J. Nonlinear Sci. Appl. 3 (2010), no. 2, 148–151

The Journal of Nonlinear Science and Applications
http://www.tjnsa.com

A GENERALIZATION OF NADLER'S FIXED POINT THEOREM

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Abstract. In this paper, we prove a generalization of Nadler's fixed point theorem [S.B. Nadler Jr., Multi-valued contraction mappings, Pacific J. Math. 30 (1969) 475-487].

1. Introduction and preliminaries

Let (X, d) be a metric space. $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, and $x \in X$, define $D(x, A) := \inf\{d(x, a); a \in A\}$, and

$$H(A, B) := \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A).$$

It is easy to see that H is a metric on $CB(X)$. H is called the Hausdorff metric induced by d.

Definition 1.1. An element $x \in X$ is said to be a fixed point of a multi–valued mapping $T : X \to CB(X)$, if such that $x \in T(x)$.

One can show that $(CB(X), H)$ is a complete metric space, whenever (X, d) is a complete metric space (see for example Lemma 8.1.4, of [6]).

In 1969, Nadler [3] extended the Banach contraction principle [1] to set–valued mappings. In this paper among other things, we give a generalization of Nadler's fixed point theorem. The following lemma has important role in the proof of main theorem.

Date: Received: 24 Jan. 2010.

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2000 *Mathematics Subject Classification.* 54H25.

Key words and phrases. Hausdorff metric; Set–valued contraction; Nadler's fixed point theorem.

Lemma 1.2. *([3]) Let (X, d) be a metric space and $A, B \in CB(X)$. Then for each $a \in A$ and $\epsilon > 0$ there exists an $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \epsilon.$$

2. Main results

We start our work with our main result, which can be regarded as an extension of Nadler's fixed point theorem.

Theorem 2.1. *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that*

$$H(Tx, Ty) \leq \alpha d(x, y) + \beta[D(x, Tx) + D(y, Ty)] + \gamma[D(x, Ty) + D(y, Tx)]$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Proof. Let $x_0 \in X, x_1 \in Tx_0$ and define $r := \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}$. If $r = 0$ then proof is clear. Now, assume $r > 0$, then it follows from Lemma 1.2 that

$$\begin{cases} \exists x_2 \in Tx_1; & d(x_1, x_2) \leq H(Tx_0, Tx_1) + r, \\ \exists x_3 \in Tx_2; & d(x_2, x_3) \leq H(Tx_1, Tx_2) + r^2, \\ . & \\ . & \\ . & \\ \exists x_{n+1} \in Tx_n; & d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + r^n. \end{cases}$$

Hence, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) + r^n \\ &\leq \alpha d(x_{n-1}, x_n) + \beta[D(x_n, Tx_n) + D(x_{n-1}, Tx_{n-1})] \\ &\quad + \gamma[D(x_n, Tx_{n-1}) + D(x_{n-1}, Tx_n))] + r^n \\ &\leq \alpha d(x_{n-1}, x_n) + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &\quad + \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + r^n \end{aligned}$$

for all $n \in \mathbf{N}$. It follows that

$$d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n) + \frac{r^n}{1 - (\beta + \gamma)}$$

for all $n \in \mathbf{N}$. It can be conclude that

$$d(x_n, x_{n+1}) \leq r^n d(x_0, x_1) + \frac{n r^n}{1 - (\beta + \gamma)}$$

for all $n \in \mathbf{N}$. Now, since $r < 1$, then $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. We are going to show that x^* is a fixed point of T. We have

$$\begin{aligned} D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + D(x_{n+1}, Tx^*) \leq d(x^*, x_{n+1}) + H(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta[D(x_n, Tx_n) + D(x^*, Tx^*)] \\ &\quad +\gamma[D(x_n, Tx^*) + D(x^*, Tx_n)] \end{aligned}$$

for all $n \in \mathbf{N}$. Therefore,

$$\begin{aligned} D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta[d(x_n, x_{n+1}) + D(x^*, Tx^*)] \\ &\quad +\gamma[D(x_n, Tx^*) + d(x_{n+1}, x^*)] \end{aligned}$$

for all $n \in \mathbf{N}$. Passing the limit $n \to \infty$ in (1), then we have

$$D(x^*, Tx^*) \leq (\beta + \gamma) D(x^*, Tx^*).$$

On the other hand $\beta + \gamma < 1$, then $D(x^*, Tx^*) = 0$. It follows that $x^* \in Tx^*$. $\square$

Corollary 2.2. ([2]; page 201) *Let (X, d) be a complete metric space and let T be a mapping from X into X such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Corollary 2.3. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \leq a_1 d(x, y) + a_2 D(x, Tx) + a_3 D(y, Ty) + a_4 D(x, Ty) + a_5 D(y, Tx)$$

for all $x, y \in X$, where $a_i \geq 0$ for each $i \in \{1, 2, \cdots, 5\}$ and $\sum_{i=1}^{5} a_i < 1$. Then T has a fixed point.

Corollary 2.4. (Nadler [3]) *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then T has a fixed point.

Corollary 2.5. ([4]; page 5 and [5]; Page 31) *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \leq \beta[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where $\beta \in [0, \frac{1}{2})$. Then T has a fixed point.

Corollary 2.6. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \leq \gamma[D(x, Ty) + D(y, Tx)]$$

for all $x, y \in X$, where $\gamma \in [0, \frac{1}{2})$. Then T has a fixed point.

Corollary 2.7. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \leq \alpha d(x, y) + \beta[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where $\alpha + 2\beta < 1$. Then T has a fixed point.

References

[1] S. Banach, Sure operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math. 3 (1922) 133–181.
[2] G. E. Hardy and T. D. Rogers, A generalization of a fixed point theorem of Reich, Canad. Math. Bull. 16 (1973), 201–206.
[3] N.B. Nadler Jr., Multi–valued contraction mappings, Pacific J. Math. 30 (1969) 475–488.
[4] S. Reich, Kannan's fixed point theorem, Boll. Un. Mat. Ital. 4 (1971), 1–11.
[5] S. Reich, Fixed points of contractive functions, Boll. Un. Mat. Ital. 5 (1972), 26–42.
[6] I.A. Rus, *Generalized Contractions and Applications*, Cluj Univercity Press, Cluj–Nappa, 2001.

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