forms:

$$B(u, w) = \sigma \int_\Omega uw + \int_\Omega \Delta u \Delta w, \quad L(w) = \int_\Omega Gw.$$

We see that B is coercive and continuous, and L is continuous on $H_0^2(\Omega)$. Then, Lax-Milgram theorem specifies that the equation

$$B(u, w) = L(w), \forall w \in H_0^2(\Omega), \tag{3.7}$$

has a unique solution $u \in H_0^2(\Omega)$. Therefore, $v = u - f \in H_0^2(\Omega)$. As a result, from (3.5), we get

$$z, z_\rho \in L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)) \text{ and } z(\cdot, 0, \cdot) = v.$$

Hence, $V \in \mathcal{H}$. By (3.7), we infer that

$$\sigma \int_\Omega uw + \int_\Omega \Delta u \Delta w = \int_\Omega Gw, \forall w \in H_0^2(\Omega).$$

By the elliptic regularity theory, we have $u \in H^4(\Omega)$. In addition, using Green's formula gives

$$\sigma \int_\Omega uw + \int_\Omega w \Delta^2 u - \int_\Omega Gw = 0, \forall w \in H_0^2(\Omega),$$

and from (3.6), we get

$$\int_\Omega \left[(1 + \mu_1) v + \Delta^2 u + \int_{\tau_1}^{\tau_2} \mu_2(s) z(\cdot, 1, s)\, ds - g \right] w = 0, \forall w \in H_0^2(\Omega).$$

Thus,

$$(1 + \mu_1) v + \Delta^2 u + \int_{\tau_1}^{\tau_2} \mu_2(s) z(\cdot, 1, s)\, ds = g \in L^2(\Omega).$$

Hence,

$$V = (u, v, z)^T \in D(\mathcal{A}).$$

As a result, $I + \mathcal{A}$ is surjective and then $\mathcal{A}$ is maximal.

As a result, we denote that $J:\mathcal{H}\longrightarrow\mathcal{H}$ is locally Lipschitz. Hence, if we set

$$f(s)=|s|^{p-2}s\ln|s|^{k},\ |s|\neq 0 \text{ and } f(s)=0,\ |s|=0,$$

then

$$f'(s)=k\left[1+(p-1)\ln|s|\right]|s|^{p-2},\ |s|\neq 0 \text{ and } f'(s)=0,\ |s|=0.$$

Therefore,

$$\begin{aligned}\left\|J(\Phi)-J\left(\tilde{\Phi}\right)\right\|_{\mathcal{H}}^{2}&=\left\|\left(0,|u|^{p-2}u\ln|u|^{k}-\left|\tilde{u}\right|^{p-2}\tilde{u}\ln\left|\tilde{u}\right|^{k},0\right)\right\|_{\mathcal{H}}^{2}\\&=\left\||u|^{p-2}u\ln|u|^{k}-\left|\tilde{u}\right|^{p-2}\tilde{u}\ln\left|\tilde{u}\right|^{k}\right\|_{L^{2}}^{2}=\left\|f(u)-f\left(\tilde{u}\right)\right\|_{L^{2}}^{2}.\end{aligned}$$

From the mean value theorem, we get, for some $\theta\in[0,1]$,

$$\begin{aligned}\left|f(u)-f\left(\tilde{u}\right)\right|&=\left|f'\left(\theta u+(1-\theta)\tilde{u}\right)\left(u-\tilde{u}\right)\right|\\&=k\left|\left[1+(p-1)\ln\left|\theta u+(1-\theta)\tilde{u}\right|\right]\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\left(u-\tilde{u}\right)\right|\\&\leqslant k\left[1+(p-1)\left|\ln\left|\theta u+(1-\theta)\tilde{u}\right|\right|\right]\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\left|u-\tilde{u}\right|\\&=k(p-1)\left|\ln\left|\theta u+(1-\theta)\tilde{u}\right|\right|\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\left|u-\tilde{u}\right|\\&\quad+k\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\left|u-\tilde{u}\right|.\end{aligned}$$

To control the logarithmic term $\ln\left|\theta u+(1-\theta)\tilde{u}\right|\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}$, seems in the last inequality we remind that, for any $\varepsilon>0$,

$$\lim_{|s|\to+\infty}\frac{\ln|s|}{|s|^{\varepsilon}}=0.$$

Then, there exists $B>0$ such that

$$\frac{\ln|s|}{|s|^{\varepsilon}}<1,\ \forall|s|>B.$$

Thus, whenever $\left|\theta u+(1-\theta)\tilde{u}\right|>B$, we obtain

$$\ln\left|\theta u+(1-\theta)\tilde{u}\right|\leqslant\left|\theta u+(1-\theta)\tilde{u}\right|^{\varepsilon},$$

and

$$\ln\left|\theta u+(1-\theta)\tilde{u}\right|\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\leqslant\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2+\varepsilon}.$$

Since $p>2$, then for some $A>0$ and $\left|\theta u+(1-\theta)\tilde{u}\right|\leqslant B$, we get

$$\ln\left|\theta u+(1-\theta)\tilde{u}\right|\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\leqslant A.$$

Therefore, we obtain

$$\ln\left|\theta u+(1-\theta)\tilde{u}\right|\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\leqslant A+\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2+\varepsilon}.$$

Then, we get the following estimation for

$$\begin{aligned}&\left|f(u)-f\left(\tilde{u}\right)\right|\\&\leqslant k(p-1)\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2+\varepsilon}\left|u-\tilde{u}\right|+k\left|\theta u+(1-\theta)\tilde{u}\right|^{p-2}\left|u-\tilde{u}\right|+kA(p-1)\left|u-\tilde{u}\right|\\&\leqslant k(p-1)\left(|u|+\left|\tilde{u}\right|\right)^{p-2+\varepsilon}\left|u-\tilde{u}\right|+k\left(|u|+\left|\tilde{u}\right|\right)^{p-2}\left|u-\tilde{u}\right|+kA(p-1)\left|u-\tilde{u}\right|.\end{aligned}$$

As $u,\ \tilde{u}\in H_0^2(\Omega)$, utilizing Hölder's inequality, (3.2), and the Sobolev embedding

$$H_0^2(\Omega)\hookrightarrow L^r(\Omega),\ 1\leqslant r\leqslant\frac{2n}{n-4},$$

we have

$$\begin{aligned}\int_\Omega\left[\left(|u|+\left|\tilde{u}\right|\right)^{p-2}\left|u-\tilde{u}\right|\right]^2dx&=\int_\Omega\left(|u|+\left|\tilde{u}\right|\right)^{2(p-2)}\left|u-\tilde{u}\right|^2dx\\&\leqslant C\left(\int_\Omega\left(|u|+\left|\tilde{u}\right|\right)^{2(p-1)}dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega\left|u-\tilde{u}\right|^{2(p-1)}dx\right)^{\frac{1}{p-1}}\\&\leqslant C\left[\|u\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}+\left\|\tilde{u}\right\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}\right]^{\frac{p-2}{p-1}}\left\|u-\tilde{u}\right\|_{L^{2(p-1)}(\Omega)}^2\\&\leqslant C\left[\|u\|_{H_0^2(\Omega)}^{2(p-1)}+\left\|\tilde{u}\right\|_{H_0^2(\Omega)}^{2(p-1)}\right]^{\frac{p-2}{p-1}}\left\|u-\tilde{u}\right\|_{H_0^2(\Omega)}^2.\end{aligned}$$

In a similar way,

$$\begin{aligned}\int_\Omega\left[\left(|u|+\left|\tilde{u}\right|\right)^{p-2+\varepsilon}\left|u-\tilde{u}\right|\right]^2dx&=\int_\Omega\left(|u|+\left|\tilde{u}\right|\right)^{2(p-2+\varepsilon)}\left|u-\tilde{u}\right|^2dx\\&\leqslant\left(\int_\Omega\left(|u|+\left|\tilde{u}\right|\right)^{\frac{2(p-2+\varepsilon)(p-1)}{p-2}}dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega\left|u-\tilde{u}\right|^{2(p-1)}dx\right)^{\frac{1}{p-1}}\\&\leqslant\left(\int_\Omega\left(|u|+\left|\tilde{u}\right|\right)^{2(p-1)+\frac{2\varepsilon(p-1)}{p-2}}dx\right)^{\frac{p-2}{p-1}}\left\|u-\tilde{u}\right\|_{L^{2(p-1)}(\Omega)}^2.\end{aligned}$$

Since $p<\frac{2(n-2)}{n-4}$, choosing $\varepsilon>0$ small enough, such that

$$p^*=2(p-1)+\frac{2\varepsilon(p-1)}{p-2}\leqslant\frac{2n}{n-4}.$$

Therefore, we obtain

$$\begin{aligned}\int_\Omega\left(|u|+\left|\tilde{u}\right|\right)^{2(p-2+\varepsilon)}\left|u-\tilde{u}\right|^2dx&\leqslant C\left[\|u\|_{L^{p^*}(\Omega)}^{p^*}+\left\|\tilde{u}\right\|_{L^{p^*}(\Omega)}^{p^*}\right]^{\frac{p-2}{p-1}}\left\|u-\tilde{u}\right\|_{L^{2(p-1)}(\Omega)}^2\\&\leqslant C\left[\|u\|_{H_0^2(\Omega)}^{p^*}+\left\|\tilde{u}\right\|_{H_0^2(\Omega)}^{p^*}\right]^{\frac{p-2}{p-1}}\left\|u-\tilde{u}\right\|_{H_0^2(\Omega)}^2.\end{aligned}$$

Thus, by combining the last three estimations, we have

$$\left\|J(\Phi)-J\left(\tilde{\Phi}\right)\right\|_{\mathcal{H}}^2\leqslant\left[k^2(p-1)^2A^2\right]\left\|u-\tilde{u}\right\|_{H_0^2(\Omega)}^2+C\left[\left(\|u\|_{H_0^2(\Omega)}^{2(p-1)}+\left\|\tilde{u}\right\|_{H_0^2(\Omega)}^{2(p-1)}\right)^{\frac{p-2}{p-1}}\right.$$

$$+\left(\|u\|^{p^*}_{H^2_0(\Omega)}+\left\|\tilde{u}\right\|^{p^*}_{H^2_0(\Omega)}\right)^{\frac{p-2}{p-1}}\Bigg]\left\|u-\tilde{u}\right\|^2_{H^2_0(\Omega)}$$
$$\leqslant C\left(\|u\|_{H^2_0(\Omega)},\left\|\tilde{u}\right\|_{H^2_0(\Omega)}\right)\left\|u-\tilde{u}\right\|^2_{H^2_0(\Omega)}.$$

Hence, J is locally Lipschitz. Then, by the Theorem 1.2 page 184, Pazy [18] (see also a remark in the begining of page 118, Komornik [13]), we complete the proof. □

4. Global existence

In this section, we establish that the solution of (2.1) is uniformly bounded and global in time. For this aim, we set

$$I(t)=\|\Delta u\|^2-\int_\Omega |u|^p\ln|u|^k\,dx,$$
$$J(t)=\frac{1}{2}\|\Delta u\|^2+\frac{k}{p^2}\|u\|_p^p+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(\xi+\mu_2(s)\right)z^2\,ds\,d\rho\,dx-\frac{1}{p}\int_\Omega |u|^p\ln|u|^k\,dx. \tag{4.1}$$

Therefore, we have $E(t)=J(t)+\frac{1}{2}\|u_t\|^2$.

Lemma 4.1. *Assume that the initial data* $u_0, u_1\in H^2_0(\Omega)\times L^2(\Omega)$ *satisfying*

$$I(0)>0 \text{ and } \beta=kC_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}}<1. \tag{4.2}$$

Then, $I(t)>0$*, for any* $t\in[0,T]$.

Proof. Since $I(0)>0$ we infer from continuity that there exists $T^*\leqslant T$ such that $I(t)\geqslant 0$ for all $t\in[0,T^*]$. This implies that, for all $t\in[0,T^*]$,

$$J(t)=\frac{p-2}{2p}\|\Delta u\|^2+\frac{k}{p^2}\|u\|_p^p+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left(\xi+\mu_2(s)\right)z^2\,ds\,d\rho\,dx+\frac{1}{p}I(t).$$

Therefore, we have

$$J(t)\geqslant\frac{p-2}{2p}\|\Delta u\|^2.$$

Hence,

$$\|\Delta u\|^2\leqslant\frac{2p}{p-2}J(t)\leqslant\frac{2p}{p-2}E(t)\leqslant\frac{2p}{p-2}E(0).$$

On the other hand, by using the fact that $\ln|u|<|u|^l$, we get

$$\int_\Omega |u|^p\ln|u|\,dx\leqslant\int_\Omega |u|^{p+l}\,dx, \tag{4.3}$$

where l is choosen to be $0<l<\frac{4}{n-4}$, such that

$$p+l<\frac{2n-4}{n-4}+l<\frac{2n}{n-4}.$$

Therefore, the embedding $H_0^2(\Omega) \hookrightarrow L^{p+l}(\Omega)$, satisfies

$$\int_\Omega |u|^p \ln|u|\,dx \leqslant C_{p+l}\|\Delta u\|^{p+l} = C_{p+l}\|\Delta u\|^2\|\Delta u\|^{p-2+l}$$
$$= C_{p+l}\|\Delta u\|^2\left(\|\Delta u\|^2\right)^{\frac{p-2+l}{2}} \leqslant C_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}}\|\Delta u\|^2,$$

where C_{p+l} is the embedding constant. As a result, by (4.1) and (4.2), we infer that

$$I(t) > \|\Delta u\|^2 - \beta\|\Delta u\|^2 > 0, \forall t \in [0, T^*].$$

By repeating this procedure, T^* can be extended to T. □

Theorem 4.2. *If the initial data u_0, u_1 satisfy the conditions of Lemma 4.1, then the solution of (2.1) is uniformly bounded and global in time.*

Proof. It suffices to show that $\|\Delta u\|^2 + \|u_t\|^2$ is bounded independently of t. We see that,

$$E(0) \geqslant E(t) = \frac{1}{2}\|u_t\|^2 + J(t) \geqslant \frac{1}{2}\|u_t\|^2 + \frac{k}{p^2}\|u\|_p^p + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) z^2 ds d\rho dx + \frac{1}{p}I(t)$$
$$\geqslant \frac{1}{2}\|u_t\|^2 + \frac{1}{p}(1-\beta)\|\Delta u\|^2.$$

Thus, $\|\Delta u\|^2 + \|u_t\|^2 \leqslant CE(0)$, where C is a positive constant depending only on k, p and C_{p+1}. □

5. Exponential decay

In this section, we establish the decay of solutions. Firstly, we give the following lemmas.

Lemma 5.1 ([10]). *The functional*

$$F_1(t) = \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}(\xi + \mu_2(s)) z|(x,\rho,s,t)|^2 ds d\rho dx$$

satisfies, along the solution of (2.1), for some c_1, $c_2 > 0$,

$$F_1'(t) \leqslant c_1\|u_t\|^2 - c_2\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) z|(x,\rho,s,t)|^2 ds d\rho dx. \quad (5.1)$$

Lemma 5.2. *The functional*

$$F_2(t) = NE(t) + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega |u|^2 dx$$

satisfies, along the solution of (2.1),

$$F_2'(t) \leqslant -(NC_0 - \varepsilon)\|u_t\|^2 - \varepsilon(1-\beta-\delta)\|\Delta u\|^2 - \left(NC_0 - \varepsilon\frac{c_*}{4\delta}\right)\int_\Omega\int_{\tau_1}^{\tau_2} z^2(x,1,s,t) ds dx, \quad (5.2)$$

where N, α, and ε are positive constants.

Proof. A direct differentiation, by using equations in (2.1), satisfies

$$\begin{aligned} F_2'(t) \leqslant & -NC_0 \int_\Omega \left(|u_t|^2 + |z(x,1,s,t)|^2 \right) dx \\ & + \varepsilon \left(\int_\Omega |u_t|^2 dx - \int_\Omega |\Delta u|^2 dx + \int_\Omega |u|^p \ln |u|^k dx \right) - \varepsilon \int_\Omega u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) \, ds dx. \end{aligned} \tag{5.3}$$

Utilizing Young's inequality and the boundness property of $\mu_2(s)$, we obtain, for any $\delta > 0$ and some $c_* > 0$,

$$-\int_\Omega u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) \, ds dx \leqslant \delta \|\Delta u\|^2 + \frac{c_*}{4\delta} \int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) \, ds dx. \tag{5.4}$$

By combining (4.3), (5.3), and (5.4), the result follows. □

Theorem 5.3. *Assume that* (4.2) *holds. Then, there exist two positive constants* c_3 *and* c_4 *such that*

$$E(t) \leqslant c_3 e^{-c_4 t}.$$

Proof. Setting

$$F_3(t) = F_1(t) + F_2(t),$$

it is easy to verify, for ε small enough, that

$$F_3(t) \sim E(t). \tag{5.5}$$

By using (5.1) and (5.2), we obtain

$$\begin{aligned} F_3'(t) \leqslant & -(NC_0 - \varepsilon - c_1) \|u_t\|^2 - \varepsilon (1 - \beta - \delta) \|\Delta u\|^2 \\ & - \left(NC_0 - \varepsilon \frac{c_*}{4\delta} \right) \int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) \, ds dx \\ & - c_2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) |z(x,\rho,s,t)|^2 \, ds d\rho dx. \end{aligned} \tag{5.6}$$

Since $\beta < 1$, choosing δ small enough, such that $\alpha = 1 - \beta - \delta > 0$. For some $\omega > 0$, the embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$ satisfies

$$\|u\|_p^p \leqslant C \|\Delta u\|_2^p \leqslant C \left(\|\Delta u\|^2 \right)^{\frac{p-2}{2}} \|\Delta u\|^2 \leqslant C (E(0))^{\frac{p-2}{2}} \|\Delta u\|^2 \leqslant \omega \|\Delta u\|^2,$$

or

$$-\frac{\varepsilon \alpha \omega^{-1}}{2} \|u\|_p^p \geqslant -\frac{\varepsilon \alpha}{2} \|\Delta u\|_2^2.$$

Hence, (5.6) takes the form

$$\begin{aligned} F_3'(t) \leqslant & -(NC_0 - \varepsilon - c_1) \|u_t\|^2 - \frac{\varepsilon \alpha}{2} \|\Delta u\|^2 - \frac{\varepsilon \alpha \omega^{-1}}{2} \|u\|_p^p \\ & - \left(NC_0 - \varepsilon \frac{c_p}{4\delta} \right) \int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) \, ds dx \\ & - c_2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) |z(x,\rho,s,t)|^2 \, ds d\rho dx. \end{aligned} \tag{5.7}$$

Whence δ is fixed, choosing N to be large enough, such that

$$NC_0 - \varepsilon - c_1 > 0 \quad \text{and} \quad NC_0 - \varepsilon \frac{c_p}{4\delta} > 0.$$

Therefore, (5.7) takes the form, for some $C > 0$,

$$F_3'(t) \leqslant -C\left[\|u_t\|^2 + \|\Delta u\|^2 + \|u\|_p^p + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\left(\xi + \mu_2(s)\right) z^2 ds d\rho dx\right] \leqslant -CE(t).$$

By using the equivalence relation (5.5) and a simple integration over $(0, t)$, the result is proved. □

6. Blow up

In this section, we prove the blow up results for negative initial energy. We have the assumption: $\mu_2 : [\tau_1, \tau_2] \to R$ is an L^∞ function such that:

$$\left(\frac{2\delta - 1}{2}\right)\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \leqslant \mu_1, \quad \delta > \frac{1}{2}. \tag{6.1}$$

We have the following lemmas to obtain the main result.

Lemma 6.1. *Suppose that* (3.2) *and* (6.1) *hold. Let u be a solution of* (2.1)*. Then, $\mathcal{K}(t)$ is nonincreasing, such that*

$$\begin{aligned} \mathcal{K}(t) = {} & \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{k}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln |u|^k dx \\ & + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\left|z^2(x, \rho, s, t)\right| ds d\rho dx, \end{aligned} \tag{6.2}$$

which satisfies

$$\mathcal{K}'(t) \leqslant -c_1\left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x, 1, s, t)\right| ds dx\right). \tag{6.3}$$

Proof. By multiplying the first equation of (2.1) by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left\{\begin{array}{c} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{k}{p^2}\|u\|_p^p \\ -\frac{1}{p}\int_\Omega |u|^p \ln |u|^k dx \end{array}\right\} = -\mu_1\|u_t\|^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,|z(x, 1, s, t)|\, ds dx, \tag{6.4}$$

and

$$\begin{aligned} & \frac{d}{dt}\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\left|z^2(x, \rho, s, t)\right| ds d\rho dx \\ & = -\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} 2\,|\mu_2(s)|\, z z_\rho\, ds d\rho dx \\ & = \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x, 0, s, t)\right| ds dx - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x, 1, s, t)\right| ds dx \\ & = \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\|u_t\|^2 - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x, 1, s, t)\right| ds dx. \end{aligned} \tag{6.5}$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) = {} & -\mu_1\|u_t\|^2 - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,|u_t z(x, 1, s, t)|\, ds dx \\ & + \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\|u_t\|^2 - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x, 1, s, t)\right| ds dx. \end{aligned} \tag{6.6}$$

From (6.4) and (6.5), we get (6.2). Utilizing the Young's inequality, (6.1), and (6.6), we get (6.3). As a result, the proof is completed. □

Lemma 6.2. *There exists a positive constant* $C > 0$ *depending on* Ω *only such that*

$$\left(\int_{\Omega} |u|^p \ln |u|^k \, dx\right)^{s/p} \leqslant C \left[\int_{\Omega} |u|^p \ln |u|^k \, dx + \|\Delta u\|_2^2\right]$$

for any $u \in L^{p+1}(\Omega)$ *and* $2 \leqslant s \leqslant p$, *provided that* $\int_{\Omega} |u|^p \ln |u|^k \, dx \geqslant 0$.

Proof. In [12], from Lemma 3.2 we know that

$$\left(\int_{\Omega} |u|^p \ln |u|^k \, dx\right)^{s/p} \leqslant C \left[\int_{\Omega} |u|^p \ln |u|^k \, dx + \|\nabla u\|_2^2\right]$$

is satisfied, utilizing Sobolev embedding theorem we obtain this result. □

Similar to the [12] and using the Sobolev Embedding Theorem, we have the lemmas as follows.

Lemma 6.3. *Depending on* Ω *only, suppose that* $C > 0$, *so that*

$$\|u\|_2^2 \leqslant C \left[\left(\int_{\Omega} |u|^p \ln |u|^k \, dx\right)^{2/p} + \|\Delta u\|_2^{4/p}\right], \tag{6.7}$$

provided that $\int_{\Omega} |u|^p \ln |u|^k \, dx \geqslant 0$.

Lemma 6.4. *Depending on* Ω *only, assume that* $C > 0$, *such that*

$$\|u\|_p^s \leqslant C \left[\|u\|_p^p + \|\Delta u\|_2^2\right],$$

for any $u \in L^p(\Omega)$ *and* $2 \leqslant s \leqslant p$.

To get the main result, we define

$$\begin{aligned} H(t) = -\mathcal{K}(t) = &-\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\|\Delta u\|^2 - \frac{k}{p^2}\|u\|_p^p + \frac{1}{p}\int_{\Omega} |u|^p \ln |u|^k \, dx \\ &-\frac{1}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,\left|z^2(x,\rho,s,t)\right| ds\,d\rho\,dx. \end{aligned}$$

Theorem 6.5. *Assume that* (6.1) *holds. Assume further that*

$$\begin{cases} p \geqslant 2, & n \leqslant 4, \\ 2 < p \leqslant \frac{2(n-2)}{n-4}, & \text{if } n > 4, \end{cases}$$

and

$$\mathcal{K}(0) < 0.$$

Thus, the solution of (2.1) *blows up in finite time.*

Proof. By (6.3), we know that

$$\mathcal{K}(t) \leqslant \mathcal{K}(0) < 0.$$

Thus,

$$\begin{aligned} H'(t) = -\mathcal{K}'(t) &\geqslant c_1 \left(\|u_t\|^2 + \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x,1,s,t)\right| ds\,dx\right) \\ &\geqslant c_1 \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\left|z^2(x,1,s,t)\right| ds\,dx \geqslant 0 \end{aligned} \tag{6.8}$$

and

$$0 \leqslant H(0) \leqslant H(t) \leqslant \frac{1}{p}\int_{\Omega} |u|^{p} \ln |u|^{k} \, dx. \tag{6.9}$$

We introduce

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t \, dx + \frac{\mu_1 \varepsilon}{2}\int_{\Omega} u^2 dx, \ t \geqslant 0, \tag{6.10}$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{6.11}$$

By multiplying the first equation in (2.1) by u and with a derivative of (6.10), we get

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 + \varepsilon \int_{\Omega} u u_{tt} \, dx + \varepsilon \mu_1 \int_{\Omega} u u_t \, dx \\ &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 \\ &\quad - \varepsilon \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, |u z(x,1,s,t)| \, ds dx + \varepsilon \int_{\Omega} |u|^{p} \ln |u|^{k} \, dx. \end{aligned} \tag{6.12}$$

Thanks to Young's inequality, we get

$$\begin{aligned} &\varepsilon \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, u z(x,1,s,t) \, ds dx \\ &\leqslant \varepsilon \left\{ \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 + \frac{1}{4\delta_1} \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \left| z^2(x,1,s,t) \right| ds dx \right\}. \end{aligned} \tag{6.13}$$

By (6.12), we obtain

$$\begin{aligned} L'(t) \geqslant{}& (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 - \varepsilon \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 \\ &- \frac{\varepsilon}{4\delta_1} \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \left| z^2(x,1,s,t) \right| ds dx + \varepsilon \int_{\Omega} |u|^{p} \ln |u|^{k} \, dx. \end{aligned}$$

By using (6.8) and setting δ_1 such that $\frac{1}{4\delta_1 c_1} = \kappa H^{-\alpha}(t)$, we obtain

$$\begin{aligned} L'(t) \geqslant{}& [(1-\alpha) - \varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ &- \varepsilon \|\Delta u\|^2 - \varepsilon \frac{H^{\alpha}(t)}{4 c_1 \kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 + \varepsilon \int_{\Omega} |u|^{p} \ln |u|^{k} \, dx. \end{aligned}$$

For $0 < a < 1$, we have

$$\begin{aligned} L'(t) \geqslant{}& [(1-\alpha) - \varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon a \int_{\Omega} |u|^{p} \ln |u|^{k} \, dx + \varepsilon \frac{p(1-a)+2}{2} \|u_t\|^2 \\ &+ \varepsilon \left(\frac{p(1-a)}{2} - 1 \right) \|\Delta u\|^2 + \frac{\varepsilon (1-a) k}{p} \|u\|_p^p \\ &- \varepsilon \frac{H^{\alpha}(t)}{4 c_1 \kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \|u\|^2 + \varepsilon p (1-a) H(t) \\ &+ \frac{\varepsilon p (1-a)}{2} \int_{\Omega}\int_{0}^{1}\int_{\tau_1}^{\tau_2} s |\mu_2(s)| \left| z^2(x,\rho,s,t) \right| ds d\rho dx. \end{aligned} \tag{6.14}$$

By using (6.7) and (6.9), we get

$$H^{\alpha}(t)\|u\|_2^2 \leqslant \left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{\alpha}\|u\|_2^2 \leqslant \left[\left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{\alpha+2/p}+\left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{\alpha}\|\Delta u\|_2^{4/p}\right].$$

From Young's inequality, we have

$$H^{\alpha}(t)\|u\|_2^2 \leqslant \left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{\alpha}\|u\|_2^2 \leqslant \left[\begin{array}{c}\left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{(p\alpha+2)/p}\\ +\frac{2}{p}\|\Delta u\|^2+\frac{p-2}{p}\left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{\alpha p/(p-2)}\end{array}\right].$$

Hence, we get

$$H^{\alpha}(t)\|u\|_2^2 \leqslant \left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{\alpha}\|u\|_2^2 \leqslant C\left[\begin{array}{c}\left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{(p\alpha+2)/p}+\|\Delta u\|^2\\ +\left(\int_{\Omega}|u|^p\ln|u|^k\,dx\right)^{\alpha p/(p-2)}\end{array}\right],$$

where $C=\max\left\{\frac{2}{p},\frac{p-2}{p}\right\}$. By exploiting (6.11), we obtain

$$2<\alpha p+2\leqslant p \quad\text{and}\quad 2<\frac{\alpha p^2}{p-2}\leqslant p.$$

Thus, Lemma 6.2 yields

$$H^{\alpha}(t)\|u\|_2^2 \leqslant c\left(\int_{\Omega}|u|^p\ln|u|^k\,dx+\|\Delta u\|_2^2\right). \tag{6.15}$$

By combining (6.14) and (6.15), we get

$$\begin{aligned} L'(t) &\geqslant [(1-\alpha)-\varepsilon\kappa]H^{-\alpha}(t)H'(t)\\ &+\varepsilon\left(a-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right)\int_{\Omega}|u|^p\ln|u|^k\,dx\\ &+\varepsilon\left(\frac{p(1-a)-2}{2}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right)\|\Delta u\|^2\\ &+\frac{\varepsilon(1-a)k}{p}\|u\|_p^p+\varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2+\varepsilon p(1-a)H(t)\\ &+\frac{\varepsilon p(1-a)}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned} \tag{6.16}$$

Since, choosing $a>0$ so small, such that

$$\frac{p(1-a)-2}{2}>0,$$

and choosing κ large enough, we get

$$\begin{cases}\frac{p(1-a)-2}{2}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0,\\ a-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0.\end{cases}$$

Once κ and a are fixed, picking ε so small, such that

$$(1-\alpha)-\varepsilon\kappa>0,\quad H(0)+\varepsilon\int_{\Omega}u_0u_1\,dx>0.$$

Thus, for some $\lambda > 0$, estimate (6.16) takes the form

$$\begin{aligned} L'(t) \geqslant \lambda \Big[& H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \|u\|_p^p + \int_\Omega |u|^p \ln|u|^k dx \\ & + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t)\, ds\, d\rho\, dx \Big], \end{aligned} \tag{6.17}$$

and

$$L(t) \geqslant L(0) > 0, \ t \geqslant 0.$$

From the embedding $\|u\|_2 \leqslant c\|u\|_p$ and Hölder's inequality, we get

$$\int_\Omega u u_t dx \leqslant \|u\|_2 \|u_t\|_2 \leqslant c\|u\|_p \|u_t\|_2,$$

then from the Young's inequality, we have

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leqslant c\left(\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right), \text{ for } 1/\mu + 1/\theta = 1. \tag{6.18}$$

From Lemma 6.4, we take $\theta = 2(1-\alpha)$ which gives $\mu/(1-\alpha) = 2/(1-2\alpha) \leqslant p$. Thus, for $s = 2/(1-2\alpha)$, estimate (6.18) yields

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leqslant c\left(\|u\|_p^s + \|u_t\|_2^2\right).$$

Therefore, Lemma 6.4 satisfies

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leqslant c\left[\|\Delta u\|^2 + \|u_t\|^2 + \|u\|_p^p\right].$$

Hence,

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx + \frac{\mu_1 \varepsilon}{2}\int_\Omega u^2 dx\right)^{1/(1-\alpha)} \\ &\leqslant c\left[H(t) + \left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)}\right] \\ &\leqslant c\left[H(t) + \left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} + \|u\|_p^{2/(1-\alpha)}\right] \\ &\leqslant c\left[H(t) + \|\Delta u\|^2 + \|u_t\|^2 + \|u\|_p^p\right], \ t \geqslant 0. \end{aligned} \tag{6.19}$$

By combining (6.17) and (6.19), we get

$$L'(t) \geqslant \Lambda L^{1/(1-\alpha)}(t), \ t \geqslant 0, \tag{6.20}$$

where Λ is a positive constant. A simple integration of (6.20) over $(0,t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geqslant \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Lambda\alpha t/(1-\alpha)}.$$

Thus, $L(t)$ blows up in time T^*,

$$T \leqslant T^* = \frac{1-\alpha}{\Lambda\alpha L^{\alpha/(1-\alpha)}(0)}.$$

As a result, the proof is completed. □

7. Conclusions

Recently, there has been published much work concerning the wave equations (Kirchhoff, Petrovsky, Bessel,...etc.) with different state of delay time (constant delay, time-varying delay,...etc.). However, to the best of our knowledge, there were no existence, exponential decay and blow up of solutions for the logarithmic Petrovsky equation with distributed delay. Under suitable conditions, we have been proved the local existence, global existence, exponential decay and blow up results of the logarithmic Petrovsky equation with distributed delay.

References

[1] R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, Elsevier/Academic Press, Amsterdam, (2003). 2

[2] M. M. Al-Gharabli, S. A. Messaoudi, *Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term*, J. Evol. Equ., **18** (2018), 105–125. 1

[3] K. Bartkowski, P. Górka, *One-dimensional Klein-Gordon equation with logarithmic nonlinearities*, J. Phys. A, **41** (2008), 11 pages. 1, 1

[4] I. Białynicki-Birula, J. Mycielski, *Wave equations with logarithmic nonlinearities*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **23** (1975), 461–466. 1

[5] I. Białynicki-Birula, J. Mycielski, *Nonlinear wave mechanics*, Ann. Phys., **100** (1976), 62–93. 1

[6] T. Cazenave, A. Haraux, *Équations d'évolution avec non linéarité logarithmique*, Ann. Fac. Sci. Toulouse Math. (5), **2** (1980), 21–51. 1

[7] W. Chen, Y. Zhou, *Global nonexistence for a semilinear Petrovsky equation*, Nonlinear Anal., **70** (2009), 3203–3208. 1

[8] R. Datko, J. Lagnese, M. P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim., **24** (1986), 152–156. 1

[9] P. Górka, *Logarithmic Klein-Gordon equation*, Acta Phys. Polon. B, **40** (2009), 59–66. 1, 1

[10] M. Kafini, *On the decay of a nonlinear wave equation with delay*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **67** (2021), 309–325. 1, 1, 5.1

[11] M. Kafini, S. A. Messaoudi, *A blow-up result in a nonlinear wave equation with delay*, Mediterr. J. Math., **13** (2016), 237–247. 1

[12] M. Kafini, S. A. Messaoudi, *Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay*, Appl. Anal., **99** (2020), 530–547. 1, 1, 6

[13] V. Komornik, *Exact controllability and stabilization*, Masson, Paris; John Wiley & Sons, Ltd., Chichester, (1994). 3, 3

[14] K. Liu, *Locally distributed control and damping for the conservative systems*, SIAM J. Control Optim., **35** (1997), 1574–1590.

[15] G. Liu, *The existence, general decay and blow-up for a plate equation with nonlinear damping and a logarithmic source term*, Electron. Res. Arch., **28** (2020), 263–289. 1

[16] S. A. Messaoudi, *Blow up in a nonlinearly damped wave equation*, Math. Nachr., **231** (2001), 105–111. 1

[17] S. Nicaise, C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim., **45** (2006), 1561–1585. 1, 2

[18] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, (1983). 3, 3

[19] E. Pişkin, B. kutmuştur, *An introduction to Sobolev spaces*, Bentham Science Publishers, Ltd., Sharjah, (2021). 2

[20] E. Pişkin, N. Polat, *On the decay of solutions for a nonlinear Petrovsky equation*, Math. Sci. Lett., **3** (2013), 43–47. 1

[21] H. Yüksekkaya, E. Pişkin, S. M. Boulaaras, B. B. Cherif, S. Ahmed Zubair, *Existence, Nonexistence, and Stability of Solutions for a Delayed Plate Equation with the Logarithmic Source*, Adv. Math. Phys., **2021** (2021), 11 pages. 1

[22] E. Zuazua, *Exponential decay for the semi-linear wave equation with locally disributed damping*, Commun. Partial Differ. Equ., **15** (1990), 205–235.