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GENERALIZED CONTRACTIONS AND COMMON FIXED POINT THEOREMS CONCERNING τ-DISTANCE

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Abstract. In this paper we consider the generalized distance, present a generalization of Ćirić's generalized contraction fixed point theorems on a complete metric space and investigate a common fixed point theorem about a sequence of mappings concerning generalized distance.

1. Introduction and preliminary

In order to generalization of Banach's contraction principle, Ćirić introduced generalized contraction([16]). In 2001 Suzuki introduced the concept of τ-distance, a generalization of both w-distance ([3]) and Tataru's distance([13]), on a metric space, and discussed it's properties and improved the generalization of Banach's contraction principle , Caristi's fixed point theorem, Downing-Kirk's theorem, Ekeland's variational principal, Hamilton-Jacobi equation, the nonconvex minimization theorem according Takahashi and several fixed point theorems for contractive mapping with respect to w-distanc, See ([7],[8],[9],[10],[11], [6],[12],[13]). In this paper using the λ-generalized contraction and τ-distance we prove some fixed point theorems. Also, we investigate a sequence of maps which satisfy a common condition of generalized contraction type.

At first we recall some definitions and lemmas which will be used later.

Definition 1.1. ([8])*Let X be a metric space with metric d. A function $p : X \times X \to [0, \infty)$ is called τ-distance on X if there exist a function $\eta : X \times [0, \infty) \to [0, \infty)$ such that the following are satisfied:*

(τ_1) $p(x, z) \le p(x, y) + p(y, z)$ *for all* $x, y, z \in X$;

(τ_2) $\eta(x, 0) = 0$ *and* $\eta(x, t) \ge t$ *for all* $x \in X$ *and* $t \in [0, \infty)$, *and* η *is*

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concave and continuous in it's second variable;

(τ_3) $\lim_n x_n = x$ *and* $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$, *imply* $p(w, x) \leq \liminf_n p(w, x_n)$ *for all* $w \in X$;

(τ_4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ *and* $\lim_n \eta(x_n, t_n) = 0$, *imply* $\lim_n \eta(y_n, t_n) = 0$;

(τ_5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ *and* $\lim_n \eta(z_n, p(z_n, y_n)) = 0$, *imply* $\lim_n d(x_n, y_n) = 0$.

It can be replaced (τ_2) *by the following* $(\tau_2)'$.

$(\tau_2)'$ $\inf\{\eta(x, t) : t > 0\} = 0$ *for all* $x \in X$, *and* η *is nondecreasing in its second variable. The best well-known* τ*-distances are the metric function* d *and* w*-distances. If* p *be a* w*-distance on the metric space* (X, d) *and a function* η *from* : $X \times [0, \infty)$ *into* $[0, \infty)$ *given by* $\eta(x, t) = t$, *for all* $x \in X$, *then it is easy to check that* p *is a* τ*-distance.*

Let (X, d) *be a metric space and* p *be a* τ*-distance on* X. *A sequence* $\{x_n\}$ *in* X *is called* p*-Cauchy if there exists a function* $\eta : X \times [0, \infty) \to [0, \infty)$ *satisfying* (τ_2)-(τ_5) *and a sequence* z_n *in* X *such that* $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$.

The following lemmas are essential for next sections.

Lemma 1.2. ([7]) *Let* (X, d) *be a metric space and* p *be a* τ*-distance on* X. *If* $\{x_n\}$ *is a* p*-Cauchy sequence, then it is a Cauchy sequence. Moreover if* $\{y_n\}$ *is a sequence satisfying* $\lim_n \sup\{p(x_n, y_m) : m > n\} = 0$, *then* $\{y_n\}$ *is also* p*-Cauchy sequence and* $\lim_n d(x_n, y_n) = 0$.

Lemma 1.3. ([7]) *Let* (X, d) *be a metric space and* p *be a* τ*-distance on* X. *If* $\{x_n\}$ *in* X *satisfies* $\lim_n p(z, x_n) = 0$ *for some* $z \in X$, *then* x_n *is a* p*-Cauchy sequence. Moreover if* $\{y_n\}$ *in* X *also satisfies* $\lim_n p(z, y_n) = 0$, *then* $\lim_n d(x_n, y_n) = 0$. *In particular, for* $x, y, z \in X$, $p(z, x) = 0$ *and* $p(z, y) = 0$ *imply* $x = y$.

Lemma 1.4. ([7]) *Let* (X, d) *be a metric space and* p *be a* τ*-distance on* X. *If a sequence* $\{x_n\}$ *in* X *satisfies* $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, *then* $\{x_n\}$ *is a* p*-Cauchy sequence. Moreover, if* $\{y_n\}$ *in* X *satisfies* $\lim_n p(x_n, y_n) = 0$, *then* $\{y_n\}$ *is also* p*-Cauchy sequence and* $\lim_n d(x_n, y_n) = 0$.

Remark 1.5. *If* $p(x, y) = 0$ *then the equality* $x = y$ *is not necessarily hold, but* $p(x, y) = p(y, x) = 0$ *imply* $x = y$ *because* $0 \leq p(x, x) \leq p(x, y) + p(y, x) = 0$ *and hence* $p(x, x) = 0$. *Now by Lemma* 1.3 $x = y$.

2. Generalized Contractions

Throughout this paper we denote by N the set of all positive integer, R real numbers with usual metric and (X, d) be a complete metric space.

Definition 2.1. *Let* f *and* g *be selfmappings on a complete metric space* X, p *be a* τ*-distance on* X *and* $g(X) \subseteq f(X)$. *We say* g *is* λ*-generalized contraction (shortly* λ*-GC) with respect to* (p, f), $\lambda \in (0, 1)$, *if and only if there exist nonnegative functions* q, r, s, t, *satisfying*

$$sup_{x,y \in X}\{q(x, y) + r(x, y) + s(x, y) + 4t(x, y)\} \leq \lambda < 1 \tag{2.1}$$

such that for each $x, y \in X$;

$$\max\{p(f(x), g(y)), p(g(y), f(x))\} \leqq q(x,y)p(x,y) + r(x,y)p(x, f(x)) \\ +s(x,y)p(y, g(y)) + t(x,y)[p(x, g(y)) + p(y, f(x))]. \tag{2.2}$$

Example 2.2. *a)Let (X, d) be a complete metric space and $p(x,y) = d(x,y)$, then every contraction selfmapping f on X is λ-GC with respect to (p, f).*
b) Let $X = [0,2] \subseteq R$ and

$$f(x) = g(x) = \begin{cases} \frac{x}{9}, & x \in [0,1] \\ \frac{x}{10}, & x \in (1,2]. \end{cases}$$

$q(x,y) = \frac{1}{10}$, $r(x,y) = s(x,y) = \frac{1}{4}$, $t(x,y) = \frac{1}{11}$ *and* $p(x,y) = |x-y|$. *Then g is λ-GC with respect to (p, f), but it is not a contraction mapping.*

We prove the following lemma which will be used in the next theorem.

Lemma 2.3. *Let $x_0 \in X$. Define the sequence $\{x_n\}$ by*

$$x_{2n+1} = f(x_{2n}), \quad x_{2n+2} = g(x_{2n+1}), \tag{2.3}$$

where f and g are selfmappings on X such that g is λ-GC with respect to (p, f). Then $\{x_n\}$ is a Cauchy sequence.

Proof. Put

$$M_1 = \max\{p(x_{2n+1}, x_{2n+2}), p(x_{2n+2}, x_{2n+1})\}$$

and

$$M_2 = \max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n})\},$$

by (2.1), (2.2) and (2.3) we have,

$$\begin{aligned} M_1 &= \max\{p(f(x_{2n}), g(x_{2n+1})), p(g(x_{2n+1}), f(x_{2n}))\} \\ &\leq \lambda \max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, f(x_{2n})), \\ &\quad p(x_{2n+1}, g(x_{2n+1})), \frac{1}{4}[p(x_{2n}, g(x_{2n+1})) + p(x_{2n+1}, f(x_{2n})]\} \\ &= \lambda \max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), \\ &\quad p(x_{2n+1}, x_{2n+2}), \frac{1}{4}[p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})]\} \\ &= \lambda M(x_{2n}, x_{2n+1}) \end{aligned}$$

where

$$M(x_{2n}, x_{2n+1}) = \max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \frac{1}{4}[p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})]\}.$$

Now if $M(x_{2n}, x_{2n+1}) = p(x_{2n+1}, x_{2n+2})$, then we have,

$$p(x_{2n+1}, x_{2n+2}) \leq \lambda p(x_{2n+1}, x_{2n+2}),$$

which implies $p(x_{2n+1}, x_{2n+2}) = 0$.
If $M(x_{2n}, x_{2n+1}) = \frac{1}{4}[p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})]$ then,

$$p(x_{2n+1}, x_{2n+2}) \leq \frac{\lambda}{4}[p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})],$$

so

$$p(x_{2n+1}, x_{2n+2}) \leq \frac{\lambda}{2} p(x_{2n}, x_{2n+2}) \quad or \quad p(x_{2n+1}, x_{2n+2}) \leq \frac{\lambda}{2} p(x_{2n+1}, x_{2n+1}).$$

If $p(x_{2n+1}, x_{2n+2}) \leq \frac{\lambda}{2} p(x_{2n}, x_{2n+2})$ since,

$$\begin{aligned} \frac{\lambda}{2} p(x_{2n}, x_{2n+2}) &\leq \frac{\lambda}{2}[p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})] \\ &\leq \frac{\lambda}{2} p(x_{2n}, x_{2n+1}) + \frac{1}{2} p(x_{2n+1}, x_{2n+2}) \end{aligned}$$

we have

$$p(x_{2n+1}, x_{2n+2}) \leq \lambda p(x_{2n}, x_{2n+1}).$$

If $p(x_{2n+1}, x_{2n+2}) \leq \frac{\lambda}{2} p(x_{2n+1}, x_{2n+1})$ since,

$$\frac{\lambda}{2} p(x_{2n+1}, x_{2n+1}) \leq \frac{\lambda}{2}[p(x_{2n+1}, x_{2n}) + p(x_{2n}, x_{2n+1})]$$

we have

$$p(x_{2n+1}, x_{2n+2}) \leq \lambda p(x_{2n+1}, x_{2n}) \quad or \quad p(x_{2n+1}, x_{2n+2}) \leq \lambda p(x_{2n}, x_{2n+1}).$$

Therefore in any cases we have;

$$M_1 \leq \lambda p(x_{2n+1}, x_{2n}) \text{ or } \quad M_1 \leq \lambda p(x_{2n}, x_{2n+1}). \tag{2.4}$$

Similarly

$$M_2 \leq \lambda p(x_{2n-1}, x_{2n}) \text{ or } \quad M_2 \leq \lambda p(x_{2n}, x_{2n-1}). \tag{2.5}$$

Continuing this process we have,

$$p(x_n, x_{n+1}) \leq \lambda \max\{p(x_{n-1}, x_n), p(x_n, x_{n-1})\} \leq ... \leq \lambda^n \max\{p(x_0, x_1), p(x_1, x_0)\}$$

Putting $r(x_0) = \max\{p(x_0, x_1), p(x_1, x_0)\}$, then for any $m > n$;

$$p(x_m, x_n) \leq \sum_{k=0}^{m-n-1} p(x_{n+k+1}, x_{n+k}) \leq \sum_{k=0}^{m-n-1} \lambda^{(n+k)} r(x_0) \leq \lambda^n r(x_0)(1-\lambda)^{-1}.$$

So $\limsup_n \{p(x_m, x_n) : m \geq n\} = 0$. Hence by Lemmas 1.2 and 1.4 $\{x_n\}$ is a Cauchy sequence. □

Theorem 2.4. *Let (X, d) be a metric space, p be a τ-distance on X and $x_0 \in X$ and f and g be selfmappings on X such that g is λ-GC with respect to (p, f). Moreover assume that the following holds:*
If $\limsup_n \{p(x_n, x_m) : m > n\} = 0$ and $\lim_n p(x_n, y) = 0$ then, $\lim_n p(x_n, f(x_n)) = 0$ implies $f(y) = y$ and $\lim_n p(x_n, g(x_n)) = 0$ implies $g(y) = y$. Then f and g have a unique common fixed point, namely z, such that $p(z, z) = 0$ and $(fg)^n(x_0) \to z$ and $(gf)^n(x_0) \to z$.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_{2n+1} = f(x_{2n})$ and $x_{2n+2} = g(x_{2n+1})$. Then by Lemma 2.3 $\{x_n\}$ is a Cauchy sequence and converges to some point $z \in X$. We show that $f(z) = z$, and $g(z) = z$.
By (τ_3) we have;

$$\begin{aligned} \limsup_n (p(x_{2n}, f(x_{2n})) + p(x_{2n}, z)) &\leq \limsup_n (p(x_{2n}, x_{2n+1}) + \liminf_{m\to\infty} p(x_{2n}, x_m) \\ &\leq 2 \limsup_{m\geq 2n} p(x_{2n}, x_m) = 0. \end{aligned}$$

Similarly $\limsup_n (p(x_{2n+1}, g(x_{2n+1})) + p(x_{2n+1}, z)) = 0$. Therefore

$$\limsup_n \{p(x_n, x_m) : m > n\} = 0 \quad \text{and} \quad lim_n(x_n, z) = 0.$$

So we have,

$$\lim_n (p(x_{2n}, f(x_{2n})) = 0$$

and

$$\lim_n p(x_{2n}, z) = 0.$$

Putting $x'_n = x_{2n}$, the hypothesis implies $f(z) = z$. With a similar computations we have $g(z) = z$.
Now if we put $x = y = z$ in (2.2) we get $p(z, z) \leq \lambda p(z, z)$ which implies $p(z, z) = 0$.

If u be another common fixed point for f and g by using (2.2) we have

$$\begin{aligned} \max\{p(z, u), p(u, z)\} &\leqq q(z, u)p(z, u) + r(z, u)p(z, z) + s(z, u)p(u, u) \\ &+ t(z, u)[p(z, u) + p(u, z)] \\ &\leq \lambda. \max\{p(z, u), p(z, z), p(u, u), \frac{1}{4}[p(z, u) + p(u, z)]\} \\ &= \lambda. \max\{p(z, u), \frac{1}{4}[p(z, u) + p(u, z)\}. \end{aligned}$$

The last equality holds because $p(z, z) = p(u, u) = 0$. In any cases this inequalities show that $p(z, u) = p(u, z) = 0$ and by Remark 1.5 $z = u$.□

Note that if f is continuous then, $\{x_n\}$ and $\{f(x_n)\}$ converge to y, implies $f(y) = y$. If $\limsup_n \{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, y) = 0$, and $\lim_n p(x_n, f(x_n)) = 0$, then by Lemma 1.4 we have $\lim_n x_n = \lim_n f(x_n) = y$, but in general it doesn't imply $f(y) = y$. For example, let $X = R$, (real numbers with usual metric), $x_n = \frac{n-1}{n}$, $p = d$, $y = 1$ and $f : R \to R$ defined by

$$f(t) = \begin{cases} t, & t \neq 1, \\ 2, & t = 1. \end{cases}$$

It is possible that g^k be λ-GC with respect to (p, f), for some $k \in N$ and $k > 1$, but g is not so.

Example 2.5. *Let $X = \{a, b, c\}$ where $a, b, c \in R$ are three distinct real numbers; $f(x) = a$, constant map on X, and $g : X \to X$ is given by $g(a) = a$, $g(b) = c$, $g(c) = a$. Put $p = d$. We have $g^2 = f$, and so g^2 is λ-GC with respect to (p, f), but since $g(X) \nsubseteq f(X)$ so g is not λ-GC with respect to (p, f).*

Corollary 2.6. *Let (X,d) be a metric space, p be a τ-distance on X and $x_0 \in X$ and f and g be selfmappings on X such that g^k is λ-GC with respect to (p,f), for some $k \in N$. Moreover assume that the following holds:*
If $\limsup_n\{p(x_n,x_m) : m > n\} = 0$ and $\lim_n p(x_n,y) = 0$ then, $\lim_n p(x_n,f(x_n)) = 0$ implies $f(y) = y$ and $\lim_n p(x_n,g^k(x_n)) = 0$ implies $g^k(y) = y$. Then f and g have a unique common fixed point.

Proof. *By Theorem 2.4 f and g^k have common fixed point, z. Now we have $g^k(g(z)) = g(g^k(z)) = g(z)$. It follows that $g(z) = z = f(z)$, by uniqueness.*$\Box$

Corollary 2.7. *Let (X,d) be a metric space, p be a τ-distance on X, $x_0 \in X$ and f and g be selfmappings on X such that g is λ-GC with respect to (p,f). Moreover assume that if $\{x_n\}, \{f(x_n)\}$ and $\{g(x_n)\}$ converges to y, it implies $f(y) = y$ and $g(y) = y$. Then f and g have a unique common fixed point, namely z, such that $p(z,z) = 0$ and $(fg)^n(x_0) \to z$ and $(gf)^n(x_0) \to z$.*

Corollary 2.8. *Let (X,d) be a metric space, p be a τ-distance on X and $x_0 \in X$. Suppose f and g are continuous selfmappings on X,and g is λ-GC with respect to (p,f). Then f and g have a unique common fixed point, namely z, such that $p(z,z) = 0$ and $(fg)^n(x_0) \to z$ and $(gf)^n(x_0) \to z$.*

3. Sequence of Generalized Contraction Maps

Throughout this section we prove a common fixed point theorem for a sequence of maps which satisfy a common condition of generalized contraction type. We begin with a lemma.

Lemma 3.1. *Let (X,d) be a metric space, p be a τ-distance on X. Let f and f_0 be selfmappings on X such that the following holds:*

$$\begin{aligned}\max\{p(f_0(x),f(y)),p(f(y),f_0(x))\} \le \lambda \max\{&p(x,y),p(x,f_0(x)),\\ &p(y,f(y)),p(x,f(y)),p(y,f_0(x))\}\end{aligned} \tag{3.1}$$

for some $\lambda \in (0,1)$ and all $x,y \in X$. If $f_0(z) = z$ and $p(z,z) = 0$, for some $z \in X$, then $f(z) = z$ and z is unique.

Proof. Since $f_0(z) = z$, by (3.1) we have

$$\begin{aligned}\max\{p(z,f(z)),p(f(z),z)\} &= \max\{p(f_0(z),f(z)),p(f(z),f_0(z))\}\\ &\leqq \lambda\max\{p(z,z),p(z,f(z))\} = \lambda p(z,f(z))\end{aligned}$$

which implies $p(z,f(z)) = 0$ and hence by Lemma 1.3 $z = f(z)$.
If $v \in X$ be such that $f_0(v) = v$ and $p(v,v) = 0$ then we have $f(v) = v$ and

$$\begin{aligned}p(z,v) = p(f_0(z),f(v)) &\le \lambda\max\{p(z,v),p(z,z),p(v,v),p(v,z)\}\\ &= \lambda\max\{p(z,v),p(v,z)\}.\end{aligned}$$

With similar computation

$$p(v,z) \le \lambda\max\{p(z,v),p(v,z)\}$$

so $p(z,v) = p(v,z) = 0$ and by Remark (1.5) $v = z$. $\Box$

Theorem 3.2. *Let (X, d) be a complete metric space, p be a τ-distance on X and $\{f_n\}$ be a sequence of selfmappings on X, such that f_0 is continuous and for each $x, y \in X$;*

$$\max\{p(f_0(x), f_n(y)), p(f_n(y), f_0(x))\} \leqq \lambda. \max\{p(x, y), p(x, f_0(x)), \tag{3.2}$$
$$p(y, f_n(y)), \tfrac{1}{4}[p(x, f_n(y)) + p(y, f_0(x))\},$$

whereas $\lambda \in (0, 1)$ and $n = 0, 1, 2, 3, \ldots$. Then $\{f_n\}$ have a unique common fixed point, namely z, such that $p(z, z) = 0$.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by

$$x_1 = f_0(x_0), \;\; x_2 = f_0(x_1) = f_0^2(x_0), \ldots, \;\; x_n = f_0^n(x_0), \ldots \tag{3.3}$$

We show that $\{x_n\}$ is a Cauchy sequence. By (3.2) we have

$$\max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n)\} = \max\{p(f_0(x_{n-1}), f_0(x_{n-2})), p(f_0(x_{n-2}), f_0(x_{n-1}))\}$$
$$\leq \lambda \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n), \tfrac{1}{4}p(x_{n-2}, x_n) + p(x_{n-1}, x_{n-1})\}.$$

We will prove that

$$\max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n)\} \leq \lambda \max\{p(x_{n-1}, x_{n-2}), p(x_{n-2}, x_{n-1})\}. \tag{3.4}$$

To show this set $M = \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n), \frac{1}{4}p(x_{n-2}, x_n) + p(x_{n-1}, x_{n-1})\}$.
If $M = p(x_{n-1}, x_n)$ then $p(x_{n-1}, x_n) = 0$ and (3.4) holds.
If $M = p(x_{n-2}, x_{n-1})$ then $\max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n)\} \leq \lambda p(x_{n-2}, x_{n-1})$ and (3.4) holds
If $M = \frac{1}{4}p(x_{n-2}, x_n) + p(x_{n-1}, x_{n-1})\}$ then
$4 \max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n)\} \leq \lambda p(x_{n-2}, x_n) + p(x_{n-1}, x_{n-1})$ hence

$$\begin{aligned} 2\max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n)\} &\leq \lambda p(x_{n-2}, x_n) \\ &\leq \lambda p(x_{n-2}, x_{n-1}) + p(x_{n-1}, x_n) \end{aligned}$$

or

$$\begin{aligned} 2\max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n)\} &\leq \lambda p(x_{n-1}, x_{n-1}) \\ &\leq \lambda p(x_{n-1}, x_{n-2}) + p(x_{n-2}, x_{n-1}) \end{aligned}$$

which implies

$$\max\{p(x_n, x_{n-1}), p(x_{n-1}, x_n)\} \leq \lambda \max\{p(x_{n-1}, x_{n-2}), p(x_{n-2}, x_{n-1})\},$$

so in any cases (3.4) holds.
Continuing this process one has,

$$p(x_{n-1}, x_n) \leq \lambda \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_{n-2})\} \leq \ldots \leq \lambda^n \max\{p(x_0, x_1), p(x_1, x_0)\}$$

Putting $r(x_0) = \max\{p(x_0, x_1), p(x_1, x_0)\}$, for any $m > n$;

$$p(x_n, x_m) \leq \sum_{k=0}^{m-n-1} p(x_{n+k}, x_{n+k+1}) \leq \sum_{k=0}^{m-n-1} \lambda^{(n+k)} r(x_0) \leq \lambda^n r(x_0)(1 - \lambda)^{-1}.$$

So $\limsup_n\{p(x_n, x_m) : m > n\} = 0$. Then by Lemma 1.4 $\{x_n\}$ is a Cauchy sequence, since X is complete metric space there exist some point $z \in X$ such that $\lim_n x_n = z$. On the other hand continuity of f_0 implies

$$f_0(z) = f_0(\lim_n x_n) = \lim_n(f_0(x_n)) = \lim_n(x_{n+1}) = z$$

therefore $f_0(z) = z$. By (3.2) we have

$$p(z, z) = p(f_0(z), z) = p(z, f_0(z)) = p(f_0(z), f_0(z)) \leqq \lambda p(z, z),$$

so $p(z, z) = 0$. Then by Lemma 3.1 z is a unique fixed point of f_0 and $f_n(z) = z$ for all $n = 1, 2, 3, ...$.□

Note that if the condition of continuity of f_0 is replaced by the lower semicontinity of p in its first variable, the theorem will be holds too. Because if p be lower semicontinuous in its first variable by (3.2) and triangle inequality we have

$$\begin{aligned} p(z, f_0(z)) &\leq p(z, x_n) + p(f_0(x_{n-1}), f_0(z)) \\ &\leq p(z, x_n) + \lambda \max\{p(z, x_{n-1}), p(z, f_0(z)), p(x_{n-1}, f_n(x_{n-1})), \\ &\quad \frac{1}{4}[p(z, f_n(x_{n-1})) + p(x_{n-1}, f_0(z))\} \\ &\leq p(z, x_n) + \lambda. \max\{p(z, x_{n-1}), p(z, f_0(z)), p(x_{n-1}, x_n), \\ &\quad \frac{1}{4}[p(z, x_n) + p(x_{n-1}, f_0(z))] \\ &\leq p(z, x_n) + \lambda.[p(z, x_{n-1}) + p(z, f_0(z)) + p(x_{n-1}, x_n) + p(x_n, z)] \end{aligned}$$

hence

$$p(z, f_0(z)) \leq \frac{1}{1-\lambda}[p(z, x_n) + \lambda[p(z, x_{n-1}) + p(x_{n-1}, x_n) + p(x_n, z)]].$$

By (τ_3)

$$(p(x_n, z)) \leq \liminf_m (p(x_n, x_m) \leq \lambda^n r(x_0)(1-\lambda)^{-1} \tag{3.5}$$

so $\lim_n(p(x_n, z)) = 0$, moreover by construction $\lim_n(p(x_{n-1}, x_n)) = 0$. Since p is lower semicontinuous in its first variable we have

$$\lim_n p(z, x_n) = \lim_n p(z, x_{n-1}) = 0,$$

therefore $p(z, f_0(z)) = 0$. On the other hand

$$\begin{aligned} p(f_0(z), z) &\leq p(x_n, z) + p(f_0(z), f_0(x_{n-1})) \\ &\leq p(x_n, z) + \lambda \max\{p(z, x_{n-1}), p(x_{n-1}, f_0(x_{n-1})), p(z, f_0(z)), \\ &\quad \frac{1}{4}[p(x_{n-1}, f_0(z)) + p(z, f_0(x_{n-1}))]\} \\ &\leq p(x_n, z) + \lambda \max\{p(z, x_{n-1}), p(x_{n-1}, x_n), p(z, f_0(z)), \\ &\quad \frac{1}{4}[p(x_{n-1}, f_0(z)) + p(z, x_n)]\} \\ &\leq p(x_n, z) + \lambda[p(z, x_{n-1}) + p(x_{n-1}, x_n) + p(x_n, z) + p(z, f_0(z))]. \end{aligned}$$

Hence $p(f_0(z), z) = 0$ and so $f_0(z) = z$ and we have the following theorem:

Theorem 3.3. *Let (X, d) be a complete metric space, p be a τ-distance on X such that p is lower semicontinuous in its first variable and $\{f_n\}$ be a sequence of selfmappings on X, satisfying*

$$\max\{p(f_0(x), f_n(y)), p(f_n(y), f_0(x))\} \leqq \lambda. \max\{p(x, y), p(x, f_0(x)), \tag{3.6}$$

$$p(y, f_n(y)), \frac{1}{4}[p(x, f_n(y)) + p(y, f_0(x))\}.$$

for each $x, y \in X$, $\lambda \in (0,1)$ and $n = 0, 1, 2, 3, \ldots$. Then $\{f_n\}$ have a unique common fixed point, namely z, such that $p(z, z) = 0$.

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