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STABILITY OF A GENERALIZED EULER-LAGRANGE TYPE ADDITIVE MAPPING AND HOMOMORPHISMS IN C^*-ALGEBRAS II

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ABSTRACT. Let X, Y be Banach modules over a C^*-algebra and let $r_1, \cdots, r_n \in \mathbb{R}$ be given. We prove the generalized Hyers-Ulam stability of the following functional equation in Banach modules over a unital C^*-algebra:

$$\sum_{j=1}^{n} f\Big(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i x_i - \frac{1}{2} r_j x_j\Big) + \sum_{i=1}^{n} r_i f(x_i) = nf\Big(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\Big) \tag{0.1}$$

We show that if $\sum_{i=1}^{n} r_i \ne 0$, $r_i \ne 0, r_j \ne 0$ for some $1 \le i < j \le n$ and a mapping $f : X \to Y$ satisfies the functional equation (0.1) then the mapping $f : X \to Y$ is additive. As an application, we investigate homomorphisms in unital C^*-algebras.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations was originated from a question of Ulam [66] concerning the stability of group homomorphisms:

Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x_1.x_2), h(x_1) * h(x_2)) < \delta$ for all $x_1, x_2 \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x_1), H(x_1)) < \epsilon$ for all $x_1 \in G_1$?

Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings

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and by Th.M. Rassias [58] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. (Th.M. Rassias [58]). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then the inequality (1.1) holds for $x, y \ne 0$ and (1.2) for $x \ne 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Theorem 1.2. (J.M. Rassias [49]–[51]). *Let X be a real normed linear space and Y a real Banach space. Assume that $f : X \to Y$ is a mapping for which there exist constants $\theta \ge 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \ne 1$ and f satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\|x\|^p\|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \le \frac{\theta}{|2^r - 2|}\|x\|^r$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is $\mathbb{R}$-linear.

The paper of Th.M. Rassias [58] has provided a lot of influence in the development of what we call the *generalized Hyers-Ulam stability* of functional equations. In 1994, a generalization of Theorems 1.1 and 1.2 was obtained by Găvruta [10], who replaced the bounds $\varepsilon(\|x\|^p + \|y\|^p)$ and $\theta\|x\|^p\|y\|^q$ by a general control function $\varphi(x, y)$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [65] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [5] proved the generalized Hyers-Ulam stability of the quadratic functional equation. J.M. Rassias [52, 53] introduced and investigated the stability problem of Ulam for the Euler-Lagrange quadratic mappings (1.3) and

$$f(a_1x_1 + a_2x_2) + f(a_2x_1 - a_1x_2) = (a_1^2 + a_2^2)[f(x_1) + f(x_2)]. \tag{1.4}$$

Grabiec [14] has generalized these results mentioned above. In addition, J.M. Rassias [54] generalized the Euler-Lagrange quadratic mapping (1.4) and investigated its stability problem. Thus these Euler-Lagrange type equations (mappings) are called as Euler-Lagrange-Rassias functional equations (mappings).

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [4], [6], [7], [9]–[13], [16]–[22], [24]–[64] and [67]).

Recently, C. Park and J. Park [46] introduced and investigated the following additive functional equation of Euler-Lagrange type

$$\begin{aligned}\sum_{i=1}^{n} r_i L\left(\sum_{j=1}^{n} r_j(x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) L\left(\sum_{i=1}^{n} r_i x_i\right) \\ = \left(\sum_{i=1}^{n} r_i\right)\sum_{i=1}^{n} r_i L(x_i), \qquad r_1, \cdots, r_n \in (0, \infty)\end{aligned} \tag{1.5}$$

whose solution is said to be a *generalized additive mapping of Euler-Lagrange type.*

In this paper, we introduce the following additive functional equation of Euler-Lagrange type which is somewhat different from (1.5):

$$\sum_{j=1}^{n} f\Big(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i x_i - \frac{1}{2}r_j x_j\Big) + \sum_{i=1}^{n} r_i f(x_i) = nf\Big(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\Big), \tag{1.6}$$

where $r_1, \cdots, r_n \in \mathbb{R}$. Every solution of the functional equation (1.6) is said to be a *generalized Euler-Lagrange type additive mapping.*

We investigate the generalized Hyers-Ulam stability of the functional equation (1.6) in Banach modules over a C^*-algebra. These results are applied to investigate C^*-algebra homomorphisms in unital C^*-algebras.

Throughout this paper, assume that A is a unital C^*-algebra with norm $\|.\|_A$ and unit e, that B is a unital C^*-algebra with norm $\|.\|_B$, and that X and Y are left Banach modules over a unital C^*-algebra A with norms $\|.\|_X$ and $\|.\|_Y$, respectively. Let $U(A)$ be the group of unitary elements in A and let $r_1, \cdots, r_n \in \mathbb{R}$.

For a given mapping $f : X \to Y$, $u \in U(A)$ and a given $\mu \in \mathbb{C}$, we define $D_{u,r_1,\cdots,r_n} f$ and $D_{\mu,r_1,\cdots,r_n} f : X^n \to Y$ by

$$\begin{aligned} D_{u,r_1,\cdots,r_n} f(x_1,\cdots,x_n) := & \sum_{j=1}^{n} f\Big(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i u x_i - \frac{1}{2} r_j u x_j\Big) + \sum_{i=1}^{n} r_i u f(x_i) \\ & - n f\Big(\frac{1}{2}\sum_{i=1}^{n} r_i u x_i\Big) \end{aligned}$$

and

$$\begin{aligned} D_{\mu,r_1,\cdots,r_n} f(x_1,\cdots,x_n) := & \sum_{j=1}^{n} f\Big(\frac{1}{2}\sum_{1\le i\le n, i\ne j} \mu r_i x_i - \frac{1}{2} \mu r_j x_j\Big) + \sum_{i=1}^{n} \mu r_i f(x_i) \\ & - n f\Big(\frac{1}{2}\sum_{i=1}^{n} \mu r_i x_i\Big) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

2. Generalized Hyers-Ulam stability of the functional equation (1.6) in Banach modules over a C^*-algebra

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \cdots, r_n$ be real numbers with $\sum_{k=1}^{n} r_k \ne 0$ and $r_i \ne 0, r_j \ne 0$ for some $1 \le i < j \le n$. Assume that a mapping $L : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (1.6) for all $x_1, \cdots, x_n \in \mathcal{X}$. Then the mapping L is additive. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in \mathcal{X}$ and all $1 \le k \le n$.*

Proof. Since $\sum_{k=1}^{n} r_k \ne 0$, putting $x_1 = \cdots = x_n = 0$ in (1.6), we get $L(0) = 0$. Without loss of generality, we may assume that $r_1, r_2 \ne 0$. Letting $x_3 = \cdots = x_n = 0$ in (1.6), we get

$$\begin{aligned} L\left(\frac{-r_1x_1 + r_2x_2}{2}\right) + L\left(\frac{r_1x_1 - r_2x_2}{2}\right) &+ r_1L(x_1) + r_2L(x_2) \\ &= 2L\left(\frac{r_1x_1 + r_2x_2}{2}\right) \end{aligned} \tag{2.1}$$

for all $x_1, x_2 \in \mathcal{X}$. Letting $x_2 = 0$ in (2.1), we get

$$r_1L(x_1) = L\left(\frac{r_1x_1}{2}\right) - L\left(-\frac{r_1x_1}{2}\right) \tag{2.2}$$

for all $x_1 \in \mathcal{X}$.

Similarly, by putting $x_1 = 0$ in (2.1), we get

$$r_2L(x_2) = L\left(\frac{r_2x_2}{2}\right) - L\left(-\frac{r_2x_2}{2}\right) \tag{2.3}$$

for all $x_1 \in \mathcal{X}$. It follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned} L\left(\frac{-r_1x_1+r_2x_2}{2}\right) + L\left(\frac{r_1x_1-r_2x_2}{2}\right) + L\left(\frac{r_1x_1}{2}\right) + L\left(\frac{r_2x_2}{2}\right) \\ - L\left(-\frac{r_1x_1}{2}\right) - L\left(-\frac{r_2x_2}{2}\right) = 2L\left(\frac{r_1x_1+r_2x_2}{2}\right) \end{aligned} \tag{2.4}$$

for all $x_1, x_2 \in \mathcal{X}$. Replacing x_1 and x_2 by $\frac{2x}{r_1}$ and $\frac{2y}{r_2}$ in (2.4), we get

$$L(-x+y) + L(x-y) + L(x) + L(y) - L(-x) - L(-y) = 2L(x+y) \tag{2.5}$$

for all $x, y \in \mathcal{X}$. Letting $y = -x$ in (2.5), we get that $L(-2x) + L(2x) = 0$ for all $x \in \mathcal{X}$. So the mapping L is odd. Therefore, it follows from (2.5) that the mapping L is additive. Moreover, let $x \in \mathcal{X}$ and $1 \le k \le n$. Setting $x_k = x$ and $x_l = 0$ for all $1 \le l \le n$, $l \ne k$, in (1.6) and using the oddness of L, we get that $L(r_kx) = r_kL(x)$. □

Using the same method as in the proof of Lemma 2.1, we have an alternative result of Lemma 2.1 when $\sum_{k=1}^{n} r_k = 0$.

Lemma 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \cdots, r_n$ be real numbers with $r_i \ne 0, r_j \ne 0$ for some $1 \le i < j \le n$. Assume that a mapping $L : \mathcal{X} \to \mathcal{Y}$ with $L(0) = 0$ satisfies the functional equation (1.6) for all $x_1, \cdots, x_n \in \mathcal{X}$. Then the mapping L is additive. Moreover, $L(r_kx) = r_kL(x)$ for all $x \in \mathcal{X}$ and all $1 \le k \le n$.*

We investigate the generalized Hyers-Ulam stability of a generalized Euler-Lagrange type additive mapping in Banach spaces.

Throughout this paper, $r_1, \cdots, r_n$ **will be real numbers such that** $r_i \ne 0, r_j \ne 0$ **for fixed** $1 \le i < j \le n$.

Theorem 2.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^n \to [0, \infty)$ such that*

$$\widetilde{\varphi_{ij}}(x, y) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi\big(0, \cdots, \underbrace{2^kx}_{ith}, 0, \cdots, \underbrace{2^ky}_{jth}, 0, \cdots, 0\big) < \infty, \tag{2.6}$$

$$\lim_{k\to\infty} \frac{1}{2^k} \varphi\big(2^kx_1, \cdots, 2^kx_n\big) = 0, \tag{2.7}$$

$$\|D_{e,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_Y \le \varphi(x_1, \cdots, x_n) \tag{2.8}$$

for all $x, x_1, \cdots, x_n \in X$ and $y \in \{0, \pm x\}$. Then there exists a unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\begin{aligned} \|f(x) - L(x)\|_Y \le \frac{1}{4}\Big\{ &\big[\widetilde{\varphi_{ij}}\big(\frac{2x}{r_i}, \frac{2x}{r_j}\big) + 2\widetilde{\varphi_{ij}}\big(\frac{x}{r_i}, -\frac{x}{r_j}\big)\big] \\ &+ \big[\widetilde{\varphi_{ij}}\big(\frac{2x}{r_i}, 0\big) + 2\widetilde{\varphi_{ij}}\big(\frac{x}{r_i}, 0\big)\big] \\ &+ \big[\widetilde{\varphi_{ij}}\big(0, \frac{2x}{r_j}\big) + 2\widetilde{\varphi_{ij}}\big(0, -\frac{x}{r_j}\big)\big] \Big\} \end{aligned} \tag{2.9}$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

Proof. For each $1 \leq k \leq n$ with $k \neq i, j$, let $x_k = 0$ in (2.8).Then we get the following inequality

$$\left\| f\left(\frac{-r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i - r_j x_j}{2}\right) - 2f\left(\frac{r_i x_i + r_j x_j}{2}\right) + r_i f(x_i) + r_j f(x_j) \right\|_Y$$
$$\leq \varphi(0, \cdots, 0, \underbrace{x_i}_{i\,th}, 0, \cdots, 0, \underbrace{x_j}_{j\,th}, 0, \cdots, 0) \tag{2.10}$$

for all $x_i, x_j \in X$. For convenience, set

$$\varphi_{ij}(x, y) := \varphi(0, \cdots, 0, \underbrace{x}_{i\,th}, 0, \cdots, 0, \underbrace{y}_{j\,th}, 0, \cdots, 0)$$

for all $x, y \in X$ and all $1 \leq i < j \leq n$. Letting $x_i = 0$ in (2.10), we get

$$\left\| f\left(-\frac{r_j x_j}{2}\right) - f\left(\frac{r_j x_j}{2}\right) + r_j f(x_j) \right\|_Y \leq \varphi_{ij}(0, x_j) \tag{2.11}$$

for all $x_j \in X$.

Similarly, letting $x_j = 0$ in (2.10), we get

$$\left\| f\left(-\frac{r_i x_i}{2}\right) - f\left(\frac{r_i x_i}{2}\right) + r_i f(x_i) \right\|_Y \leq \varphi_{ij}(x_i, 0) \tag{2.12}$$

for all $x_i \in X$. It follows from (2.10), (2.11) and (2.12) that

$$\begin{aligned} &\left\| f\left(\frac{-r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i - r_j x_j}{2}\right) - 2f\left(\frac{r_i x_i + r_j x_j}{2}\right) \right. \\ &\quad \left. + f\left(\frac{r_i x_i}{2}\right) + f\left(\frac{r_j x_j}{2}\right) - f\left(-\frac{r_i x_i}{2}\right) - f\left(-\frac{r_j x_j}{2}\right) \right\|_Y \\ &\leq \varphi_{ij}(x_i, x_j) + \varphi_{ij}(x_i, 0) + \varphi_{ij}(0, x_j) \end{aligned} \tag{2.13}$$

for all $x_i, x_j \in X$. Replacing x_i and x_j by $\frac{2x}{r_i}$ and $\frac{2y}{r_j}$ in (2.13), we get that

$$\begin{aligned} &\|f(-x + y) + f(x - y) - 2f(x + y) \\ &\quad + f(x) + f(y) - f(-x) - f(-y)\|_Y \\ &\qquad \leq \varphi_{ij}\left(\frac{2x}{r_i}, \frac{2y}{r_j}\right) + \varphi_{ij}\left(\frac{2x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{2y}{r_j}\right) \end{aligned} \tag{2.14}$$

for all $x, y \in X$. Putting $y = x$ in (2.14), we get

$$\|2f(x) - 2f(-x) - 2f(2x)\|_Y \tag{2.15}$$
$$\leq \varphi_{ij}\left(\frac{2x}{r_i}, \frac{2x}{r_j}\right) + \varphi_{ij}\left(\frac{2x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{2x}{r_j}\right)$$

for all $x \in X$. Replacing x and y by $\frac{x}{2}$ and $-\frac{x}{2}$ in (2.14), respectively, we get

$$\|f(x) + f(-x)\|_Y \leq \varphi_{ij}\left(\frac{x}{r_i}, -\frac{x}{r_j}\right) + \varphi_{ij}\left(\frac{x}{r_i}, 0\right) + \varphi_{ij}\left(0, -\frac{x}{r_j}\right) \tag{2.16}$$

for all $x \in X$. It follows from (2.15) and (2.16) that

$$\|f(2x) - 2f(x)\|_Y \leq \psi(x) \tag{2.17}$$

for all $x \in X$, where

$$\begin{aligned}\psi(x) := \frac{1}{2}\Big\{ &\big[\varphi_{ij}\big(\frac{2x}{r_i}, \frac{2x}{r_j}\big) + 2\varphi_{ij}\big(\frac{x}{r_i}, -\frac{x}{r_j}\big)\big] \\ &+ \big[\varphi_{ij}\big(\frac{2x}{r_i}, 0\big) + 2\varphi_{ij}\big(\frac{x}{r_i}, 0\big)\big] \\ &+ \big[\varphi_{ij}\big(0, \frac{2x}{r_j}\big) + 2\varphi_{ij}\big(0, -\frac{x}{r_j}\big)\big]\Big\}.\end{aligned}$$

It follows from (2.6) that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{2^k}\psi(2^k x) = \frac{1}{2}\Big\{ &\big[\widetilde{\varphi_{ij}}\big(\frac{2x}{r_i}, \frac{2x}{r_j}\big) + 2\widetilde{\varphi_{ij}}\big(\frac{x}{r_i}, -\frac{x}{r_j}\big)\big] \\ &+ \big[\widetilde{\varphi_{ij}}\big(\frac{2x}{r_i}, 0\big) + 2\widetilde{\varphi_{ij}}\big(\frac{x}{r_i}, 0\big)\big] \\ &+ \big[\widetilde{\varphi_{ij}}\big(0, \frac{2x}{r_j}\big) + 2\widetilde{\varphi_{ij}}\big(0, -\frac{x}{r_j}\big)\big]\Big\} < \infty\end{aligned} \tag{2.18}$$

for all $x \in X$. Replacing x by $2^k x$ in (2.17) and dividing both sides of (2.17) by 2^{k+1}, we get

$$\Big\|\frac{1}{2^{k+1}}f(2^{k+1}x) - \frac{1}{2^k}f(2^k x)\Big\|_Y \leq \frac{1}{2^{k+1}}\psi(2^k x)$$

for all $x \in X$ and all $k \in \mathbb{Z}$. Therefore, we have

$$\begin{aligned}\Big\|\frac{1}{2^{k+1}}f(2^{k+1}x) - \frac{1}{2^m}f(2^m x)\Big\|_Y &\leq \sum_{l=m}^{k}\Big\|\frac{1}{2^{l+1}}f(2^{l+1}x) - \frac{1}{2^l}f(2^l x)\Big\|_Y \\ &\leq \frac{1}{2}\sum_{l=m}^{k}\frac{1}{2^l}\psi(2^l x)\end{aligned} \tag{2.19}$$

for all $x \in X$ and all integers $k \geq m$. It follows from (2.18) and (2.19) that the sequence $\{\frac{f(2^k x)}{2^k}\}$ is Cauchy in Y for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty}\frac{f(2^k x)}{2^k}$$

for all $x \in X$. Letting $m = 0$ in (2.19) and taking the limit as $k \to \infty$ in (2.19), we obtain the desired inequality (2.9).

It follows from (2.7) and (2.8) that

$$\begin{aligned}\|D_{e,r_1,\cdots,r_n}L(x_1,\cdots,x_n)\|_Y &= \lim_{k\to\infty}\frac{1}{2^k}\|D_{e,r_1,\cdots,r_n}f(2^k x_1,\cdots,2^k x_n)\|_Y \\ &\leq \lim_{k\to\infty}\frac{1}{2^k}\varphi(2^k x_1,\cdots,2^k x_n) = 0\end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Therefore, the mapping $L : X \to Y$ satisfies the equation (1.6) and $L(0) = 0$. Hence by Lemma 2.2, L is a generalized Euler-Lagrange type additive mapping and $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

To prove the uniqueness, let $T : X \to Y$ be another generalized Euler-Lagrange type additive mapping with $T(0) = 0$ satisfying (2.9). By Lemma 2.2, the mapping T is additive. Therefore, it follows from (2.9) and (2.18) that

$$\begin{aligned} \|L(x) - T(x)\|_Y &= \lim_{k\to\infty} \frac{1}{2^k} \left\| f(2^k x) - T(2^k x) \right\|_Y \leq \frac{1}{2} \lim_{k\to\infty} \frac{1}{2^k} \sum_{l=0}^{\infty} \frac{1}{2^l} \psi(2^{l+k} x) \\ &= \frac{1}{2} \lim_{k\to\infty} \sum_{l=k}^{\infty} \frac{1}{2^l} \psi(2^l x) = 0. \end{aligned}$$

So $L(x) = T(x)$ for all $x \in X$. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^n \to [0, \infty)$ satisfying (2.6), (2.7) and*

$$\|D_{u,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \tag{2.20}$$

for all $x_1, \cdots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ satisfying (2.9) for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

Proof. By Theorem 2.3, there exists a unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ satisfying (2.9) and moreover $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

By the assumption, for each $u \in U(A)$, we get

$$\begin{aligned} &\left\| D_{u,r_1,\cdots,r_n} L(0, \cdots, 0, \underbrace{x}_{i \text{ th}}, 0 \cdots, 0) \right\|_Y \\ &= \lim_{k\to\infty} \frac{1}{2^k} \left\| D_{u,r_1,\cdots,r_n} f(0, \cdots, 0, \underbrace{2^k x}_{i \text{ th}}, 0 \cdots, 0) \right\|_Y \\ &\leq \lim_{k\to\infty} \frac{1}{2^k} \varphi(0, \cdots, 0, \underbrace{2^k x}_{i \text{ th}}, 0 \cdots, 0) = 0 \end{aligned}$$

for all $x \in X$. So

$$r_i u L(x) = L(r_i u x)$$

for all $u \in U(A)$ and all $x \in X$. Since $L(r_i x) = r_i L(x)$ for all $x \in X$ and $r_i \neq 0$,

$$L(ux) = uL(x)$$

for all $u \in U(A)$ and all $x \in X$.

By the same reasoning as in the proofs of [40] and [42],

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in A$ $(a, b \neq 0)$ and all $x, y \in X$. Since $L(0x) = 0 = 0L(x)$ for all $x \in X$, the unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ is an A-linear mapping. □

Corollary 2.5. *Let $\delta \geq 0$, $\{\epsilon_k\}_{k\in J}$ and $\{p_k\}_{k\in J}$ be real numbers such that $\epsilon_k \geq 0$ and $0 < p_k < 1$ for all $k \in J$, where $J \subseteq \{1, 2, \cdots, n\}$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{u,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_Y \leq \delta + \sum_{k\in J} \epsilon_k \|x_k\|_X^{p_k}$$

for all $x_1, \cdots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \begin{cases} M_{ij}(x), & i, j \in J; \\ M_i(x), & i \in J,\, j \notin J; \\ M_j(x), & j \in J,\, i \notin J; \\ M, & i, j \notin J. \end{cases}$$

for all $x \in X$, where

$$M_{ij}(x) = \frac{9}{2}\delta + \sum_{k\in\{i,j\}} \frac{(2 + 2^{p_k})\epsilon_k}{(2 - 2^{p_k})r_k^{p_k}} \|x\|_X^{p_k}$$

$$M_i(x) = \frac{9}{2}\delta + \frac{(2 + 2^{p_i})\epsilon_i}{(2 - 2^{p_i})r_i^{p_i}} \|x\|_X^{p_i}$$

$$M_j(x) = \frac{9}{2}\delta + \frac{(2 + 2^{p_j})\epsilon_j}{(2 - 2^{p_j})r_j^{p_j}} \|x\|_X^{p_j}, \quad M = \frac{9}{2}\delta.$$

Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

Proof. Define $\varphi(x_1, \cdots, x_n) := \delta + \sum_{k\in J} \epsilon_k \|x_k\|_X^{p_k}$, and apply Theorem 2.4. Then we get the desired result. □

Corollary 2.6. *Let $\delta, \epsilon \geq 0$, $p, q > 0$ with $\lambda = p + q < 1$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{u,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_Y \leq \delta + \epsilon \|x_i\|_X^p \|x_j\|_X^q$$

for all $x_1, \cdots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{9}{2}\delta + \frac{(2 + 2^\lambda)\epsilon}{2(2 - 2^\lambda) r_i^p r_j^q} \|x\|_X^\lambda$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \leq k \leq n$.

Proof. Define $\varphi(x_1, \cdots, x_n) := \delta + \epsilon \|x_i\|_X^p \|x_j\|_X^q$. Applying Theorem 2.4, we obtain the desired result. □

Theorem 2.7. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi_{ij}}(x,y) := \sum_{k=1}^{\infty} 2^k \phi\big(0, \cdots, \underbrace{\frac{x}{2^k}}_{i\,th}, 0, \cdots, \underbrace{\frac{y}{2^k}}_{j\,th}, 0, \cdots, 0\big) < \infty, \tag{2.21}$$

$$\lim_{k\to\infty} 2^k \phi\left(\frac{x_1}{2^k}, \cdots, \frac{x_n}{2^k}\right) = 0, \tag{2.22}$$

$$\|D_{e,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_Y \le \phi(x_1, \cdots, x_n) \tag{2.23}$$

for all $x, x_1, \cdots, x_n \in X$ and $y \in \{0, \pm x\}$. Then there exists a unique generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\begin{aligned}\|f(x) - L(x)\|_Y \le \frac{1}{4}\Big\{ &\big[\widetilde{\phi_{ij}}\big(\frac{2x}{r_i}, \frac{2x}{r_j}\big) + 2\widetilde{\phi_{ij}}\big(\frac{x}{r_i}, -\frac{x}{r_j}\big)\big] \\ &+ \big[\widetilde{\phi_{ij}}\big(\frac{2x}{r_i}, 0\big) + 2\widetilde{\phi_{ij}}\big(\frac{x}{r_i}, 0\big)\big] \\ &+ \big[\widetilde{\phi_{ij}}\big(0, \frac{2x}{r_j}\big) + 2\widetilde{\phi_{ij}}\big(0, -\frac{x}{r_j}\big)\big]\Big\}\end{aligned} \tag{2.24}$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. By a similar method to the proof of Theorem 2.3, we have the following inequality

$$\|f(2x) - 2f(x)\|_Y \le \Psi(x) \tag{2.25}$$

for all $x \in X$, where

$$\begin{aligned}\Psi(x) : = \frac{1}{2}\Big\{ &\big[\phi_{ij}\big(\frac{2x}{r_i}, \frac{2x}{r_j}\big) + 2\phi_{ij}\big(\frac{x}{r_i}, -\frac{x}{r_j}\big)\big] \\ &+ \big[\phi_{ij}\big(\frac{2x}{r_i}, 0\big) + 2\phi_{ij}\big(\frac{x}{r_i}, 0\big)\big] \\ &+ \big[\phi_{ij}\big(0, \frac{2x}{r_j}\big) + 2\phi_{ij}\big(0, -\frac{x}{r_j}\big)\big]\Big\}.\end{aligned}$$

It follows from (2.21) that

$$\begin{aligned}\sum_{k=1}^{\infty} 2^k \Psi\left(\frac{x}{2^k}\right) = \frac{1}{2}\Big\{ &\big[\widetilde{\phi_{ij}}\big(\frac{2x}{r_i}, \frac{2x}{r_j}\big) + 2\widetilde{\phi_{ij}}\big(\frac{x}{r_i}, -\frac{x}{r_j}\big)\big] \\ &+ \big[\widetilde{\phi_{ij}}\big(\frac{2x}{r_i}, 0\big) + 2\widetilde{\phi_{ij}}\big(\frac{x}{r_i}, 0\big)\big] \\ &+ \big[\widetilde{\phi_{ij}}\big(0, \frac{2x}{r_j}\big) + 2\widetilde{\phi_{ij}}\big(0, -\frac{x}{r_j}\big)\big]\Big\} < \infty\end{aligned} \tag{2.26}$$

for all $x \in X$. Replacing x by $\frac{x}{2^{k+1}}$ in (2.25) and multiplying both sides of (2.25) by 2^k, we get

$$\Big\|2^{k+1} f\big(\frac{x}{2^{k+1}}\big) - 2^k f\big(\frac{x}{2^k}\big)\Big\|_Y \le 2^k \Psi\big(\frac{x}{2^{k+1}}\big)$$

for all $x \in X$ and all $k \in \mathbb{Z}$. Therefore, we have

$$\begin{aligned}\Big\|2^{k+1}f\big(\frac{x}{2^{k+1}}\big)-2^m f\big(\frac{x}{2^m}\big)\Big\|_Y &\le \sum_{l=m}^{k}\Big\|2^{l+1}f\big(\frac{x}{2^{l+1}}\big)-2^l f\big(\frac{x}{2^l}\big)\Big\|_Y \\ &\le \sum_{l=m}^{k} 2^l\Psi\big(\frac{x}{2^{l+1}}\big)\end{aligned} \tag{2.27}$$

for all $x \in X$ and all integers $k \ge m$. It follows from (2.26) and (2.27) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy in Y for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Letting $m = 0$ in (2.27) and taking the limit as $k \to \infty$ in (2.27), we obtain the desired inequality (2.24).

The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 2.8. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\phi : X^n \to [0,\infty)$ satisfying (2.21), (2.22) and*

$$\|D_{u,r_1,\cdots,r_n} f(x_1,\cdots,x_n)\| \le \phi(x_1,\cdots,x_n) \tag{2.28}$$

for all $x_1,\cdots,x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ satisfying (2.24) for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. The proof is similar to the proof of Theorem 2.4. □

Corollary 2.9. *Let $\{\epsilon_k\}_{k\in J}$ and $\{p_k\}_{k\in J}$ be real numbers such that $\epsilon_k \ge 0$ and $p_k > 1$ for all $k \in J$, where $J \subseteq \{1,2,\cdots,n\}$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{u,r_1,\cdots,r_n} f(x_1,\cdots,x_n)\|_Y \le \sum_{k\in J} \epsilon_k \|x_k\|_X^{p_k}$$

for all $x_1,\cdots,x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x)-L(x)\|_Y \le \begin{cases} N_{ij}(x), & i,j \in J; \\ N_i(x), & i \in J,\ j \notin J; \\ N_j(x), & j \in J,\ i \notin J; \\ N, & i,j \notin J. \end{cases}$$

for all $x \in X$, where

$$\begin{aligned} N_{ij}(x) &= \sum_{k\in\{i,j\}} \frac{(2^{p_k}+2)\epsilon_k}{(2^{p_k}-2)r_k^{p_k}}\|x\|_X^{p_k} \\ N_i(x) &= \frac{(2^{p_i}+2)\epsilon_i}{(2^{p_i}-2)r_i^{p_i}}\|x\|_X^{p_i} \\ N_j(x) &= \frac{(2^{p_j}+2)\epsilon_j}{(2^{p_j}-2)r_j^{p_j}}\|x\|_X^{p_j}. \end{aligned}$$

Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. Define $\phi(x_1, \cdots, x_n) := \sum_{k\in J} \epsilon_k \|x_k\|_X^{p_k}$. Applying Theorem 2.8, we obtain the desired result. □

Corollary 2.10. *Let $\epsilon \ge 0$, $p, q > 0$ with $\lambda = p + q > 1$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{u,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_Y \le \epsilon\|x_i\|_X^p \|x_j\|_X^q$$

for all $x_1, \cdots, x_n \in X$ and all $u \in U(A)$. Then there exists a unique A-linear generalized Euler-Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{(2^\lambda + 2)\epsilon}{2(2^\lambda - 2)r_i^p r_j^q}\|x\|_X^\lambda$$

for all $x \in X$. Moreover, $L(r_k x) = r_k L(x)$ for all $x \in X$ and all $1 \le k \le n$.

Proof. Define $\phi(x_1, \cdots, x_n) := \epsilon\|x_i\|_X^p \|x_j\|_X^q$. Applying Theorem 2.8, we obtain the desired result. □

Remark 2.11. In Theorems 2.7, 2.8 and Corollaries 2.9, 2.10 one can assume that $\sum_{k=1}^n r_k \ne 0$ instead of $f(0) = 0$.

3. Homomorphisms in unital C^*-algebras

In this section, we investigate C^*-algebra homomorphisms in unital C^*-algebras. We will use the following lemma in the proof of the next theorem.

Lemma 3.1. [42] *Let $f : A \to B$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{S}^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$. Then the mapping $f : A \to B$ is $\mathbb{C}$-linear.*

Theorem 3.2. *Let $\epsilon \geq 0$ and $\{p_k\}_{k\in J}$ be real numbers such that $p_k > 0$ for all $k \in J$, where $J \subseteq \{1, 2, \cdots, n\}$ and $|J| \geq 3$. Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying (2.7) and*

$$\|D_{\mu,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_B \leq \epsilon \prod_{k\in J} \|x_k\|_A^{p_k} \tag{3.1}$$

$$\left\|f(2^k u^*) - f(2^k u)^*\right\|_B \leq \varphi(\underbrace{2^k u, \cdots, 2^k u}_{n \text{ times}}), \tag{3.2}$$

$$\left\|f(2^k ux) - f(2^k u)f(x)\right\|_B \leq \varphi(\underbrace{2^k ux, \cdots, 2^k ux}_{n \text{ times}}) \tag{3.3}$$

for all $x, x_1, \cdots, x_n \in A$, all $u \in U(A)$, all $k \in \mathbb{N}$ and all $\mu \in \mathbb{S}^1$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. Since $|J| \geq 3$, letting $\mu = 1$ and $x_k = 0$ for all $1 \leq k \leq n$, $k \neq i, j$, in (3.1), we get

$$f\left(\frac{-r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i - r_j x_j}{2}\right) + r_i f(x_i) + r_j f(x_j) = 2f\left(\frac{r_i x_i + r_j x_j}{2}\right)$$

for all $x_i, x_j \in A$. By the same reasoning as in the proof of Lemma 2.1, the mapping f is additive and $f(r_k x) = r_k f(x)$ for all $x \in A$ and $k = i, j$. So by letting $x_i = x$ and $x_k = 0$ for all $1 \leq k \leq n$, $k \neq i$, in (3.1), we get that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{S}^1$. Therefore, by Lemma 3.1, the mapping f is $\mathbb{C}$-linear. Hence it follows from (2.7), (3.2) and (3.3) that

$$\begin{aligned}
\|f(u^*) - f(u)^*\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \left\|f(2^k u^*) - f(2^k u)^*\right\|_B \\
&\leq \lim_{k\to\infty} \frac{1}{2^k} \varphi(\underbrace{2^k u, \cdots, 2^k u}_{n \text{ times}}) = 0,
\end{aligned}$$

$$\begin{aligned}
\|f(ux) - f(u)f(x)\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \left\|f(2^k ux) - f(2^k u)f(x)\right\|_B \\
&\leq \lim_{k\to\infty} \frac{1}{2^k} \varphi(\underbrace{2^k ux, \cdots, 2^k ux}_{n \text{ times}}) = 0
\end{aligned}$$

for all $x \in A$ and all $u \in U(A)$. So $f(u^*) = f(u)^*$ and $f(ux) = f(u)f(x)$ for all $x \in A$ and all $u \in U(A)$. Since f is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [23]), i.e., $x = \sum_{k=1}^m \lambda_k u_k$, where $\lambda_k \in \mathbb{C}$ and $u_k \in U(A)$ for all $1 \leq k \leq n$, we have

$$\begin{aligned}
f(x^*) &= f\left(\sum_{k=1}^m \overline{\lambda_k} u_k^*\right) = \sum_{k=1}^m \overline{\lambda_k} f\left(u_k^*\right) = \sum_{k=1}^m \overline{\lambda_k} f(u_k)^* \\
&= \left(\sum_{k=1}^m \lambda_k f(u_k)\right)^* = f\left(\sum_{k=1}^m \lambda_k u_k\right)^* \\
&= f(x)^*,
\end{aligned}$$

$$\begin{aligned} f(xy) &= f\left(\sum_{k=1}^{m}\lambda_k u_k y\right) = \sum_{k=1}^{m}\lambda_k f(u_k y) \\ &= \sum_{k=1}^{m}\lambda_k f(u_k) f(y) = f\left(\sum_{k=1}^{m}\lambda_k u_k\right) f(y) \\ &= f(x)f(y) \end{aligned}$$

for all $x, y \in A$.

Therefore, the mapping $f : A \to B$ is a C^*-algebra homomorphism. □

The following theorem is an alternative result of Theorem 3.2.

Theorem 3.3. *Let $\epsilon \geq 0$ and $\{p_k\}_{k\in J}$ be real numbers such that $p_k > 0$ for all $k \in J$, where $J \subseteq \{1, 2, \cdots, n\}$ and $|J| \geq 3$. Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying (2.22) and*

$$\|D_{\mu,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_B \leq \epsilon \prod_{k\in J} \|x_k\|_A^{p_k}$$

$$\Big\|f\big(\frac{u^*}{2^k}\big) - f\big(\frac{u}{2^k}\big)^*\Big\|_B \leq \phi\big(\underbrace{\frac{u}{2^k}, \cdots, \frac{u}{2^k}}_{n \ times}\big), \tag{3.4}$$

$$\Big\|f\big(\frac{ux}{2^k}\big) - f\big(\frac{u}{2^k}\big)f(x)\Big\|_B \leq \phi\big(\underbrace{\frac{ux}{2^k}, \cdots, \frac{ux}{2^k}}_{n \ times}\big) \tag{3.5}$$

for all $x, x_1, \cdots, x_n \in A$, all $u \in U(A)$, all $k \in \mathbb{N}$ and all $\mu \in \mathbb{S}^1$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Remark 3.4. In Theorems 3.2 and 3.3, one can assume that $\sum_{k=1}^{n} r_k \neq 0$ instead of $f(0) = 0$.

Theorem 3.5. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying (2.6), (2.7), (3.2), (3.3) and*

$$\|D_{\mu,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_B \leq \varphi(x_1, \cdots, x_n), \tag{3.6}$$

for all $x_1, \cdots, x_n \in A$ and all $\mu \in \mathbb{S}^1$. Assume that $\lim_{k\to\infty} \frac{1}{2^k} f(2^k e)$ is invertible. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. Consider the C^*-algebras A and B as left Banach modules over the unital C^*-algebra $\mathbb{C}$. By Theorem 2.4, there exists a unique $\mathbb{C}$-linear generalized Euler-Lagrange type additive mapping $H : A \to B$ defined by

$$H(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$$

for all $x \in A$. By (2.7), (3.2) and (3.3), we get

$$\begin{aligned} \|H(u^*) - H(u)^*\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \left\| f\left(2^k u^*\right) - f\left(2^k u\right)^* \right\|_B \\ &\le \lim_{k\to\infty} \frac{1}{2^k} \varphi(\underbrace{2^k u, \cdots, 2^k u}_{n \text{ times}}) = 0, \end{aligned}$$

$$\begin{aligned} \|H(ux) - H(u)f(x)\|_B &= \lim_{k\to\infty} \frac{1}{2^k} \left\| f\left(2^k ux\right) - f(2^k u)f(x) \right\|_B \\ &\le \lim_{k\to\infty} \frac{1}{2^k} \varphi(\underbrace{2^k ux, \cdots, 2^k ux}_{n \text{ times}}) = 0 \end{aligned}$$

for all $u \in U(A)$ and all $x \in A$. So $H(u^*) = H(u)^*$ and $H(ux) = H(u)f(x)$ for all $u \in U(A)$ and all $x \in A$. Therefore, by the additivity of H, we have

$$H(ux) = \lim_{k\to\infty} \frac{1}{2^k} H\left(2^k ux\right) = H(u) \lim_{k\to\infty} \frac{1}{2^k} f\left(2^k x\right) = H(u)H(x) \tag{3.7}$$

for all $u \in U(A)$ and all $x \in A$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{k=1}^m \lambda_k u_k$, where $\lambda_k \in \mathbb{C}$ and $u_k \in U(A)$ for all $1 \le k \le n$, it follows from (3.7) that

$$\begin{aligned} H(xy) &= H\left(\sum_{k=1}^m \lambda_k u_k y\right) = \sum_{k=1}^m \lambda_k H(u_k y) \\ &= \sum_{k=1}^m \lambda_k H(u_k) H(y) = H\left(\sum_{k=1}^m \lambda_k u_k\right) H(y) \\ &= H(x)H(y), \end{aligned}$$

$$\begin{aligned} H(x^*) &= H\left(\sum_{k=1}^m \overline{\lambda_k} u_k^*\right) = \sum_{k=1}^m \overline{\lambda_k} H(u_k^*) = \sum_{k=1}^m \overline{\lambda_k} H(u_k)^* \\ &= \left(\sum_{k=1}^m \lambda_k H(u_k)\right)^* = H\big(\sum_{k=1}^m \lambda_k u_k\big)^* \\ &= H(x)^* \end{aligned}$$

for all $x, y \in A$. Since $H(e) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k e)$ is invertible and

$$H(e)H(y) = H(ey) = H(e)f(y)$$

for all $y \in A$, $H(y) = f(y)$ for all $y \in A$.

Therefore, the mapping $f : A \to B$ is a C^*-algebra homomorphism. □

The following theorem is an alternative result of Theorem 3.5.

Theorem 3.6. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\phi : A^n \to [0, \infty)$ satisfying (2.21), (2.22), (3.4), (3.5) and*

$$\|D_{\mu, r_1, \cdots, r_n} f(x_1, \cdots, x_n)\|_B \le \phi(x_1, \cdots, x_n),$$

for all $x_1, \cdots, x_n \in A$ and all $\mu \in \mathbb{S}^1$. Assume that $\lim_{k\to\infty} 2^k f(\frac{e}{2^k})$ is invertible. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Corollary 3.7. *Let $\{\epsilon_k\}_{k\in J}$ and $\{p_k\}_{k\in J}$ be real numbers such that $\epsilon_k \geq 0$ and $p_k > 1\,(0 < p_k < 1)$ for all $k \in J$, where $J \subseteq \{1, 2, \cdots, n\}$. Assume that a mapping $f : A \to B$ with $f(0) = 0$ satisfies the inequalities*

$$\|D_{\mu,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_B \leq \sum_{k\in J} \epsilon_k \|x_k\|_A^{p_k},$$

$$\Big\| f\big(\frac{u^*}{2^m}\big) - f\big(\frac{u}{2^m}\big)^* \Big\|_B \leq \sum_{k\in J} \frac{\epsilon_k}{2^{mp_k}}$$

$$\Big(\textit{respectively,}\ \|f(2^m u^*) - f(2^m u)^*\|_B \leq \sum_{k\in J} \epsilon_k 2^{mp_k}\Big),$$

$$\Big\| f\big(\frac{ux}{2^m}\big) - f\big(\frac{u}{2^m}\big) f(x) \Big\|_B \leq \sum_{k\in J} \frac{\epsilon_k}{2^{mp_k}} \|x\|_A^{p_k}$$

$$\Big(\textit{respectively,}\ \|f(2^m ux) - f(2^m u) f(x)\|_B \leq \sum_{k\in J} \epsilon_k 2^{mp_k} \|x\|_A^{p_k}\Big)$$

for all $x_1, \cdots, x_n \in A$, all $u \in U(A)$, all $m \in \mathbb{N}$ and all $\mu \in \mathbb{S}^1$. Assume that $\lim_{k\to\infty} 2^k f(\frac{e}{2^k})$ $\big($respectively, $\lim_{k\to\infty} \frac{1}{2^k} f(2^k e)\big)$ is invertible. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. The result follows from Theorem 3.6 (respectively, Theorem 3.5). □

Remark 3.8. In Theorem 3.6 and Corollary 3.7, one can assume that $\sum_{k=1}^n r_k \neq 0$ instead of $f(0) = 0$.

Theorem 3.9. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : A^n \to [0, \infty)$ satisfying (2.6), (2.7), (3.2), (3.3) and*

$$\|D_{\mu,r_1,\cdots,r_n} f(x_1, \cdots, x_n)\|_B \leq \varphi(x_1, \cdots, x_n) \tag{3.8}$$

for $\mu = i, 1$ and all $x_1, \cdots, x_n \in A$. Assume that $\lim_{k\to\infty} \frac{1}{2^k} f(2^k e)$ is invertible and for each fixed $x \in A$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. Put $\mu = 1$ in (3.8). By the same reasoning as in the proof of Theorem 2.3, there exists a unique generalized Euler–Lagrange type additive mapping $H : A \to B$ defined by

$$H(x) = \lim_{k\to\infty} \frac{f(2^k x)}{2^k}$$

for all $x \in A$. By the same reasoning as in the proof of [58], the generalized Euler-Lagrange type additive mapping $H : A \to B$ is $\mathbb{R}$-linear.

By the same method as in the proof of Theorem 2.4, we have

$$\begin{aligned}
&\Big\| D_{\mu,r_1,\cdots,r_n} H(0,\cdots,0,\underbrace{x}_{j\text{ th}},0\cdots,0)\Big\|_Y \\
&= \lim_{k\to\infty} \frac{1}{2^k} \Big\| D_{\mu,r_1,\cdots,r_n} f(0,\cdots,0,\underbrace{2^k x}_{j\text{ th}},0\cdots,0)\Big\|_Y \\
&\le \lim_{k\to\infty} \frac{1}{2^k} \varphi(0,\cdots,0,\underbrace{2^k x}_{j\text{ th}},0\cdots,0) = 0
\end{aligned}$$

for all $x \in A$. So

$$r_j \mu H(x) = H(r_j \mu x)$$

for all $x \in A$. Since $H(r_j x) = r_j H(x)$ for all $x \in X$ and $r_j \neq 0$,

$$H(\mu x) = \mu H(x)$$

for $\mu = i, 1$ and all $x \in A$.

For each element $\lambda \in \mathbb{C}$ we have $\lambda = s + it$, where $s, t \in \mathbb{R}$. Thus

$$\begin{aligned}
H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) \\
&= sH(x) + itH(x) = (s+it)H(x) \\
&= \lambda H(x)
\end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in A$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$ and all $x, y \in A$. Hence the generalized Euler-Lagrange type additive mapping $H : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 3.5. □

The following theorem is an alternative result of Theorem 3.9.

Theorem 3.10. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there is a function $\phi : A^n \to [0, \infty)$ satisfying (2.21), (2.22), (3.4), (3.5) and*

$$\|D_{\mu,r_1,\cdots,r_n} f(x_1,\cdots,x_n)\|_B \le \phi(x_1,\cdots,x_n), \tag{3.9}$$

for $\mu = i, 1$ and all $x, x_1, \cdots, x_n \in A$. Assume that $\lim_{k\to\infty} 2^k f(\frac{e}{2^k})$ is invertible and for each fixed $x \in A$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Remark 3.11. In Theorem 3.10, one can assume that $\sum_{k=1}^{n} r_k \neq 0$ instead of $f(0) = 0$.

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