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APPROXIMATION OF MIXED TYPE FUNCTIONAL EQUATIONS IN p–BANACH SPACES

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of the functional equation

$$\sum_{i=1}^{n} f(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf(\frac{1}{n}\sum_{i=1}^{n} x_i) \qquad (n \geq 2),$$

in p–Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [31] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D.H. Hyers [15] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : X \longrightarrow Y$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

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for all $x, y \in X$, and for some $\delta > 0$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in X$. Aoki [3] and Rassias [25] provided a generalization of the Hyers theorem for additive and linear functions, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. (*Th.M. Rassias*). *Let $f : X \to Y$ be a function from a normed vector space X into a Banach space Y subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \varepsilon\|x\|^p/(1 - 2^{p-1}) \tag{1.2}$$

for all $x \in X$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each fixed $x \in X$ the function $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then A is linear.

The above Theorem has provided a lot of influence during the last three decades in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability of functional equations (see [6, 16]). Furthermore, a generalization of Rassias theorem was obtained by Găvruta, who replaced $\varepsilon(\| x \|^p + \| y \|^p)$ by a general control function $\varphi(x, y)$ [13]. The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.3}$$

is related to a symmetric bi-additive function [1, 22]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B_1 such that $f(x) = B_1(x, x)$ for all x. The bi-additive function B_1 is given by

$$B_1(x, y) = \frac{1}{4}(f(x + y) - f(x - y))$$

In the paper [6], Czerwik proved the Hyers–Ulam–Rassias stability of the equation (1.3).

It was shown by Rassias [26] that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$

$$n\|\frac{1}{n}\sum_{i=1}^{n} x_i\|^2 + \sum_{i=1}^{n} \|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

for all $x_1, ..., x_n \in X$ (see also [2, 19]). During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and functions (see [5]–[14], [17, 18, 21, 22] and [26]–[29]). We also refer the readers to the books [1, 6, 16, 20, 27].

We consider some basic concepts concerning p–normed spaces.

Definition 1.2. (See [4, 30]). Let X be a real linear space. A function $\| \, . \, \| : X \to \mathbb{R}$ is a quasi-norm (valuation) if it satisfies the following conditions:

(QN_1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
(QN_2) $\|\lambda. \, x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;
(QN_3) There is a constant $M \geq 1$: $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.
Then $(X, \| \, . \, \|)$ is called a quasi-normed space. The smallest possible M is called the modulus of concavity of $\| \, . \, \|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\| \, . \, \|$ is called a p–norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p–Banach space.

By the Aoki-Rolewicz Theorem [30], each quasi-norm is equivalent to some p–norm (see also [4]). Since it is much easier to work with p–norms, henceforth we restrict our attention mainly to p–norms.

Employing the above identity, we introduce the following functional equation deriving from additive and quadratic functions:

$$\sum_{i=1}^{n} f(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf(\frac{1}{n}\sum_{i=1}^{n} x_i) \tag{1.4}$$

where $n \geq 2$ is a fixed integer. It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional equation (1.4). A. Najati and Th. M. Rassias [23] investigated the general solution of the functional equation (1.4).

This paper is organized as follows: In Section 2, we prove the generalized Hyers–Ulam stability of the functional equation (1.4) in p–Banach spaces, for odd functions. The generalized Hyers–Ulam stability of the functional equation (1.4) in p–Banach spaces, for even functions is discussed in Section 3. Finally, in Section 4, we show that the generalized Hyers–Ulam stability of a mixed additive and quadratic functional equation (1.4) in p–Banach spaces.

2. Stability of the functional equation (1.4) in p–Banach spaces: for odd functions

In the rest of this paper, we will assume that X be a p–normed space and Y be a p–Banach space. For convenience, we use the following abbreviation for a given function $f : X \to Y$,

$$D_f(x_1, ..., x_n) = \sum_{i=1}^{n} f(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j) - \sum_{i=1}^{n} f(x_i) + nf(\frac{1}{n}\sum_{i=1}^{n} x_i)$$

for all $x_1, ..., x_n \in X$, where $n \geq 2$ is a fixed integer. We now investigate the generalized Hyers-Ulam stability problem for functional equation (1.4).

Lemma 2.1. ([24]) *Let $0 < p \leq 1$ and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Then*

$$(\sum_{i=1}^{n} x_i)^p \leq \sum_{i=1}^{n} {x_i}^p.$$

Theorem 2.2. *Let $\ell \in \{-1, 1\}$ be fixed, X be a p–normed space, Y be a p–Banach space and $\varphi : X^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} 2^{m\ell}\varphi(\frac{x_1}{2^{m\ell}}, ..., \frac{x_n}{2^{m\ell}}) = 0 \tag{2.1}$$

for all $x_1, ..., x_n \in X$, and

$$\sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{\imath p\ell}\varphi^p(\frac{u_1}{2^{\imath\ell}}, ..., \frac{u_n}{2^{\imath\ell}}) < \infty \tag{2.2}$$

for all $u_1 \in \{-x, x, 2x\}$ and all $u_2, ..., u_n \in \{-x, 0, x\}$ (denoted $(\varphi(x_1, ..., x_n))^p$ by $\varphi^p(x_1, ..., x_n)$). Suppose that an odd function $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, ..., x_n)\| \leq \varphi(x_1, ..., x_n) \tag{2.3}$$

for all $x_1, ..., x_n \in X$. Furthermore, assume that $f(0) = 0$ in (2.3) for the case $\ell = 1$. Then the limit

$$A(x) := \lim_{m\to\infty} 2^{m\ell} f(\frac{x}{2^{m\ell}}) \tag{2.4}$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying

$$\|f(x) - A(x)\| \leq \frac{1}{2}(\widetilde{\psi}_o(x))^{\frac{1}{p}} \tag{2.5}$$

for all $x \in X$, where

$$\begin{aligned}\widetilde{\psi}_o(x) := \sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{\imath p\ell}\{\varphi^p(\frac{2x}{2^{\imath\ell}}, 0, ..., 0) + \frac{1}{2^p}[n^p\varphi^p(\frac{x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}, 0, ..., 0) \\ + \varphi^p(\frac{-x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}, ..., \frac{x}{2^{\imath\ell}}) + \varphi^p(\frac{x}{2^{\imath\ell}}, \frac{-x}{2^{\imath\ell}}, ..., \frac{-x}{2^{\imath\ell}})]\}.\end{aligned} \tag{2.6}$$

Proof. For $\ell = 1$, letting $x_1 = nx$, $x_2 = -ny$ and $x_i = 0$ $(i = 3, ..., n)$ in (2.3) and using the oddness of f, we get

$$\begin{aligned}&\|f((n-1)x + y) - f(x + (n-1)y) - f(nx) + f(ny) + 2f(x - y)\| \\ &\quad \leq \varphi(nx, -ny, 0, ..., 0)\end{aligned} \tag{2.7}$$

for all $x, y \in X$. Letting $y = 0$ in (2.7), we get

$$\|f(nx) - f((n-1)x) - f(x)\| \leq \varphi(nx, 0, ..., 0) \tag{2.8}$$

for all $x \in X$. Setting $x_1 = ny$, $x_2 = ... = x_n = nx$ in (2.3) and using the oddness of f, we get

$$\begin{aligned}&\|(n-1)f(x - y) - f((n-1)(x - y)) - (n-1)f(nx) + nf((n-1)x + y) \\ &\qquad - f(ny)\| \leq \varphi(ny, nx, ..., nx)\end{aligned} \tag{2.9}$$

for all $x, y \in X$. Interchange x with y in (2.9) and using the oddness of f, we get

$$\begin{aligned}\|f((n-1)(x-y)) - (n-1)f(x-y) - (n-1)f(ny) - f(nx)\\ + nf(x+(n-1)y)\| \leq \varphi(nx, ny, ..., ny)\end{aligned} \quad (2.10)$$

for all $x, y \in X$. Using (2.7), we get from (2.9) and (2.10) that

$$\begin{aligned}&\|f((n-1)(x-y)) + f(x-y) - f(nx) + f(ny)\|\\ &\leq \frac{1}{2}[n\varphi(nx, -ny, 0, ..., 0) + \varphi(ny, nx, ..., nx) + \varphi(nx, ny, ..., ny)]\end{aligned} \quad (2.11)$$

for all $x, y \in X$. It follows from (2.8) and (2.11) that

$$\begin{aligned}&\|f(n(x-y)) - f(nx) + f(ny)\| \leq \varphi(n(x-y), 0, ..., 0)\\ &+ \frac{1}{2}[n\varphi(nx, -ny, 0, ..., 0) + \varphi(ny, nx, ..., nx) + \varphi(nx, ny, ..., ny)]\end{aligned} \quad (2.12)$$

for all $x, y \in X$. Replacing x by $\frac{x}{n}$ and y by $\frac{-x}{n}$ in (2.12) and using the oddness of f, we get

$$\begin{aligned}\|f(2x) - 2f(x)\| \leq \frac{1}{2}[n\varphi(x, x, 0, ..., 0) + \varphi(-x, x, ..., x) + \varphi(x, -x, ..., -x)]\\ + \varphi(2x, 0, ..., 0)\end{aligned} \quad (2.13)$$

for all $x \in X$. Let

$$\begin{aligned}\psi_o(x) :=& \frac{1}{2}[n\varphi(x, x, 0, ..., 0) + \varphi(-x, x, ..., x) + \varphi(x, -x, ..., -x)]\\ &+ \varphi(2x, 0, ..., 0)\end{aligned} \quad (2.14)$$

for all $x \in X$. Thus (2.13) means that

$$\|f(2x) - 2f(x)\| \leq \psi_o(x) \quad (2.15)$$

for all $x \in X$. If we replace x in (2.15) by $\frac{x}{2^{m+1}}$ and multiply both sides of (2.15) by 2^m, we see that

$$\|2^{m+1}f(\frac{x}{2^{m+1}}) - 2^m f(\frac{x}{2^m})\| \leq 2^m \psi_o(\frac{x}{2^{m+1}}) \quad (2.16)$$

for all $x \in X$ and all non-negative integers m. Hence

$$\begin{aligned}\|2^{m+1}f(\frac{x}{2^{m+1}}) - 2^k f(\frac{x}{2^k})\|^p &\leq \sum_{\imath=k}^{m} \|2^{\imath+1}f(\frac{x}{2^{\imath+1}}) - 2^\imath f(\frac{x}{2^\imath})\|^p\\ &\leq \sum_{\imath=k}^{m} 2^{\imath p}\psi_o^p(\frac{x}{2^{\imath+1}})\end{aligned} \quad (2.17)$$

for all non-negative integers m and k with $m \geq k$ and all $x \in X$. Since $0 < p \leq 1$, so by Lemma 2.1 and (2.14), we get

$$\begin{aligned}\psi_o^p(x) \leq& \frac{1}{2^p}[n^p\varphi^p(x, x, 0, ..., 0) + \varphi^p(-x, x, ..., x) + \varphi^p(x, -x, ..., -x)]\\ &+ \varphi^p(2x, 0, ..., 0)\end{aligned} \quad (2.18)$$

for all $x \in X$. Therefore it follows from (2.1), (2.2) and (2.18) that

$$\sum_{\imath=1}^{\infty} 2^{\imath p}\psi_o^p(\frac{x}{2^\imath}) < \infty, \qquad \lim_{m\to\infty} 2^m \psi_o(\frac{x}{2^m}) = 0 \tag{2.19}$$

for all $x \in X$. It follows from (2.17) and (2.19) that the sequence $\{2^m f(\frac{x}{2^m})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^m f(\frac{x}{2^m})\}$ converges for all $x \in X$. Therefore, one can define a function $A : X \to Y$ by

$$A(x) := \lim_{m\to\infty} 2^m f(\frac{x}{2^m}) \tag{2.20}$$

for all $x \in X$. Letting $k = 0$ and passing the limit $m \to \infty$ in (2.17), we get

$$\|f(x) - A(x)\|^p \le \sum_{\imath=0}^{\infty} 2^{\imath p}\psi_o^p(\frac{x}{2^{\imath+1}}) = \frac{1}{2^p}\sum_{\imath=1}^{\infty} 2^{\imath p}\psi_o^p(\frac{x}{2^\imath}) \tag{2.21}$$

for all $x \in X$. Therefore (2.5) follows from (2.18) and (2.21). Now we show that A is additive. It follows from (2.1), (2.3) and (2.20) that

$$\|DA(x_1, ..., x_n)\| = \lim_{m\to\infty} 2^m \|Df(\frac{x_1}{2^m}, ..., \frac{x_n}{2^m})\| \le \lim_{m\to\infty} 2^m \varphi(\frac{x_1}{2^m}, ..., \frac{x_n}{2^m}) = 0$$

for all $x_1, ..., x_n \in X$. Hence the function A satisfies (1.4). Since f is an odd function, then (2.20) implies that the function $A : X \to Y$ is odd. Therefore by Lemma 2.1 of [23], we see that the function $A : X \to Y$ is additive.

To prove the uniqueness property of A, let $A' : X \to Y$ be another additive function satisfying (2.5). Since

$$\lim_{m\to\infty} 2^{mp}\sum_{\imath=1}^{\infty} 2^{\imath p}\varphi^p(\frac{u_1}{2^{m+\imath}}, ..., \frac{u_n}{2^{m+\imath}}) = \lim_{m\to\infty}\sum_{\imath=m+1}^{\infty} 2^{\imath p}\varphi^p(\frac{u_1}{2^\imath}, ..., \frac{u_n}{2^\imath}) = 0$$

for all $u_1 \in \{-x, x, 2x\}$ and all $u_2, ..., u_n \in \{-x, 0, x\}$. Hence

$$\lim_{m\to\infty} 2^{mp}\widetilde{\psi}_o(\frac{x}{2^m}) = 0 \tag{2.22}$$

for all $x \in X$. It follows from (2.5) and (2.22) that

$$\|A(x) - A'(x)\|^p = \lim_{m\to\infty} 2^{mp}\|f(\frac{x}{2^m}) - A'(\frac{x}{2^m})\|^p \le \frac{1}{2^p}\lim_{m\to\infty} 2^{mp}\widetilde{\psi}_o(\frac{x}{2^m}) = 0$$

for all $x \in X$. So we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A.

For $\ell = -1$, we can prove the theorem by a similar technique. □

Corollary 2.3. *Let ε, λ_i $(1 \le i \le n)$ be non-negative real numbers such that $\lambda_i < 1$ or $\lambda_i > 1$ $(1 \le i \le n)$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|Df(x_1, ..., x_n)\| \le \varepsilon \sum_{i=1}^{n} \|x_i\|^{\lambda_i} \tag{2.23}$$

for all $x_1, ..., x_n \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{\varepsilon}{2}[\alpha_1^p\|x\|^{\lambda_1 p} + \alpha_2^p\|x\|^{\lambda_2 p} + \alpha_3^p\|x\|^{\lambda_3 p} + ... + \alpha_n^p\|x\|^{\lambda_n p}]^{\frac{1}{p}}$$

for all $x \in X$, *where*

$$\alpha_1 = [\frac{2^{p(1+\lambda_1)} + n^p + 2}{|2^p - 2^{\lambda_1 p}|}]^{\frac{1}{p}}, \quad \alpha_2 = [\frac{n^p + 2}{|2^p - 2^{\lambda_2 p}|}]^{\frac{1}{p}}, \quad \alpha_i = [\frac{2^{p+1}}{|2^p - 2^{\lambda_i p}|}]^{\frac{1}{p}} \quad (3 \leq i \leq n).$$

Proof. In Theorem 2.2, put $\varphi(x_1, ..., x_n) := \varepsilon \sum_{i=1}^{n} \|x_i\|^{\lambda_i}$ for all $x_1, ..., x_n \in X$. □

3. Stability of the functional equation (1.4) in p–Banach spaces: for even functions

In this section, we prove the generalized Hyers–Ulam–Rassias stability of the functional equation (1.4) in p–Banach spaces for quadratic functions.

Theorem 3.1. *Let* $\ell \in \{-1, 1\}$ *be fixed,* X *be a* p*–normed space,* Y *be a* p*–Banach space and* $\varphi : X^n \to [0, \infty)$ *be a function such that*

$$\lim_{m\to\infty} 2^{2m\ell}\varphi(\frac{x_1}{2^{m\ell}}, ..., \frac{x_n}{2^{m\ell}}) = 0 \tag{3.1}$$

for all $x_1, ..., x_n \in X$, *and*

$$\sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{2\imath p\ell}\varphi^p(\frac{u_1}{2^{\imath\ell}}, ..., \frac{u_n}{2^{\imath\ell}}) < \infty \tag{3.2}$$

for all $u_1 \in \{0, x, nx\}$, $u_2 \in \{0, (n-1)x, nx\}$ *and all* $u_3, ..., u_n \in \{0, nx\}$ *(denoted* $(\varphi(x_1, ..., x_n))^p$ *by* $\varphi^p(x_1, ..., x_n)$*). Suppose that an even function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, ..., x_n)\| \leq \varphi(x_1, ..., x_n) \tag{3.3}$$

for all $x_1, ..., x_n \in X$. *Furthermore, assume that* $f(0) = 0$ *in* (3.3) *for the case* $\ell = 1$. *Then the limit*

$$Q(x) := \lim_{m\to\infty} 2^{2m\ell} f(\frac{x}{2^{m\ell}}) \tag{3.4}$$

exists for all $x \in X$ *and* $Q : X \to Y$ *is a unique quadratic function satisfying*

$$\|f(x) - Q(x)\| \leq \frac{1}{2^2}(\widetilde{\psi}_e(x))^{\frac{1}{p}} \tag{3.5}$$

for all $x \in X$, *where*

$$\widetilde{\psi}_e(x) := \sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{2\imath p\ell}\{\frac{1}{(2n-2)^p}[\varphi^p(\frac{nx}{2^{\imath\ell}}, \frac{nx}{2^{\imath\ell}}, 0, ..., 0) + (2n+4)^p\varphi^p(\frac{nx}{2^{\imath\ell}}, 0, ..., 0)$$
$$+ 2^p\varphi^p(0, \frac{nx}{2^{\imath\ell}}, ..., \frac{nx}{2^{\imath\ell}}) + 2^p\varphi^p(\frac{x}{2^{\imath\ell}}, \frac{(n-1)x}{2^{\imath\ell}}, 0, ..., 0)]\}. \tag{3.6}$$

Proof. For $\ell = 1$, letting $x_1 = nx$, $x_2 = -ny$ and $x_i = 0$ $(i = 3, ..., n)$ in (3.3) and using the evenness of f, we get

$$\|f((n-1)x + y) + f(x + (n-1)y) - f(nx) - f(ny) + (2n-2)f(x-y)\|$$
$$\leq \varphi(nx, -ny, 0, ..., 0) \tag{3.7}$$

for all $x, y \in X$. Putting $y = 0$ in (3.7) and using the evenness of f, we get

$$\|f(nx) - f((n-1)x) - (2n-1)f(x)\| \leq \varphi(nx, 0, ..., 0) \tag{3.8}$$

for all $x \in X$. Letting $y = (1-n)x$ in (3.7) and replacing x by $\frac{x}{n}$ in the obtained inequality, we get

$$\|f((n-1)x) - f((n-2)x) - (2n-3)f(x)\| \le \varphi(x, (n-1)x, 0, ..., 0) \tag{3.9}$$

for all $x, y \in X$. Letting $x_1 = nx$, $x_2 = ... = x_n = ny$ in (3.3) and using the evenness of f, we get

$$\begin{aligned}\|f((n-1)(x-y)) + (n-1)f(x-y) - (n-1)f(ny) - f(nx)\\ + nf(x + (n-1)y)\| \le \varphi(nx, ny, ..., ny)\end{aligned} \tag{3.10}$$

for all $x, y \in X$. Since f is even, it follows from (3.10) that

$$\begin{aligned}\|f((n-1)(x-y)) + (n-1)f(x-y) - (n-1)f(nx) - f(ny)\\ + nf((n-1)x + y)\| \le \varphi(ny, nx, ..., nx)\end{aligned} \tag{3.11}$$

for all $x, y \in X$. Applying (3.7), (3.10) and (3.11), we get

$$\begin{aligned}\|f((n-1)(x-y)) - (n-1)^2 f(x-y)\| \le \frac{1}{2}[n\varphi(nx, -ny, 0, ..., 0)\\ + \varphi(nx, ny, ..., ny) + \varphi(ny, nx, ..., nx)]\end{aligned}$$

for all $x, y \in X$. Therefore

$$\|f((n-1)x) - (n-1)^2 f(x)\| \le \frac{1}{2}[(n+1)\varphi(nx, 0, ..., 0) + \varphi(0, nx, ..., nx)] \tag{3.12}$$

for all $x \in X$. So we get from (3.8) and (3.9)

$$\|f(nx) - n^2 f(x)\| \le \frac{1}{2}[(n+3)\varphi(nx, 0, ..., 0) + \varphi(0, nx, ..., nx)] \tag{3.13}$$

and

$$\begin{aligned}\|f((n-2)x) - (n-2)^2 f(x)\| \le \frac{1}{2}[(n+1)\varphi(nx, 0, ..., 0) + \varphi(0, nx, ..., nx)]\\ + \varphi(x, (n-1)x, 0, ..., 0)\end{aligned} \tag{3.14}$$

for all $x \in X$. Letting $y = -x$ in (3.7) and using (3.13) and (3.14), we get

$$\begin{aligned}\|f(2x) - 4f(x)\| \le \frac{1}{(2n-2)}[\varphi(nx, nx, 0, ..., 0) + (2n+4)\varphi(nx, 0, ..., 0)\\ + 2\varphi(0, nx, ..., nx) + 2\varphi(x, (n-1)x, 0, ..., 0)]\end{aligned} \tag{3.15}$$

for all $x \in X$. Let

$$\begin{aligned}\psi_e(x) := \frac{1}{(2n-2)}[\varphi(nx, nx, 0, ..., 0) + (2n+4)\varphi(nx, 0, ..., 0) + 2\varphi(0, nx, ..., nx)\\ + 2\varphi(x, (n-1)x, 0, ..., 0)]\end{aligned} \tag{3.16}$$

for all $x \in X$. Thus (3.15) means that

$$\|f(2x) - 4f(x)\| \le \psi_e(x) \tag{3.17}$$

for all $x \in X$. If we replace x in (3.17) by $\frac{x}{2^{m+1}}$ and multiply both sides of (3.17) by 2^{2m}, then we have

$$\|2^{2(m+1)} f(\frac{x}{2^{m+1}}) - 2^{2m} f(\frac{x}{2^m})\| \le 2^{2m}\psi_e(\frac{x}{2^{m+1}}) \tag{3.18}$$

for all $x \in X$ and all non-negative integers m. Hence

$$\begin{aligned}\|2^{2(m+1)}f(\frac{x}{2^{m+1}}) - 2^{2k}f(\frac{x}{2^k})\|^p &\leq \sum_{\imath=k}^{m} \|2^{2(\imath+1)}f(\frac{x}{2^{\imath+1}}) - 2^{2\imath}f(\frac{x}{2^\imath})\|^p \\ &\leq \sum_{\imath=k}^{m} 2^{2\imath p}\psi_e^p(\frac{x}{2^{\imath+1}}) \end{aligned} \tag{3.19}$$

for all non-negative integers m and k with $m \geq k$ and all $x \in X$. Since $0 < p \leq 1$, so by Lemma 2.1 and (3.16), we get

$$\begin{aligned}\psi_e^p(x) \leq \frac{1}{(2n-2)^p}[&\varphi^p(nx, nx, 0, ..., 0) + (2n+4)^p\varphi^p(nx, 0, ..., 0) \\ &+ 2^p\varphi^p(0, nx, ..., nx) + 2^p\varphi^p(x, (n-1)x, 0, ..., 0)]\end{aligned} \tag{3.20}$$

for all $x \in X$. Therefore by (3.1), (3.2) and (3.20) we have

$$\sum_{\imath=1}^{\infty} 2^{2\imath p}\psi_e^p(\frac{x}{2^\imath}) < \infty, \qquad \lim_{m\to\infty} 2^{2m}\psi_e(\frac{x}{2^m}) = 0 \tag{3.21}$$

for all $x \in X$. Therefore we conclude from (3.19) and (3.21) that the sequence $\{2^{2m}f(\frac{x}{2^m})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^{2m}f(\frac{x}{2^m})\}$ converges for all $x \in X$. So one can define the function $Q : X \to Y$ by (3.4) for all $x \in X$. Letting $k = 0$ and passing the limit $m \to \infty$ in (3.19), we get

$$\|f(x) - Q(x)\|^p \leq \sum_{\imath=0}^{\infty} 2^{2\imath p}\psi_e^p(\frac{x}{2^{\imath+1}}) = \frac{1}{2^{2p}} \sum_{\imath=1}^{\infty} 2^{2\imath p}\psi_e^p(\frac{x}{2^\imath}) \tag{3.22}$$

for all $x \in X$. Therefore (3.5) follows from (3.20) and (3.22). Now we show that Q is quadratic. It follows from (3.1), (3.3) and (3.4) that

$$\|DQ(x_1, ..., x_n)\| = \lim_{m\to\infty} 2^{2m}\|Df(\frac{x_1}{2^m}, ..., \frac{x_n}{2^m})\| \leq \lim_{m\to\infty} 2^{2m}\varphi(\frac{x_1}{2^m}, ..., \frac{x_n}{2^m}) = 0$$

for all $x_1, ..., x_n \in X$. Therefore the function Q satisfies (1.4). Since f is an even function, then (3.4) implies that the function $Q : X \to Y$ is even. Therefore by Lemma 2.2 of [23], we get that the function $Q : X \to Y$ is quadratic.

To prove the uniqueness property of Q, let $Q' : X \to Y$ be another quadratic function satisfying (3.5). Since

$$\lim_{m\to\infty} 2^{2mp} \sum_{\imath=1}^{\infty} 2^{2\imath p}\varphi^p(\frac{u_1}{2^{m+\imath}}, ..., \frac{u_n}{2^{m+\imath}}) = \lim_{m\to\infty} \sum_{\imath=m+1}^{\infty} 2^{2\imath p}\varphi^p(\frac{u_1}{2^\imath}, ..., \frac{u_n}{2^\imath}) = 0$$

for all $u_1 \in \{0, x, nx\}$, $u_2 \in \{0, (n-1)x, nx\}$ and all $u_3, ..., u_n \in \{0, nx\}$, then

$$\lim_{m\to\infty} 2^{2mp}\widetilde{\psi}_e(\frac{x}{2^m}) = 0$$

for all $x \in X$. Therefore it from (3.5) and the last equation that

$$\|Q(x) - Q'(x)\|^p = \lim_{m\to\infty} 2^{2mp}\|f(\frac{x}{2^m}) - Q'(\frac{x}{2^m})\|^p \leq \frac{1}{2^{2p}} \lim_{m\to\infty} 2^{2mp}\widetilde{\psi}_e(\frac{x}{2^m}) = 0$$

for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q.

For $\ell = -1$, we can prove the theorem by a similar technique. $\square$

Corollary 3.2. *Let ε, λ_i $(1 \le i \le n)$ be non-negative real numbers such that $\lambda_i < 2$ or $\lambda_i > 2$ $(1 \le i \le n)$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|Df(x_1, ..., x_n)\| \le \varepsilon \sum_{i=1}^{n} \|x_i\|^{\lambda_i} \tag{3.23}$$

for all $x_1, ..., x_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{\varepsilon}{(2n-2)}[\beta_1^p \|x\|^{\lambda_1 p} + ... + \beta_n^p \|x\|^{\lambda_n p}]^{\frac{1}{p}}$$

for all $x \in X$, where

$$\beta_1 = [\frac{n^{p\lambda_1} + (2n+4)^p n^{p\lambda_1} + 2^p}{|2^{2p} - 2^{p\lambda_1}|}]^{\frac{1}{p}}, \quad \beta_2 = [\frac{n^{p\lambda_2} + 2^p n^{p\lambda_2} + 2^p(n-1)^{p\lambda_2}}{|2^{2p} - 2^{p\lambda_1}|}]^{\frac{1}{p}},$$

$$\beta_i = [\frac{2^p n^{p\lambda_i}}{|2^{2p} - 2^{p\lambda_1}|}]^{\frac{1}{p}} \quad (3 \le i \le n).$$

4. Stability of a mixed quadratic and additive functional equation (1.4) in p–Banach space

Now, we are ready to prove the main theorem concerning the stability problem for functional equation (1.4) in p–Banach spaces.

Theorem 4.1. *Let $\varphi : X^n \to [0, \infty)$ be a function which satisfies (2.1) for all $x_1, ..., x_n \in X$ and (2.2) for all $u_1 \in \{-x, x, 2x\}$, $u_2, ..., u_n \in \{-x, 0, x\}$ and satisfies (3.1) for all $x_1, ..., x_n \in X$ and (3.2) for all $u_1 \in \{0, x, nx\}$, $u_2 \in \{0, (n-1)x, nx\}$ and all $u_3, ..., u_n \in \{0, nx\}$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (2.3) for all $x_1, ..., x_n \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that*

$$\|f(x) - A(x) - Q(x)\| \le \frac{1}{2^3}\{[\widetilde{\psi}_e(x) + \widetilde{\psi}_e(-x)]^{\frac{1}{p}}\} + \frac{1}{2^2}\{[\widetilde{\psi}_o(x) + \widetilde{\psi}_o(-x)]^{\frac{1}{p}}\} \tag{4.1}$$

for all $x \in X$, where $\widetilde{\psi}_e(x)$ and $\widetilde{\psi}_o(x)$ are defined as in equations (2.6) and (3.6).

Proof. Assume that $\varphi : X^n \to [0, \infty)$ satisfies (3.1) for all $x_1, ..., x_n \in X$ and (3.2) for all $u_1 \in \{0, x, nx\}$, $u_2 \in \{0, (n-1)x, nx\}$ and all $u_3, ..., u_n \in \{0, nx\}$. Let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$, then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\|Df_e(x_1, ..., x_n)\| \le \widetilde{\varphi}(x_1, ..., x_n)$$

for all $x_1, ..., x_n \in X$, where $\widetilde{\varphi}(x_1, ..., x_n) := \frac{1}{2}(\varphi(x_1, ..., x_n) + \varphi(-x_1, ..., -x_n))$. So

$$\lim_{m\to\infty} 2^{2m\ell} \widetilde{\varphi}(\frac{x_1}{2^{m\ell}}, ..., \frac{x_n}{2^{m\ell}}) = 0$$

for all $x_1, ..., x_n \in X$. Since

$$\widetilde{\varphi}^p(x_1, ..., x_n) \le \frac{1}{2^p}(\varphi^p(x_1, ..., x_n) + \varphi^p(-x_1, ..., -x_n))$$

for all $x_1, ..., x_n \in X$, then

$$\sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{2\imath p\ell}\widetilde{\varphi}^p(\frac{u_1}{2^{\imath\ell}}, ..., \frac{u_n}{2^{\imath\ell}}) < \infty$$

for all $u_1 \in \{0, x, nx\}$, $u_2 \in \{0, (n-1)x, nx\}$ and all $u_3, ..., u_n \in \{0, nx\}$. Hence from Theorem 3.1, there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2^2}(\tilde{\tilde{\psi}}_e(x))^{\frac{1}{p}} \tag{4.2}$$

for all $x \in X$, where

$$\tilde{\tilde{\psi}}_e(x) := \sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{2\imath p\ell}\{\frac{1}{(2n-2)^p}[\widetilde{\varphi}^p(\frac{nx}{2^{\imath\ell}}, \frac{nx}{2^{\imath\ell}}, 0, ..., 0) + (2n+4)^p\widetilde{\varphi}^p(\frac{nx}{2^{\imath\ell}}, 0, ..., 0)$$
$$+ 2^p\widetilde{\varphi}^p(0, \frac{nx}{2^{\imath\ell}}, ..., \frac{nx}{2^{\imath\ell}}) + 2^p\widetilde{\varphi}^p(\frac{x}{2^{\imath\ell}}, \frac{(n-1)x}{2^{\imath\ell}}, 0, ..., 0)]\}$$

for all $x \in X$. It is clear that

$$\tilde{\tilde{\psi}}_e(x) \leq \frac{1}{2^p}\ [\widetilde{\psi}_e(x) + \widetilde{\psi}_e(-x)]$$

for all $x \in X$. Therefore it follows from (4.2) that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2^3}[\widetilde{\psi}_e(x) + \widetilde{\psi}_e(-x)]^{\frac{1}{p}} \tag{4.3}$$

for all $x \in X$.

Also, let $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$, by using the above method and Theorem 2.2, it follows that there exist a unique additive function $A : X \to Y$ such that

$$\|f_o(x) - A(x)\| \leq \frac{1}{2^2}[\widetilde{\psi}_o(x) + \widetilde{\psi}_o(-x)]^{\frac{1}{p}} \tag{4.4}$$

for all $x \in X$. Hence (4.1) follows from (4.3) and (4.4). Now, if $\varphi : X^n \to [0, \infty)$ satisfies (2.1) for all $x_1, ..., x_n \in X$ and (2.2) for all $u_1 \in \{-x, x, 2x\}$ and all $u_2, ...u_n \in \{-x, 0, x\}$, we can prove the theorem by a similar technique. □

Corollary 4.2. *Let ε, λ_i $(1 \leq i \leq n)$ be non-negative real numbers such that $1 < \lambda_i < 2$ or $\lambda_i > 2$ or $\lambda_i < 1$ $(1 \leq i \leq n)$. Suppose that a function $f : X \to Y$ satisfies the inequality $\|Df(x_1, ..., x_n)\| \leq \varepsilon\ \sum_{i=1}^{n} \|x_i\|^{\lambda_i}$ for all $x_1, ..., x_n \in X$. Furthermore, assume that $f(0) = 0$ for the case f is even. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that*

$$\|f(x) - Q(x) - A(x)\| \leq \frac{\varepsilon}{(2n-2)}[\beta_1^p\|x\|^{\lambda_1 p} + ... + \beta_n^p\|x\|^{\lambda_n p}]^{\frac{1}{p}}$$
$$+ \frac{\varepsilon}{2}[\alpha_1^p\|x\|^{\lambda_1 p} + ... + \alpha_n^p\|x\|^{\lambda_n p}]^{\frac{1}{p}}$$

for all $x \in X$, where α_i and β_i $(1 \leq i \leq n)$ are defined as in Corollaries (2.3) and (3.2).

Proof. put $\varphi(x_1, ..., x_n) := \varepsilon\ \sum_{i=1}^{n} \|x_i\|^{\lambda_i}$, Since

$$\|D_{f_e}(x_1, ..., x_n)\| \leq \varphi(x_1, ..., x_n), \qquad \|D_{f_o}(x_1, ..., x_n)\| \leq \varphi(x_1, ..., x_n)$$

for all $x_1, ..., x_n \in X$. Thus the result follows from Corollaries (2.3) and (3.2). $\square$

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