

SOME COMMON FIXED POINT RESULTS IN NON-NORMAL CONE METRIC SPACES

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ABSTRACT. The aim of this paper is to obtain extended variants of some common fixed point results in cone metric spaces in the case that the underlying cone is not normal. The first result concerns g -quasicontractions of D. Ilić and V. Rakočević [*Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882], and the second is concerned with Hardy-Rogers-type conditions and extends some recent results of M. Abbas, B. E. Rhoades and T. Nazir [*Common fixed points for four maps in cone metric spaces*, Appl. Math. Comput. **216** (2010), 80–86].

1. INTRODUCTION AND PRELIMINARIES

In 2007, Huang and Zhang [7] reintroduced the concept of a cone metric space, replacing the set of real numbers by an ordered Banach space in the definition of metric, and obtained some fixed point theorems for contractive type mappings. Since then, there has been a lot of activity in this area and several interesting fixed point results have appeared in cone metric spaces (e.g., [1, 2, 3, 7, 8, 9, 11, 12, 13, 14, 15]). Most of the papers dealt with fixed point results in normal cone metric spaces ([1, 2, 7, 8, 9]). In this paper, unlike other authors, we do not impose the normality condition on cones. Our proofs only require the assumption that the interior of the cone be non-empty (such cones are usually called *solid*). It is worth mentioning that we neither use continuity of the vector metric d nor the Sandwich Theorem.

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The following definitions and results will be needed in the sequel.

Definition 1.1. [6] Let E be a real Banach space. A subset P of E is called a *cone* if:

- (a) P is closed, non-empty and $P \neq \{\theta\}$;
- (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$;
- (c) $P \cap (-P) = \{\theta\}$.

For a given cone P , we define the partial ordering \preceq with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in \text{int } P$, where $\text{int } P$ stands for the interior of P and use $x \prec y$ for $x \preceq y$ and $x \neq y$.

There exist two kinds of cones, normal and non-normal ones. The cone P in the normed space E is called *normal* whenever there is a number $k > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. For details see [6].

Definition 1.2. [7] Let X be a non-empty set. Suppose that a mapping $d : X \times X \rightarrow E$ satisfies:

- (d₁) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (d₂) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d₃) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The function d is called a *cone metric* and (X, d) is called a *cone metric space*.

The concept of a cone metric space is more general than that of a metric space. For basic notions concerning cone metric spaces we refer to [7].

Let (X, d) be a cone metric space. The following properties are often used, particularly in the case when the underlying cone is non-normal, so that some standard tools as the continuity of the metric and the Sandwich Theorem cannot be used. The only assumption is that the cone P is solid, i.e., its interior is nonempty.

- (p₁) If $a \preceq ha$ where $a \in P$ and $h \in [0, 1)$, then $a = \theta$.
- (p₂) If $\theta \preceq u \ll c$ for each c , $\theta \ll c$, then $u = \theta$.
- (p₃) If $a \preceq b + c$ for each c , $\theta \ll c$, then $a \preceq b$.
- (p₄) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
- (p₅) If $c \in \text{int } P$, $\theta \preceq a_n$ and $a_n \rightarrow \theta$, then there exists $k \in \mathbb{N}$ such that for all $n > k$ we have $a_n \ll c$.

For details about these properties see [11, 13].

The aim of this paper is to obtain extended variants of some common fixed point results in cone metric spaces in the case that the underlying cone is not normal. The first result concerns so-called g -quasicontractions of Ilić and Rakočević [8], and the second is concerned with Hardy-Rogers-type conditions and extends some recent results of Abbas, Rhoades and Nazir [3].

2. RESULTS

2.1. Common fixed points of g -quasicontractions. The following theorem gives a proper generalization of results from [4, 5, 8, 9, 14, 15] (see Example 2.4 given below). Our main result is obtained without using the normality condition and without making use of the notions of commutativity and continuity.

Recall the definition of g -quasicontractions on cone metric spaces from [8]. Such mappings are generalizations of Das-Naik's mappings from the frame of metric spaces [5].

Definition 2.1. [8] Let (X, d) be a cone metric space, and $f, g : X \rightarrow X$. Then, f is called a g -quasi-contraction if for some constant $\lambda \in (0, 1)$ and for all $x, y \in X$, there exists

$$u \in \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\},$$

such that

$$d(fx, fy) \preceq \lambda \cdot u. \quad (2.1)$$

It was proved in [8], using normality of the cone, that f and g have a unique common fixed point in X , if $fX \subset gX$, $g(X)$ is complete, f commutes with g , f or g is continuous, and they satisfy (2.1). The normality condition was later removed in the case $g = i_X$ in [13] and [15]. The case of g -quasicontractions under the assumption $0 \leq \lambda < \frac{1}{2}$ was treated in [14].

Theorem 2.2. Let (X, d) be a cone metric space over a solid cone P , and let $f, g : X \rightarrow X$ be such that $fX \subset gX$ and gX is complete. If f is a g -quasi-contraction, then f and g have a coincidence point in X . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Recall that a point $y \in X$ is called a point of coincidence for mappings $f, g : X \rightarrow X$ if there exists $x \in X$ such that $f(x) = g(x) = y$. Then, x is called a coincidence point. Mappings f and g are called weakly compatible if they commute at their coincidence points.

Proof. The condition $fX \subset gX$ implies that starting with an arbitrary $x_0 \in X$, we can construct a sequence $\{y_n\}$ of points in X such that $y_n = fx_n = gx_{n+1}$ for all $n \geq 0$. We shall prove that $\{y_n\}$ is a Cauchy sequence in X . Consider two cases.

1. First, suppose that there exists an n_0 such that $y_{n_0} = y_{n_0+1}$. Then sequence $\{y_n\}$ is constant for $n \geq n_0$. Indeed, from (2.1) it follows that

$$d(y_{n_0+1}, y_{n_0+2}) = d(fx_{n_0+1}, fx_{n_0+2}) \preceq \lambda \cdot u, \quad (2.2)$$

where

$$\begin{aligned} u &\in \{d(gx_{n_0+1}, gx_{n_0+2}), d(gx_{n_0+1}, fx_{n_0+1}), d(gx_{n_0+2}, fx_{n_0+2}), \\ &\quad d(gx_{n_0+1}, fx_{n_0+2}), d(gx_{n_0+2}, fx_{n_0+1})\} \\ &= \{d(y_{n_0}, y_{n_0+1}), d(y_{n_0}, y_{n_0+1}), d(y_{n_0+1}, y_{n_0+2}), d(y_{n_0}, y_{n_0+2}), d(y_{n_0+1}, y_{n_0+1})\} \\ &= \{\theta, \theta, d(y_{n_0+1}, y_{n_0+2}), d(y_{n_0}, y_{n_0+2}), \theta\} = \{\theta, d(y_{n_0+1}, y_{n_0+2}), d(y_{n_0}, y_{n_0+2})\}. \end{aligned}$$

Taking into account (2.2), we have to consider the following cases:

- 1⁰ $d(y_{n_0+1}, y_{n_0+2}) \preceq \lambda \cdot \theta = \theta$, which implies that $y_{n_0+1} = y_{n_0+2}$,
- 2⁰ $d(y_{n_0+1}, y_{n_0+2}) \preceq \lambda \cdot d(y_{n_0+1}, y_{n_0+2}) \Rightarrow d(y_{n_0+1}, y_{n_0+2}) = \theta$, that is, $y_{n_0+1} = y_{n_0+2}$,
- 3⁰ $d(y_{n_0+1}, y_{n_0+2}) \preceq \lambda \cdot d(y_{n_0}, y_{n_0+2}) \preceq \lambda \cdot d(y_{n_0}, y_{n_0+1}) + \lambda \cdot d(y_{n_0+1}, y_{n_0+2}) = \lambda d(y_{n_0+1}, y_{n_0+2}) \Rightarrow d(y_{n_0+1}, y_{n_0+2}) = \theta$, that is, $y_{n_0+1} = y_{n_0+2}$.

Continuing this process, we obtain that $y_n = y_{n_0}$ for each $n \geq n_0$. Therefore, in this case $\{y_n\}$ turns out to be an eventually constant sequence and hence a Cauchy one.

2. Suppose now that $y_n \neq y_{n+1}$ for all $n \geq 0$. We shall show that for each $n \in \mathbb{N}$, $n \geq 2$, there exist $1 \leq m \leq n$ such that

$$d(y_n, y_{n-1}) \preceq \lambda^{n-1} d(y_0, y_m). \quad (2.3)$$

For the proof of inequality (2.3), we use the following relation:

$$d(y_n, y_{n-1}) \preceq \lambda^{n-1} d(y_{n-1}, y_{n-2}) \text{ or } d(y_n, y_{n-1}) \preceq \lambda^{n-1} d(y_{n-2}, y_n). \quad (2.4)$$

Indeed, according to (2.1), we have

$$d(y_n, y_{n-1}) = d(fx_n, fx_{n-1}) \preceq \lambda \cdot u_{n,n-1},$$

where

$$\begin{aligned} & u_{n,n-1} \\ \in & \{d(gx_n, gx_{n-1}), d(gx_n, fx_n), d(gx_{n-1}, fx_{n-1}), d(gx_n, fx_{n-1}), d(gx_{n-1}, fx_n)\} \\ = & \{d(y_{n-1}, y_{n-2}), d(y_{n-1}, y_n), d(y_{n-2}, y_{n-1}), d(y_{n-1}, y_{n-1}), d(y_{n-2}, y_n)\} \\ = & \{d(y_{n-1}, y_{n-2}), d(y_{n-1}, y_n), \theta, d(y_{n-2}, y_n)\}. \end{aligned}$$

Since, $y_n \neq y_{n+1}$ for all $n \geq 0$ it follows that $u_{n,n-1} \in \{d(y_{n-1}, y_{n-2}), d(y_{n-2}, y_n)\}$. Hence, (2.4) holds.

Following the arguments given in the proof of [15, Theorem 2.1], we can prove (2.3). This means that for each $a, b \in \{y_0, y_1, y_2, \dots\}$ there exists $\delta(x_0) \in P$ such that $d(a, b) \preceq \delta(x_0)$ where $\delta(x_0) = \left(\prod_{i=1}^{\infty} \frac{1+\lambda^i}{1-\lambda^i}\right) d(y_0, y_1)$.

Based on this we get that $\{y_n\}$ is a Cauchy sequence. Indeed, for $k > n \geq 1$

$$\begin{aligned} d(y_n, y_k) & \preceq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{k-1}, y_k) \\ & \preceq \lambda^n d(y_0, y_{m_1}) + \lambda^{n+1} d(y_0, y_{m_2}) + \dots + \lambda^{k-1} d(y_0, y_{m_k}) \\ & \preceq (\lambda^n + \lambda^{n+1} + \dots + \lambda^{k-1}) \delta(x_0) \\ & = \lambda^n \frac{1 - \lambda^{k-n}}{1 - \lambda} \delta(x_0) \preceq \frac{\lambda^n}{1 - \lambda} \delta(x_0) \rightarrow \theta, \text{ as } n \rightarrow \infty. \end{aligned}$$

By properties (p₅) and (p₄) it follows that $\{y_n\}$ is a Cauchy sequence. Therefore, $y_n \rightarrow gz \in gX$ for some $z \in X$.

We shall prove that $fx = gz$. Taking $x = x_n, y = z$ in (2.1) we get

$$d(fx_n, fz) \preceq \lambda \cdot u_n,$$

where

$$u_n \in \{d(gx_n, gz), d(gx_n, fx_n), d(gz, fz), d(gx_n, fz), d(gz, fx_n)\}.$$

Let $c, \theta \ll c$, be given. Then we have the following cases:

- 1^o $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, gz) \ll \lambda \frac{c}{\lambda} = c$, for $n > n_0 = n_0(c)$.
- 2^o $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fx_n) \ll \lambda \frac{c}{\lambda} = c$, for $n > n_1 = n_1(c)$.
- 3^o $d(fx_n, fz) \preceq \lambda \cdot d(gz, fz) \preceq \lambda d(gz, fx_n) + \lambda d(fx_n, fz) \Rightarrow d(fx_n, fz) \preceq \frac{1}{1-\lambda} d(gz, fx_n) \ll \frac{1}{1-\lambda} \cdot (1-\lambda)c = c$, for $n > n_2 = n_2(c)$.

$$4^0 \quad d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fz) \preceq \lambda d(gx_n, fx_n) + \lambda d(fx_n, fz) \Rightarrow d(fx_n, fz) \preceq \frac{\lambda}{1-\lambda} d(gx_n, fx_n) \ll \frac{\lambda}{1-\lambda} \cdot \frac{(1-\lambda)c}{\lambda} = c, \text{ for } n > n_3 = n_3(c).$$

$$5^0 \quad d(fx_n, fz) \preceq \lambda \cdot d(gz, fx_n) \ll \lambda \frac{c}{\lambda} = c, \text{ for } n > n_4 = n_4(c).$$

Hence, $fx_n \rightarrow fz$, i.e., $fz = gz$ and f, g have a coincidence point. Let $fz = w = gz$. Since f and g are weakly compatible, then $gw = gfz = fgz = fw$.

Now we shall show that $fw = w$. For this we have

$$d(w, fw) = d(fz, fw) \preceq \lambda \cdot u(z, w),$$

where

$$\begin{aligned} u(z, w) &\in \{d(gz, gw), d(gz, fz), d(gw, fw), d(gz, fw), d(gw, fz)\} \\ &= \{d(fz, fw), \theta, \theta, d(fz, fw), d(fw, fz)\} = \{d(fz, fw), \theta\}. \end{aligned}$$

Hence, $d(fz, fw) \preceq \lambda \cdot d(fz, fw)$ or $d(fz, fw) \preceq \lambda \cdot \theta = \theta$, which implies that $fz = fw$, and w is a common fixed point of f and g .

For the proof of uniqueness, suppose that w_1 is also a fixed point of f and g . Then from (2.1),

$$d(w, w_1) = d(fw, fw_1) \preceq \lambda \cdot u(w, w_1),$$

where

$$\begin{aligned} u(w, w_1) &\in \{d(gw, gw_1), d(gw, fw), d(gw_1, fw_1), d(gw, fw_1), d(gw_1, fw)\} \\ &= \{d(w, w_1), d(w, w), d(w_1, w_1), d(w, w_1), d(w_1, w)\} = \{d(w, w_1), \theta\}. \end{aligned}$$

Hence, we obtain that $d(w, w_1) \preceq \lambda \cdot d(w, w_1)$ or $d(w, w_1) \preceq \theta$, which implies that $w = w_1$. \square

Remark 2.3. Since $fz = w = gz$, it follows that w is a point of coincidence of maps f and g . If this point of coincidence is unique, then, by [1, Proposition 1.4], w is the unique fixed point of f and g .

To prove that this is the case, let w_1 be another point of coincidence of f and g . This means that there exist z and z_1 such that $fz = gz = w$ and $fz_1 = gz_1 = w_1$. Hence, we have

$$d(w, w_1) = d(fz, fz_1) \preceq \lambda \cdot u(z, z_1),$$

where

$$\begin{aligned} u(z, z_1) &\in \{d(gz, gz_1), d(gz, fz), d(gz_1, fz_1), d(gz, fz_1), d(gz_1, fz)\} \\ &= \{d(w, w_1), d(w, w), d(w_1, w_1), d(w, w_1), d(w_1, w)\} = \{d(w, w_1), \theta\}. \end{aligned}$$

Now, we get that $d(w, w_1) \preceq \lambda \cdot d(w, w_1)$ or $d(w, w_1) \preceq \lambda \cdot \theta = \theta$, which implies that $w = w_1$.

In the previous theorem the cone P in ordered Banach space E is only supposed to have non-empty interior (it is not necessarily normal). If $g = i_X$, identity mapping on X , we obtain a result from [15]. If $E = \mathbb{R}$, $P = [0, +\infty)$, $\|\cdot\| = |\cdot|$, we obtain the results of Ćirić [4] if $g = i_X$ and Das-Naik [5] if $g \neq i_X$, for metric spaces.

Example 2.4. This example verifies that Theorem 2.2 is a proper generalization of results from [5] and [8].

Let $X = [0, 1]$, $E = C_{\mathbb{R}}^1[0, 1]$ with $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$ and $P = \{x \in E : x(t) \geq 0\}$. Define $d : X \times X \rightarrow E$ by $d(x, y)(t) := |x - y|e^t$. It is easy to see that d is a cone metric on X and (X, d) is a complete cone metric space. Take functions $fx = ax$, $gx = bx$, $0 < a < b < 1$ which map the set X into X . The mapping f is a g -quasicontraction, but it is not a quasicontraction. Indeed, if $\lambda \in [\frac{a}{b}, 1)$ then for each $x, y \in X$ there exists $u(x, y) = d(gx, gy)$ such that

$$d(fx, fy) \preceq \lambda \cdot u(x, y) = \lambda d(gx, gy) \Leftrightarrow |fx - fy|e^t \leq \lambda |gx - gy|e^t \Leftrightarrow a \leq \lambda b.$$

The mappings f and g are weakly compatible since they commute at their fixed point $x = 0$. All the conditions of Theorem 2.2 are fulfilled, and so the mappings f and g have a unique common fixed point $x = 0$. On the other hand, since the cone P is non-normal, in this case it is not possible to apply results from [5] and [8].

Remark 2.5. Let $\bar{y} = gz = \lim_{k \rightarrow \infty} y_k$. Then we have the following estimate for the distance between y_n , $n \in \mathbb{N}$ and \bar{y} :

$$d(y_n, \bar{y}) \preceq \frac{\lambda^n}{1 - \lambda} \delta(x_0).$$

Indeed,

$$d(y_n, \bar{y}) \preceq d(y_n, y_k) + d(y_k, \bar{y}) \preceq \frac{\lambda^n}{1 - \lambda} \delta(x_0) + d(y_k, \bar{y}).$$

Since, $y_k \rightarrow \bar{y}$ as $k \rightarrow \infty$ then for each $c \in \text{int } P$ there exists k_0 such that $k > k_0$: $d(y_k, \bar{y}) \ll c$. Now, according to property (p₃), we obtain that

$$d(y_n, \bar{y}) \preceq \frac{\lambda^n}{1 - \lambda} \delta(x_0).$$

2.2. Hardy-Rogers-type results. Our next theorem extends and slightly improves Theorem 2.8 and Corollaries 2.10–2.12 from [3].

Theorem 2.6. *Let f, g, S and T be self-maps on a cone metric space (X, d) with a solid cone, satisfying that $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$d(fx, gy) \preceq pd(Sx, Ty) + qd(fx, Sx) + rd(gy, Ty) + sd(fx, Ty) + td(gy, Sx) \quad (2.5)$$

for all $x, y \in X$, where p, q, r, s, t are nonnegative real numbers with $p + q + r + s + t < 1$, and $q = r$ or $s = t$. If one of $f(X)$, $g(X)$, $S(X)$, or $T(X)$ is a complete subspace of X , then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X . Moreover, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. For an arbitrary point $x_0 \in X$, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$fx_{2n-2} = Tx_{2n-1} = y_{2n-1}, \text{ and } gx_{2n-1} = Sx_{2n} = y_{2n}.$$

We have from (2.5),

$$\begin{aligned}
d(y_{2n+1}, y_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\
&\preceq pd(Sx_{2n}, Tx_{2n+1}) + qd(fx_{2n}, Sx_{2n}) + rd(gx_{2n+1}, Tx_{2n+1}) \\
&\quad + sd(fx_{2n}, Tx_{2n+1}) + td(gx_{2n+1}, Sx_{2n}) \\
&= pd(y_{2n}, y_{2n+1}) + qd(y_{2n+1}, y_{2n}) + rd(y_{2n+2}, y_{2n+1}) \\
&\quad + sd(y_{2n+1}, y_{2n+1}) + td(y_{2n+2}, y_{2n}),
\end{aligned}$$

and so

$$d(y_{2n+1}, y_{2n+2}) \preceq \frac{p+q+t}{1-r-t} d(y_{2n}, y_{2n+1}).$$

Also

$$\begin{aligned}
d(y_{2n+1}, y_{2n}) &= d(fx_{2n}, gx_{2n-1}) \\
&\preceq pd(Sx_{2n}, Tx_{2n-1}) + qd(fx_{2n}, Sx_{2n}) + rd(gx_{2n-1}, Tx_{2n-1}) \\
&\quad + sd(fx_{2n}, Tx_{2n-1}) + td(gx_{2n-1}, Sx_{2n}) \\
&= pd(y_{2n}, y_{2n-1}) + qd(y_{2n+1}, y_{2n}) + rd(y_{2n}, y_{2n-1}) \\
&\quad + sd(y_{2n+1}, y_{2n-1}) + td(y_{2n}, y_{2n}),
\end{aligned}$$

i.e.,

$$d(y_{2n+1}, y_{2n}) \preceq \frac{p+r+s}{1-q-s} d(y_{2n}, y_{2n-1}).$$

Now by induction, we get that, for each n

$$\begin{aligned}
d(y_{2n+1}, y_{2n+2}) &\preceq \frac{p+q+t}{1-r-t} d(y_{2n}, y_{2n+1}) \preceq \frac{p+q+t}{1-r-t} \frac{p+r+s}{1-q-s} d(y_{2n-1}, y_{2n}) \\
&\preceq \frac{p+q+t}{1-r-t} \frac{p+r+s}{1-q-s} \frac{p+q+t}{1-r-t} d(y_{2n-2}, y_{2n/1}) \\
&\preceq \dots \preceq \frac{p+q+t}{1-r-t} \left(\frac{p+r+s}{1-q-s} \frac{p+q+t}{1-r-t} \right)^n d(y_0, y_1),
\end{aligned}$$

and

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &\preceq \frac{p+r+s}{1-q-s} d(y_{2n}, y_{2n-1}) \preceq \frac{p+r+s}{1-q-s} \frac{p+q+t}{1-r-t} d(y_{2n-1}, y_{2n-2}) \\
&\preceq \dots \preceq \left(\frac{p+r+s}{1-q-s} \frac{p+q+t}{1-r-t} \right)^n d(y_0, y_1).
\end{aligned}$$

Let $a = \frac{p+q+t}{1-r-t}$, $b = \frac{p+r+s}{1-q-s}$. If $q = r$, then

$$a \cdot b = \frac{p+q+t}{1-q-t} \frac{p+r+s}{1-q-s} = \frac{p+q+t}{1-r-s} \frac{p+q+s}{1-r-t} < 1 \cdot 1 = 1,$$

and if $s = t$, then

$$a \cdot b = \frac{p+q+t}{1-q-t} \frac{p+r+s}{1-q-s} = \frac{p+q+t}{1-q-s} \frac{p+r+s}{1-q-t} < 1 \cdot 1 = 1.$$

Now, for $n < m$, we have

$$\begin{aligned} d(y_{2n+1}, y_{2m+1}) &\preceq d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, y_{2n+3}) + \cdots + d(y_{2m}, y_{2m+1}) \\ &\preceq \left(a \sum_{i=n}^{m-1} (ab)^i + \sum_{i=n+1}^m (ab)^i \right) d(y_0, y_1) \\ &\preceq \left(\frac{a(ab)^n}{1-ab} + \frac{(ab)^{n+1}}{1-ab} \right) d(y_0, y_1) = (1+b) \frac{a(ab)^n}{1-ab} d(y_0, y_1). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} d(y_{2n}, y_{2m+1}) &\preceq (1+a) \frac{(ab)^n}{1-ab} d(y_0, y_1), \\ d(y_{2n}, y_{2m}) &\preceq (1+a) \frac{(ab)^n}{1-ab} d(y_0, y_1), \\ d(y_{2n+1}, y_{2m}) &\preceq (1+b) \frac{a(ab)^n}{1-ab} d(y_0, y_1). \end{aligned}$$

Hence, there exists $p > 0$ such that for $p < n < m$,

$$d(y_n, y_m) \preceq \max \left\{ (1+a) \frac{(ab)^p}{1-ab}, (1+b) \frac{a(ab)^p}{1-ab} \right\} d(y_0, y_1) \rightarrow \theta, \text{ as } p \rightarrow \infty.$$

According to properties (p₅) and (p₄) $\{y_n\}$ is a Cauchy sequence. Suppose that $S(X)$ is complete. Then there exists a point u in $S(X)$ such that $Sx_{2n} = y_{2n} \rightarrow u$ as $n \rightarrow \infty$. Consequently, we can find v in X such that $Sv = u$.

Now we shall show that $fv = u$. Using (2.5),

$$\begin{aligned} d(fv, u) &\preceq d(fv, gx_{2n-1}) + d(gx_{2n-1}, u) \\ &\preceq pd(Sv, Tx_{2n-1}) + qd(fv, Sv) + rd(gx_{2n-1}, Tx_{2n-1}) \\ &\quad + sd(fv, Tx_{2n-1}) + td(gx_{2n-1}, Sv) + d(gx_{2n-1}, u). \end{aligned}$$

This implies that

$$d(fv, u) \preceq \frac{1+r+s}{1-q-s} d(u, Tx_{2n-1}) + \frac{1+r+t}{1-q-s} d(gx_{2n-1}, u).$$

Let $\theta \ll c$ be given. Since $y_n \rightarrow u$, there exists $N = N(c)$ such that for each $n > N$ we have that

$$d(u, Tx_{2n-1}) \ll \frac{1-q-s}{1+r+s} \frac{c}{2} \text{ and } d(gx_{2n-1}, u) \ll \frac{1-q-s}{1+r+t} \frac{c}{2},$$

which implies that $d(fv, u) \ll c$. By property (p₂) it follows that $fv = u$.

Since $u \in T(X)$, we can find a point w in X such that $Tw = u$. Now we shall show that $gw = u$. Using (2.5), we have

$$\begin{aligned} d(gw, u) &\preceq d(fx_{2n}, gw) + d(fx_{2n}, u) \\ &\preceq d(fx_{2n}, u) + pd(Sx_{2n}, Tw) + qd(fx_{2n}, Sx_{2n}) \\ &\quad + rd(gw, Tw) + sd(fx_{2n}, Tw) + td(gw, Sx_{2n}) \end{aligned}$$

and so

$$d(gw, u) \preceq \frac{1+q+s}{1-r-t} d(fx_{2n}, u) + \frac{p+q+t}{1-r-t} d(Sx_{2n}, u).$$

Following arguments similar to those given above, we obtain $d(gw, u) \ll c$ for each $\theta \ll c$. Hence, $gw = Tw = fv = Sv = u$. If the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then

$$\begin{aligned} fu &= fSv = Sfv = Su = w_1 \text{ (say), and} \\ gu &= gTw = Tgw = Tu = w_2 \text{ (say).} \end{aligned}$$

Again, using (2.5),

$$\begin{aligned} d(w_1, w_2) &= d(fu, gu) \\ &\preceq pd(Su, Tu) + qd(fu, Su) + rd(gu, Tu) + sd(fu, Tu) + td(gu, Su) \\ &= pd(w_1, w_2) + q \cdot \theta + r \cdot \theta + sd(w_1, w_2) + td(w_1, w_2) \\ &= (p + s + t)d(w_1, w_2), \end{aligned}$$

which implies $w_1 = w_2$. Therefore $fu = gu = Su = Tu$. Now we show that $u = gu$. Using (2.5),

$$\begin{aligned} d(u, gu) &= d(fv, gu) \\ &\preceq pd(Sv, Tu) + qd(fv, Sv) + rd(gu, Tu) + sd(fv, Tu) + td(gu, Sv) \\ &= (p + s + t)d(u, gu), \end{aligned}$$

which implies that $gu = u$. Hence, u is a common fixed point of f, g, S and T . Clearly, (2.5) implies the uniqueness of common fixed point. The proofs for the cases when $g(X)$, $f(X)$ or $T(X)$ is complete are similar, and therefore are omitted. \square

The following corollary extends and improves results in [2] and [14].

Corollary 2.7. *Let f, g and T be self-maps on a cone metric space (X, d) over a solid cone, satisfying $f(X) \cup g(X) \subset T(X)$ and*

$$d(fx, gy) \preceq pd(Tx, Ty) + qd(fx, Tx) + rd(gy, Ty) + sd(fx, Ty) + td(gy, Tx)$$

for all $x, y \in X$, where p, q, r, s, t are nonnegative real numbers with $p + q + r + s + t < 1$, and $q = r$ or $s = t$. If one of $f(X)$, $g(X)$ or $T(X)$ is a complete subspace of X , then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence in X . Moreover, if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g and T have a unique common fixed point.

The following two corollaries extend and improve results in [1] and [11]. Note that the case of ordered cone metric spaces and weakly increasing functions was treated in [12].

Corollary 2.8. *Let f and T be self-maps on a cone metric space (X, d) over a solid cone, satisfying $f(X) \subset T(X)$ and*

$$d(fx, fy) \preceq pd(Tx, Ty) + qd(fx, Tx) + rd(fy, Ty) + sd(fx, Ty) + td(fy, Tx)$$

for all $x, y \in X$, where p, q, r, s, t are nonnegative real numbers with $p + q + r + s + t < 1$. If one of $f(X)$ or $T(X)$ is a complete subspace of X , then f and T have a unique point of coincidence in X . Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point.

Corollary 2.9. *Let f be a self-map on a cone metric space (X, d) over a solid cone, satisfying*

$$d(fx, fy) \preceq pd(x, y) + qd(fx, x) + rd(fy, y) + sd(fx, y) + td(fy, x)$$

for all $x, y \in X$, where p, q, r, s, t are nonnegative real numbers with $p + q + r + s + t < 1$. If $f(X)$ is a complete subspace of X , then f has a unique fixed point in X .

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