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CONVERGENCE THEOREMS OF A SCHEME WITH ERRORS FOR I-ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we prove weak and strong convergence of the Ishikawa iterative scheme with errors to common fixed point I-asymptotically quasi-nonexpansive mappings in a Banach space. The results obtained in this paper improve and generalize the corresponding results in the existing literature.

1. Introduction

Let K be a nonempty subset of uniformly convex Banach space X. Let T be a self-mapping of K. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

A mapping $T : K \longrightarrow K$ is called nonexpansive provided

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$. T is called asymptotically nonexpansive mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda_n)\|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

T is called quasi-nonexpansive mapping provided

$$\|T^n x - p\| \leq \|x - p\|$$

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for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

T is called asymptotically quasi-nonexpansive mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda_n)\|x - p\|$$

for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

Remark 1.1. From above definitions, it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive. But the converse does not hold.

Let $T, I : K \longrightarrow K$. Then T is called I-nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\|$$

for all $x, y \in K$.

T is called I-asymptotically nonexpansive on K if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda'_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda'_n)\|I^n x - I^n y\|$$

for all $x, y \in K$ and $n = 1, 2, \ldots.$

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$ the following inequality holds:

$$\|T^n x - T^n y\| \leq L\|I^n x - I^n y\|$$

and I is uniformly Γ-Lipschitzian if there exists a constant $\Gamma > 0$ such that for all $x, y \in K$ the following inequality holds:

$$\|I^n x - I^n y\| \leq \Gamma\|x - y\|$$

T is called I-asymptotically quasi-nonexpansive on K if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda'_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda'_n)\|I^n x - p\|$$

for all $x \in K$ and $p \in F(T) \cap F(I)$ and $n = 1, 2, \ldots.$

The above definitions were given in [12]. Furthermore, in [12], the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved.

Remark 1.2. From the above definitions it follows that if $F(T) \cap F(I)$ is nonempty, a I-nonexpansive mapping must be I-quasi-nonexpansive, and linear I-quasi-nonexpansive mappings are I-nonexpansive mappings. But it is easily seen that there exist nonlinear continuous I-quasi-nonexpansive mappings which are not I-nonexpansive.

The class of asymptotically nonexpansive maps which an important generalization of the class nonexpansive maps was introduced by Goebel and Kirk [2]. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point. Also in [2], they extended this result to broader class of uniformly Lipschitzian mappings. In 1973, Petryshyn and Williamson [7] proved a necessary and sufficient condition for a Mann iterative sequence to convergence to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [1] extended the results of [7] and gave some necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings. Subsequently, in 2001 Liu Qihou [5] extended the above results and gave some necessary and sufficient conditions for Ishikawa iterative sequence of asymptotically quasi-nonexpansive mappings with error member to converge to fixed points. In [10], the necessary and sufficient condition for the convergence of the Ishikawa-type iterative sequences for two asymptotically quasi-nonexpansive mappings to common fixed point of the mappings defined on a nonempty closed convex subset of a Banach space were established. Their results were improved some above mentioned results.

Recently, Rhoades and Temir [8] and Yao and Wang [13] introduced a class of I-nonexpansive mapping. Rhoades and Temir [8] proved weak convergence of iterative sequence for I-nonexpansive mapping to common fixed point. Yao and Wang [13] proved strong convergence of iterative sequence for I-quasi-nonexpansive mapping to common fixed point.

Let X be a normed linear space, T be self-mapping on X. Let $\{x_n\}$ be of the Ishikawa iterative scheme [3] associated with T, $x_0 \in X$,

$$\begin{cases} y_n = (1-b_n)x_n + b_n T x_n \\ x_{n+1} = (1-a_n)x_n + a_n T y_n \end{cases} \tag{1.1}$$

for every $n \in \mathbb{N}$, where $0 \leq a_n, b_n \leq 1$.

Define the Ishikawa iterative process of the I-asymptotically quasi-nonexpansive mappings in uniformly convex Banach space X as follows

$$\begin{cases} x_{n+1} = b_n x_n + a_n I y_n + c_n u_n \\ y_n = \bar{b}_n x_n + \bar{a}_n T x_n + \bar{c}_n v_n \end{cases} \tag{1.2}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, $\{\bar{c}_n\}$ are sequences in $[0,1]$ with $0 < \delta \leq a_n, \bar{a}_n \leq 1-\delta < 1, a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K.

In this paper, we consider T and I self-mappings of K, where T is an I-asymptotically quasi-nonexpansive mapping and $I : K \to K$ be an asymptotically quasi-nonexpansive mapping.

We establish the weak and strong convergence of the sequence of Ishikawa iterates to a common fixed point of T and I. The aim of this paper is to study iterative process for convergence to common fixed point of I-asymptotically quasi-nonexpansive mappings and to prove some sufficient and necessary conditions for Ishikawa iterative sequences of I-asymptotically quasi-nonexpansive mappings to converge to common fixed point.

Recall some definitions and notations.

2. Preliminaries and Notations

Let X be a Banach space and K be a nonempty subset of X. Let T be a mapping of K into itself. For every ε with $0 \le \varepsilon \le 2$, we define the modulus $\delta(\varepsilon)$ of convexity of X by

$$\delta(\varepsilon) = inf\{1 - \frac{\|x-y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\}.$$

A Banach space X is said to satisfy uniformly convex if $\delta(\varepsilon) > 0$. Recall that a Banach space X is said to satisfy Opial's condition [6] if, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \ne x$. It is well known from [6] that all l_r spaces for $1 < r < \infty$ have this property. However, the L_r space do not have unless $r = 2$.

Lemma 2.1. [11] *Let $\{a_n\}$, $\{b_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \ge 1$, $a_{n+1} \le (1+\sigma_n)a_n + b_n$, where $\sum_{n=0}^{\infty} \sigma_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.2. [9] *Let K be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $\{\alpha_n\}$ a sequence $[\epsilon, 1-\epsilon]$, for some $\epsilon \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in K such that*

$$\limsup_{n\to\infty} \|x_n\| \le k,$$

$$\limsup_{n\to\infty} \|y_n\| \le k$$

and

$$\limsup_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = k$$

holds for some $k \ge 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Definition 2.3. The mappings $T, I : K \to K$ are said to satisfying condition (A) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Ix\|) \ge f(d(x,F))$ for all $x \in K$, where $d(x,F) = inf\{\|x - p\| : p \in F = F(T) \cap F(I)\}$.

3. CONVERGENCE THEOREMS FOR I-ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

Lemma 3.1. *Let X be a Banach space, K be a nonempty closed convex subset of X, T is I-asymptotically quasi-nonexpansive self-mappings on K and I is asymptotically quasi-nonexpansive self-mappings on K with constant λ_n, μ_n, respectively and $\sum\limits_{n=1}^{\infty} \lambda_n < \infty$, $\sum\limits_{n=1}^{\infty} \mu_n < \infty$ such that $F(T) \cap F(I) \neq \emptyset$ in K.*

$$\begin{cases} x_{n+1} = b_n x_n + a_n I y_n + c_n u_n \\ y_n = \bar{b}_n x_n + \bar{a}_n T x_n + \bar{c}_n v_n \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, $\{\bar{c}_n\}$ are sequences in $[0,1]$ with $0 < \delta \leq a_n, \bar{a}_n \leq 1-\delta < 1, a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K. $\sum\limits_{n=1}^{\infty} c_n < \infty$, $\sum\limits_{n=1}^{\infty} \bar{c}_n < \infty$. If $F = F(T) \cap F(I) \neq \emptyset$ then $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for common fixed point p of T and I.

Proof. Let $p \in F(T) \cap F(I)$. Since $\{u_n\}$, $\{v_n\}$ are bounded sequences in K,there exists $M > 0$ such that $max\{\sup\limits_{n\geq 1} \|u_n - p\|, \sup\limits_{n\geq 1} \|v_n - p\|\} \leq M$. Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|a_n I^n y_n + b_n x_n + c_n u_n - p\| \\ &\leq b_n \|x_n - p\| + a_n \|I^n y_n - p\| + c_n \|u_n - p\| \\ &\leq a_n \|x_n - p\| + b_n (1+\mu_n) \|y_n - p\| + c_n \|u_n - p\| \\ &\leq a_n \|x_n - p\| + b_n (1+\mu_n) \|y_n - p\| + c_n M. \end{aligned} \tag{3.1}$$

$$\begin{aligned} \|y_n - p\| &= \|\bar{b}_n x_n + \bar{a}_n T^n x_n + \bar{c}_n v_n - p\| \\ &\leq \bar{b}_n \|x_n - p\| + \bar{a}_n \|T^n x_n - p\| + \bar{c}_n \|v_n - p\| \\ &\leq \bar{b}_n \|x_n - p\| + \bar{a}_n [(1+\lambda_n)\|I^n x_n - p\|] + \bar{c}_n \|v_n - p\| \\ &\leq \bar{b}_n \|x_n - p\| + \bar{a}_n [(1+\lambda_n)(1+\mu_n)\|x_n - p\|] + \bar{c}_n \|v_n - p\| \\ &\leq \bar{b}_n \|x_n - p\| + \bar{a}_n [(1+\lambda_n)(1+\mu_n)\|x_n - p\|] + \bar{c}_n M \end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1),

$$\begin{aligned} \|x_{n+1} - p\| &\leq a_n \|x_n - p\| + b_n (1+\mu_n) \|y_n - p\| + c_n M. \\ &\leq a_n \|x_n - p\| + b_n (1+\mu_n) [\bar{b}_n \|x_n - p\| + \bar{a}_n (1+\lambda_n)(1+\mu_n) \|x_n - p\|] \\ &+ b_n (1+\mu_n) \bar{c}_n M + c_n M \\ &= a_n \|x_n - p\| + b_n (1+\mu_n) \bar{b}_n \|x_n - p\| + b_n \bar{a}_n (1+\mu_n)^2 (1+\lambda_n) \|x_n - p\| \\ &+ M(b_n (1+\mu_n) \bar{c}_n + c_n) \\ &\leq \|x_n - p\| [1 + b_n \bar{b}_n (1+\mu_n) + b_n \bar{a}_n (1+\mu_n)^2 (1+\lambda_n)] \\ &+ M(b_n (1+\mu_n) \bar{c}_n + c_n) \end{aligned}$$

Thus we obtain

$$\|x_{n+1} - p\| \leq (1+\kappa_n) \|x_n - p\| + t_n$$

where

$$\kappa_n = b_n\bar{b}_n(1+\mu_n) + b_n\bar{a}_n(1+\mu_n)^2(1+\lambda_n)$$

with $\sum_{n=1}^{\infty} \kappa_n < \infty$. $t_n = M(b_n(1+\mu_n)\bar{c}_n + c_n)$ with $\sum_{n=1}^{\infty} t_n < \infty$. By Lemma 2.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T) \cap F(I)$.

□

Lemma 3.2. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X. Let T be uniformly L-Lipschitzian, I-asymptotically quasi-nonexpansive mappings on K and I be uniformly Γ-Lipschitzian, asymptotically quasi-nonexpansive mappings on K such that $F(T) \cap F(I) \neq \emptyset$ in K. Suppose that for any given $x \in K$, the sequence $\{x_n\}$ is generated by (1.2). If $F = F(T) \cap F(I) \neq \emptyset$, then*

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ix_n - x_n\| = 0.$$

Proof. By Lemma 3.1 for any $p \in F(T)\cap F(I)$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = k$. If $k = 0$ by continuity of T and I, then the proof is completed.

Now suppose $k > 0$ and choose $N > 0$ such that $max\{\sup_{n\geq 1} \|x_n - u_n\|, \sup_{n\geq 1} \|x_n - v_n\|\} \leq N$.

$$\begin{aligned}
\|y_n - p\| &= \|\bar{b}_n x_n + \bar{a}_n T^n x_n + \bar{c}_n v_n - p\| \\
&\leq (1-\bar{a}_n)\|x_n - p\| + \bar{a}_n\|T^n x_n - p\| + \bar{c}_n\|v_n - x_n\| \\
&\leq (1-\bar{a}_n)\|x_n - p\| + \bar{a}_n[(1+\lambda_n)\|I^n x_n - p\|] + \bar{c}_n\|v_n - x_n\| \\
&\leq (1-\bar{a}_n)\|x_n - p\| + \bar{a}_n[(1+\lambda_n)(1+\mu_n)\|x_n - p\|] + \bar{c}_n\|v_n - x_n\| \\
&\leq (1-\bar{a}_n)\|x_n - p\| + \bar{a}_n[(1+\lambda_n)(1+\mu_n)\|x_n - p\|] + \bar{c}_n N \\
&= \|x_n - p\|(1 - \bar{a}_n + \bar{a}_n[(1+\lambda_n)(1+\mu_n)]) + \bar{c}_n N \\
&\leq \|x_n - p\|(1 + \bar{a}_n\lambda_n + \bar{a}_n\mu_n + \bar{a}_n\lambda_n\mu_n) + \bar{c}_n N.
\end{aligned}$$

Taking lim sup on both sides in the above inequality,

$$\limsup_{n\to\infty} \|y_n - p\| \leq k. \tag{3.3}$$

Since I is asymptotically quasi-nonexpansive mappings on K, we can get that, $\|I^n y_n - p\| \leq (1+\mu_n)\|y_n - p\|$, which on taking $\limsup_{n\to\infty}$ and using (3.3) gives

$$\limsup_{n\to\infty} \|I^n y_n - p\| \leq k.$$

Further,

$$\lim_{n\to\infty} \|x_{n+1} - p\| = k$$

means that

$$\lim_{n\to\infty} \|a_n I^n y_n + b_n x_n + c_n u_n - p\| = k$$

$$\lim_{n\to\infty}(1-a_n)\|x_n-p\|+a_n\|I^ny_n-p\|+c_n\|u_n-x_n\|=k.$$

It follows from Lemma 2.2

$$\lim_{n\to\infty}\|I^ny_n-x_n\|=0. \tag{3.4}$$

Now,

$$\begin{aligned}\|x_n-p\| &\le \|x_n-I^ny_n\|+\|I^ny_n-p\|\\ &\le \|x_n-I^ny_n\|+(1+\mu_n)\|y_n-p\|\end{aligned}$$

which on taking $\lim\limits_{n\to\infty}$ implies

$$\begin{aligned}k &= \lim_{n\to\infty}\|x_n-p\|\\ &\le \limsup_{n\to\infty}(\|x_n-I^ny_n\|+(1+\mu_n)\|y_n-p\|)\\ &= \limsup_{n\to\infty}\|y_n-p\|\le k.\end{aligned}$$

Then we obtain,

$$\limsup_{n\to\infty}\|y_n-p\|=k$$

Next,

$$\begin{aligned}\|T^nx_n-p\| &\le (1+\lambda_n)\|I^nx_n-p\|\\ &\le (1+\lambda_n)(1+\mu_n)\|x_n-p\|.\end{aligned}$$

Taking $\lim\limits_{n\to\infty}$ on both sides in the above inequality,

$$\begin{aligned}\lim_{n\to\infty}\|T^nx_n-p\| &\le \lim_{n\to\infty}[(1+\lambda_n)(1+\mu_n)\|x_n-p\|]\\ &\le \lim_{n\to\infty}\|x_n-p\|\le k.\end{aligned}$$

Further,

$$\lim_{n\to\infty}(1-\bar{a}_n)\|x_n-p\|+\bar{a}_n\|T^nx_n-p\|+\bar{c}_n\|v_n-x_n\|=\lim_{n\to\infty}\|y_n-p\|=k.$$

By Lemma 2.2, we have

$$\lim_{n\to\infty}\|T^nx_n-x_n\|=0. \tag{3.5}$$

We have also,

$$\begin{aligned}
\|I^n x_n - x_n\| &\leq \|I^n x_n - I^n y_n\| + \|I^n y_n - x_n\| \\
&\leq \Gamma\|x_n - y_n\| + \|I^n y_n - x_n\| \\
&= \Gamma\|x_n - [\bar{b}_n x_n + \bar{a}_n T^n x_n + \bar{c}_n v_n]\| + \|I^n y_n - x_n\| \\
&\leq \Gamma\|\bar{a}_n(T^n x_n - x_n)\| + \|I^n y_n - x_n\| + \Gamma\bar{c}_n\|v_n - x_n\| \\
&\leq \Gamma\bar{a}_n\|T^n x_n - x_n\| + \|I^n y_n - x_n\| + \Gamma\bar{c}_n N.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, from (3.4) and (3.5), we deduce that

$$\lim_{n\to\infty} \|I^n x_n - x_n\| = 0. \tag{3.6}$$

$$\begin{aligned}
\|x_{n+1} - Ix_{n+1}\| &\leq \|x_{n+1} - I^{n+1}x_{n+1}\| + \|I^{n+1}x_{n+1} - Ix_{n+1}\| \\
&\leq \|x_{n+1} - I^n x_{n+1}\| + \Gamma\|I^n x_{n+1} - x_{n+1}\|.
\end{aligned}$$

Taking lim sup on both sides in the above inequality and from (3.6), we obtain

$$\limsup_{n\to\infty} \|x_{n+1} - Ix_{n+1}\| \leq 0.$$

That is,

$$\lim_{n\to\infty} \|x_n - Ix_n\| = 0.$$

Next,

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\
&\leq \|x_{n+1} - T^n x_{n+1}\| + L\|I^n x_{n+1} - x_{n+1}\|.
\end{aligned}$$

Also, taking lim sup on both sides in the above inequality, from (3.6) and (3.7), we obtain

$$\limsup_{n\to\infty} \|x_{n+1} - Tx_{n+1}\| \leq 0.$$

That is,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Then the proof is completed.

□

Theorem 3.3. *Let X be uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of X. Let T, I and $\{x_n\}$ be the same as Lemma 3.1. If $F = F(T) \cap F(I) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T and I.*

Proof. Let $p \in F = F(T) \cap F(I)$. Then, as in Lemma 3.1, it follows $\lim_{n\to\infty} \|x_n - p\|$ exists and so for $n \geq 1$, $\{x_n\}$ is bounded on K. Then by the reflexivity of X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ weakly. If $F(T) \cap F(I)$ is a singleton, then the proof is complete. For $p \in F(T) \cap F(I)$, T is I-asymptotically quasi-nonexpansive on K and I is asymptotically nonexpansive on K. The proof is completed if $\{x_n\}$ converges weakly to a common fixed point of T and I, i.e., it suffices to show that the weak limit set of the sequence $\{x_n\}$ consists of exactly one point. We assume that $F(T) \cap F(I)$ is not singleton. Suppose $p, q \in w(\{x_n\})$, where $w(\{x_n\})$ denotes the weak limit set of $\{x_n\}$. Let $\{x_{n_k}\}$ and $\{x_{m_j}\}$ be two subsequences of $\{x_n\}$ which converge weakly to p and q, respectively. By Lemma 3.2 and Lemma 2.1 guarantees that $Ip = p$ and $Tp = p$. In the same way $Iq = q$ and $Tq = q$.

Next we prove the uniqeness.Assume that $p \neq q$ and $\{x_{n_k}\} \rightharpoonup p$, $\{x_{m_j}\} \rightharpoonup q$. By Opial's condition, we conclude that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\
&= \lim_{n\to\infty} \|x_n - q\| = \lim_{j\to\infty} \|x_{m_j} - q\| \\
&< \lim_{j\to\infty} \|x_{m_j} - p\| = \lim_{n\to\infty} \|x_n - p\|.
\end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T) \cap F(I)$.

□

Theorem 3.4. *Let X be a Banach space, K be a nonempty closed convex subset of X. Let T, I and $\{x_n\}$ be the same as Lemma 3.1. If $T, I : K \longrightarrow K$ satisfy condition (A) and T and I are continuous mapping and $F = F(T) \cap F(I) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T and I.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F = F(T) \cap F(I)$. Let $k = \sup_{n\geq 1} \|x_n - p\|$. From Lemma 3.2, we obtain

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ix_n - x_n\| = 0.$$

In the proof of Lemma 3.1, we obtain

$$\|x_{n+1} - p\| \leq \|x_n - p\|(1 + \kappa_n) + t_n \tag{3.7}$$

where

$$\kappa_n = b_n \bar{b}_n (1 + \mu_n) + b_n \bar{a}_n (1 + \mu_n)^2 (1 + \lambda_n)$$

with $\sum_{n=1}^{\infty} \kappa_n < \infty$. $t_n = M(b_n(1 + \mu_n)\bar{c}_n + c_n)$ with $\sum_{n=1}^{\infty} t_n < \infty$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T) \cap F(I)$. By (3.4), we get

$$d(x_{n+1}, F) \leq d(x_n, F)(1 + \kappa_n) + t_n$$

Then by Lemma 2.1, $\lim_{n\to\infty} d(x_n, F)$ exists and the condition (A) guarantees that

$$\lim_{n\to\infty} f(d(x_n, F)) = 0 \tag{3.8}$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that

$$\lim_{n\to\infty} d(x_n, F) = 0$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. Notice that from (3.7), (3.8) for any $p \in F = F(T) \cap F(I)$, we obtain

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + \kappa_{n+m-1})\|x_{n+m-1} - p\| + t_{n+m-1} \\
&\leq \exp(\kappa_{n+m-1} + \kappa_{n+m-2})\|x_{n+m-2} - p\| + \exp(\kappa_{n+m-1})(t_{n+m-1} + t_{n+m-2}) \\
&\vdots \\
&\leq \exp(\sum_{1=n}^{n+m-1} \kappa_i)\|x_n - p\| + \exp(\sum_{1=n}^{n+m-1} \kappa_i) \sum_{1=n}^{n+m-1} t_n \\
&\leq Q\|x_n - p\| + Q \sum_{1=n}^{n+m-1} t_n
\end{aligned}$$

where $Q = \exp(\sum_{1=n}^{\infty} \kappa_i)$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, for any given $\varepsilon > 0$, there exists a positive integer N_0 such that for all $n \geq N_0$, $d(x_n, F) < \frac{\varepsilon}{3Q}$ and $\sum_{1=n}^{\infty} \kappa_i < \frac{\varepsilon}{6Q}$. There exists $p_0 \in F = F(T) \cap F(I)$ such that $\|x_{n_0} - p_0\| < \frac{\varepsilon}{6}$.
Hence, for all $n \geq N_0$ and $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_0\| + \|x_n - p_0\| \\
&\leq Q\|x_{n_0} - p_0\| + Q(\sum_{1=n_0}^{n+m-1} t_i) + Q\|x_{n_0} - p_0\| + Q(\sum_{1=n_0}^{n-1} t_i) \\
&\leq 2Q\|x_{n_0} - p_0\| + Q(\sum_{1=n_0}^{n+m-1} t_i) + Q(\sum_{1=n_0}^{n-1} t_i) \\
&\leq 2Q\frac{\varepsilon}{6Q} + Q\frac{\varepsilon}{3Q} + Q\frac{\varepsilon}{3Q} = \varepsilon
\end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence in X. Thus, the completeness of X implies that $\{x_n\}$ is convergent. Assume that $\{x_n\}$ converges to a point p.

Let $\lim_{n\to\infty} x_n = p$. It will be proven that p is a common fixed point of T and I.

For all $\varepsilon_2 > 0$, $\lim_{n\to\infty} x_n = p$; thus, there exists a natural number N_2 such that when $n \geq N_2$,

$$\|x_n - p\| \leq \frac{\varepsilon_2}{2\max\{2 + \mu_1, 2 + \lambda_1\}}. \tag{3.9}$$

$\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exists a natural number $N_3 \geq N_2$ such that

$$d(x_{N_3}, F) \leq \frac{\varepsilon_2}{2\max\{2+\mu_1, 2+\lambda_1\}}. \tag{3.10}$$

Thus there exists a $p_2 \in F$ such that

$$\|x_{N_3} - p_2\| = d(x_{N_3}, p_2) \leq \frac{\varepsilon_2}{2\max\{2+\mu_1, 2+\lambda_1\}}.$$

From (3.9) and (3.10) we have

$$\begin{aligned}
\|Ip - p\| &\leq \|Ip - p_2 + p_2 - x_{N_3} + x_{N_3} - p\| \\
&\leq \|Ip - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (1+\mu_1)\|p - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (1+\mu_1)\|x_{N_3} - p\| + (1+\mu_1)\|x_{N_3} - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (2+\mu_1)\|x_{N_3} - p\| + (2+\mu_1)\|x_{N_3} - p_2\| \\
&\leq (2+\mu_1)\frac{\varepsilon_2}{2\max\{2+\mu_1, 2+\lambda_1\}} + (2+\mu_1)\frac{\varepsilon_2}{2\max\{2+\mu_1, 2+\lambda_1\}} \\
&\leq \varepsilon_2.
\end{aligned}$$

ε_2 is an arbitrary positive number. Thus $Ip = p$. This implies that $p \in F(I)$.

For all $\varepsilon_3 > 0$, $\lim_{n\to\infty} x_n = p$; thus, there exists a natural number N_2 such that when $n \geq N_2$,

$$\|x_n - p\| \leq \frac{\varepsilon_3}{2\{2+\mu_1+\lambda_1+\mu_1\lambda_1\}}. \tag{3.11}$$

$\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exists a natural number $N_3 \geq N_2$ such that

$$d(x_{N_3}, F) \leq \frac{\varepsilon_3}{2\{2+\mu_1+\lambda_1+\mu_1\lambda_1\}}. \tag{3.12}$$

Thus there exists a $p_2 \in F$ such that

$$\|x_{N_3} - p_2\| = d(x_{N_3}, p_2) \leq \frac{\varepsilon_3}{2\max\{2+\mu_1, 2+\lambda_1\}}.$$

From (3.11) and (3.12) we have

$$\begin{aligned}
\|Tp - p\| &\leq \|Tp - p_2 + p_2 - x_{N_3} + x_{N_3} - p\| \\
&\leq \|Tp - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (1+\lambda_1)\|Ip - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (1+\lambda_1)(1+\mu_1)\|x_{N_3} - p\| + (1+\lambda_1)\|x_{N_3} - p_2\| + \|x_{N_3} - p_2\| + \|x_{N_3} - p\| \\
&\leq (1+\mu_1+\lambda_1+\mu_1\lambda_1)\|x_{N_3} - p\| + (2+\mu_1)\|x_{N_3} - p_2\| \\
&\leq (2+\mu_1+\lambda_1+\mu_1\lambda_1)\frac{\varepsilon_3}{2(2+\mu_1+\lambda_1+\mu_1\lambda_1)} + (2+\lambda_1)\frac{\varepsilon_3}{2\max\{2+\mu_1, 2+\lambda_1\}} \\
&\leq \varepsilon_3.
\end{aligned}$$

ε_3 is an arbitrary positive number. Thus, also $Tp = p$. Therefore, p is a common fixed point of T and I. This implies that $\{x_n\}$ converges strongly to the fixed point of T and I. This completes the proof. $\square$

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