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GLOBAL EXISTENCE AND L^∞ ESTIMATES OF SOLUTIONS FOR A QUASILINEAR PARABOLIC SYSTEM

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Abstract. In this paper, we study the global existence, L^∞ estimates and decay estimates of solutions for the quasilinear parabolic system $u_t = \nabla \cdot (|\nabla u|^m \nabla u) + f(u,v)$, $v_t = \nabla \cdot (|\nabla v|^n \nabla v) + g(u,v)$ with zero Dirichlet boundary condition in a bounded domain $\Omega \subset R^N$.

1. Introduction

In this paper, we are concerned with the global existence, L^∞ estimates and decay estimates of solutions for the quasilinear parabolic system

$$\begin{aligned} u_t &= \nabla \cdot (|\nabla u|^m \nabla u) + f(u,v), && x \in \Omega,\ t > 0, \\ v_t &= \nabla \cdot (|\nabla v|^n \nabla v) + g(u,v), && x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), &\quad v(x,0) = v_0(x), && x \in \Omega, \\ u(x,t) &= v(x,t) = 0, && x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $R^N (N > 1)$ with smooth boundary $\partial\Omega$ and $m, n > 0$.

For $m = n = 0, f(u,v) = u^\alpha v^p, g(u,v) = u^q v^\beta$ and $u_0(x), v_0(x) \geq 0$, the problem (1.1) has been investigated extensively and the existence and nonexistence of solutions for (1.1) are well understood (see [3, 5, 6, 13] and the references cited there). We summarize some of the results. Suppose that the initial data $u_0(x), v_0(x) \geq 0$ and $u_0, v_0 \in L^\infty(\Omega)$. Then

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(A1) let $\alpha > 1$ or $\beta > 1$ or $s_0 = (1-\alpha)(1-\beta) - pq < 0$. Problem (1.1) admits a global solution for small initial data and the solution for (1.1) must blow up in finite time for large initial data;

(A2) all solutions of (1.1) are global if $\alpha, \beta \leq 1$ and $s_0 \geq 0$.

The case $m > 0$ for the single equation

$$
\begin{aligned}
u_t = \nabla \cdot (|\nabla u|^m \nabla u) + f(x,u), \qquad & x \in \Omega,\ t > 0, \\
u(x,0) = u_0(x), \qquad & x \in \Omega, \\
u(x,t) = 0, \qquad & x \in \partial\Omega
\end{aligned}
\tag{1.2}
$$

has been widely investigated in [1, 2, 4, 7, 9, 11, 12] and the references therein. But the problem (1.1) is not considered sufficiently and there seems to be little results on global existence, L^∞ estimates and blow-up of solutions for (1.1).

In this paper we are interested in extending the previous results A1 and A2 for $m = n = 0$ to $m, n > 0$. We consider problem (1.1) for general initial data (try to be more specific here) and obtain sufficient conditions for the global existence of solutions. Furthermore, we obtain L^∞ and decay estimates for solutions of (1.1), that give the behavior of solutions as $t \to 0$ and $t \to \infty$. Our method, very different from that on the basis of comparison principle used in [3, 5, 6, 13, 14, 15, 16], is based on a priori estimates and an improved Moser's technique as in [2, 10]. In contrast with other results (which results [2, 4, 7, 9, 11]), our initial data u_0, v_0 is neither restricted to be bounded nor nonnegative. To drive the L^∞ estimates for solutions of (1.1), we must treat carefully the parameters m, n, p, q, α and β.

Definition 1.1. A pair of functions $(u(x,t), v(x,t))$ is a global weak solution of (1.1) if $(u(x,t), v(x,t)) \in \left(L^\infty_{loc}\left((0,\infty), W_0^{1,m+1}(\Omega)\right) \cap L^{m+1}_{loc}\left(R^+, W_0^{1,m+1}(\Omega)\right)\right)$ $\times \left(L^\infty_{loc}\left((0,\infty), W_0^{1,n+1}(\Omega)\right) \cap L^{n+1}_{loc}\left(R^+, W_0^{1,n+1}(\Omega)\right)\right)$ and the following equalities

$$
\begin{aligned}
&\int_0^t \int_\Omega \{-u\varphi_t + |\nabla u|^m \nabla u \nabla \varphi - f(u,v)\varphi\}\, dxdt \\
&\qquad = \int_\Omega \{u_0(x)\varphi(x,0) - u(x,t)\varphi(x,t)\}\, dx, \\
&\int_0^t \int_\Omega \{-v\varphi_t + |\nabla v|^n \nabla v \nabla \varphi - g(u,v)\varphi\}\, dxdt \\
&\qquad = \int_\Omega \{v_0(x)\varphi(x,0) - u(x,t)\varphi(x,t)\}\, dx
\end{aligned}
$$

are valid for any $t > 0$ and $\varphi \in C^1\left(R^+, C_0^1(\Omega)\right)$, where $R^+ = [0, \infty)$.

Our results read as follows.

Theorem 1.2. *Suppose that*

(H_1) *The functions* $f(u,v), g(u,v) \in C^0(R^2) \cap C^1(R^2 \setminus (0,0))$ *and*

$$
|f(u,v)| \leq K_1 |u|^\alpha |v|^p, \quad |g(u,v)| \leq K_2 |u|^q |v|^\beta, \qquad (u,v) \in R^2, \tag{1.3}
$$

where the parameters α, β, p, q satisfy

$$\begin{aligned} &0 \le \alpha < 1 + m, \quad 0 \le \beta < 1 + n; \quad m, n, p, q > 0; \\ &s = (m + 1 - \alpha)(n + 1 - \beta) - pq > 0. \end{aligned} \tag{1.4}$$

(H_2) $u_0(x) \in L^{p_0}(\Omega)$, $v_0(x) \in L^{q_0}(\Omega)$ *with*

$$p_0 > \max\{1, q + 1 - \alpha\}, \quad q_0 > \max\{1, p + 1 - \beta\}.$$

Then problem (1.1) admits a global weak solution $u(x,t), v(x,t)$ which satisfies

$$u \in L^\infty\left(R^+, L^{p_0}(\Omega)\right), \quad v \in L^\infty\left(R^+, L^{q_0}(\Omega)\right)$$

and the following estimates hold for any $T > 0$

$$\|u\|_\infty \le Ct^{-\sigma}, \quad \|v\|_\infty \le Ct^{-\sigma}, \qquad 0 \le t \le T, \tag{1.5}$$

$$\|u\|_{m+2}^{m+2} + \|v\|_{n+2}^{n+2} \le C\left(t^{-1-\sigma} + t^{1-2(p+\alpha)\sigma} + t^{1-2(q+\beta)\sigma}\right), \quad 0 \le t \le T, \tag{1.6}$$

where $C = C(T, \|u_0\|_{p_0}, \|v\|_{q_0})$, $\sigma = \min\left\{\frac{N}{p_0(m+2)+mN}, \frac{N}{q_0(n+2)+nN}\right\}$.

Theorem 1.3. *Suppose $s < 0$. Then there exist $p_0, q_0 > 1$, $d_0 > 0$ such that if $u_0(x) \in L^{p_0}(\Omega)$, $v_0(x) \in L^{q_0}(\Omega)$ and $\|u_0\|_{p_0} + \|v_0\|_{q_0} < d_0$ the problem (1.1) admits a global weak solution $(u(x,t), v(x,t))$ that*

$$\begin{aligned} u(x,t) &\in L^\infty_{loc}\left((0,\infty), W_0^{1,m+1}(\Omega)\right) \cap L^{m+1}_{loc}\left(R^+, W_0^{1,m+1}(\Omega)\right) \\ v(x,t) &\in L^\infty_{loc}\left((0,\infty), W_0^{1,n+1}(\Omega)\right) \cap L^{n+1}_{loc}\left(R^+, W_0^{1,n+1}(\Omega)\right) \end{aligned} \tag{1.7}$$

satisfying

$$\|u\|_{p_0} \le C(1+t)^{-\frac{1}{\vartheta}}, \quad \|v\|_{q_0} \le C(1+t)^{-\frac{1}{\vartheta}}, \qquad t \ge 0, \tag{1.8}$$

where $\vartheta = \min\{m/p_0, n/q_0\}$.

To derive Theorem 1.2 and 1.3, we will use the following lemmas.

Lemma 1.4. [9] *Let $\beta \ge 0$, $N > p \ge 1$, $\beta + 1 \le q$, and $1 \le r \le q \le (\beta+1)Np/(N-p)$. Then for $|u|^\beta u \in W^{1,p}(\Omega)$, we have*

$$\|u\|_q \le C^{1/(\beta+1)} \|u\|_r^{1-\theta} \||u|^\beta u\|_{1,p}^{\theta/(\beta+1)},$$

with $\theta = (\beta+1)(r^{-1} - q^{-1})/(N^{-1} - p^{-1} + (\beta+1)r^{-1})^{-1}$, where C is a constant depending only on N, p and r.

Lemma 1.5. [11] *Let $y(t)$ be a nonnegative differentiable function on $(0,T]$ satisfying*

$$y'(t) + At^{\lambda\theta-1}y^{1+\theta}(t) \le Bt^{-k}y(t) + Ct^\delta$$

with $A, \theta > 0, \lambda\theta \ge 1$, $B, C \ge 0$, $k \le 1$. Then we have

$$y(t) \le A^{-1/\theta}(2A + 2BT^{1-k})^{1/\theta}t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1}t^{1-\delta} \quad 0 < t \le T$$

This paper is organized as follows. In Section 2, we apply Lemmas 1.4 and 1.5 to establish L^∞ estimates for solutions of problem (1.1). The proof of Theorem 1.3 will be given in Section 3.

2. Proof of Theorem 1.2

For $j = 1, 2, \ldots$, we choose $f_j(u,v), g_j(u,v) \in C^1$ in such a way $f_j(u,v) = f(u,v), g_j(u,v) = g(u,v)$ when $u^2 + v^2 \geq j^{-2}$, $|f_j(u,v)| \leq \eta, |g_j(u,v)| \leq \eta$ when $u^2 + v^2 \leq j^{-2}$ with some $\eta > 0$ and $(f_j(u,v), g_j(u,v)) \to (f(u,v), g(u,v))$ uniformly in R^2 as $j \to \infty$.

Let $(u_{0,j}, v_{0,j}) \in C_0^2(\Omega)$ and $u_{0,j} \to u_0$ in $L^{p_0}(\Omega), v_{0,j} \to v_0$ in $L^{q_0}(\Omega)$ as $j \to \infty$. We consider the approximate problem of (1.1)

$$\begin{aligned} u_t &= \nabla \cdot ((|\nabla u|^2 + j^{-1})^{m/2} \nabla u) + f_j(u,v), && x \in \Omega,\ t > 0, \\ v_t &= \nabla \cdot ((|\nabla v|^2 + j^{-1})^{n/2} \nabla v) + g_j(u,v), && x \in \Omega,\ t > 0, \\ u(x,0) &= u_{0,j}(x), \quad v(x,0) = v_{0,j}(x), && x \in \Omega, \\ &u(x,t) = v(x,t) = 0, && x \in \partial\Omega, \end{aligned} \tag{2.1}$$

The problem (2.1) is a standard quasilinear parabolic system and admits a unique smooth solution $(u_j(x,t), v_j(x,t))$ on $[0,T)$ for each $j = 1, 2, \ldots$, see [7, 8]. Furthermore, if $T < \infty$, then

$$\limsup_{t \to T} (\|u_j(\cdot,t)\|_\infty + \|v_j(\cdot,t)\|_\infty) = +\infty.$$

In the sequel, we will always write (u,v) instead of (u_j, v_j) and (u^p, v^p) for $(|u|^{p-1}u, |v|^{p-1}v)$ where $p > 0$. Also, let C and C_i be the generic constants independent of j and p changeable from line to line.

Lemma 2.1. *Let (H_1) and (H_2) hold. If $(u(x,t), v(x,t))$ is the solution of problem (2.1). Then $u \in L^\infty(R^+, L^{p_0}(\Omega))$, $v \in L^\infty(R^+, L^{q_0}(\Omega))$.*

Proof. Let $p_0, q_0 > 1$. Multiplying the first equation in (2.1) by $|u|^{p_0-2}u$, we obtain that

$$\frac{1}{p_0} \frac{d}{dt} \|u\|_{p_0}^{p_0} + \frac{(p_0-1)(m+2)^{m+2}}{(p_0+m)^{m+2}} \|\nabla u^{\frac{p_0+m}{m+2}}\|_{m+2}^{m+2} \leq \int_\Omega f_j(u,v)|u|^{p_0-2}u dx. \tag{2.2}$$

Notice that

$$\int_\Omega f_j(u,v)|u|^{p_0-2}u dx \leq \eta j^{1-p_0}|\Omega| + C_1 \int_\Omega |u|^{\alpha+p_0-1}|v|^p dx. \tag{2.3}$$

Similarly, we have

$$\begin{aligned} &\frac{1}{q_0} \frac{d}{dt} \|v\|_{q_0}^{q_0} + \frac{(q_0-1)(n+2)^{n+2}}{(q_0+n)^{n+2}} \|\nabla v^{\frac{q_0+n}{n+2}}\|_{n+2}^{n+2} \\ &\leq \eta j^{1-q_0}|\Omega| + C_2 \int_\Omega |v|^{\beta+q_0-1}|u|^q dx, \end{aligned} \tag{2.4}$$

with $C_1, C_2 > 0$.

By Young's inequality, we obtain

$$|u|^\gamma |v|^p + |u|^q |v|^\rho \leq \frac{|v|^{pp_1}}{p_1} + \frac{|u|^{p_2\gamma}}{p_2} + \frac{|u|^{qq_1}}{q_1} + \frac{|v|^{\rho q_2}}{q_2}, \tag{2.5}$$

where $\gamma = \alpha + p_0 - 1$, $\rho = \beta + q_0 - 1$, $t_0 = \gamma\rho - pq > 0$ and

$$p_1 = \frac{t_0}{p(\gamma - q)},\ p_2 = \frac{t_0}{\gamma(\rho - p)},\ q_1 = \frac{t_0}{q(\rho - p)},\ q_2 = \frac{t_0}{\rho(\gamma - q)}. \tag{2.6}$$

The assumption (H_2) on p_0, q_0 and (1.4) imply that $pp_1 < q_0 + n, qq_1 < p_0 + m$. Thus we have from (2.2)-(2.5) and a Sobolev's inequality that

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0}\right) + C_3\left(p_0^{-m}\|u\|_{p_0+m}^{p_0+m} + q_0^{-n}\|v\|_{q_0+n}^{q_0+n}\right) \\ &\quad \le \eta|\Omega|\left(p_0 j^{1-p_0} + q_0 j^{1-q_0}\right) + C_4\int_\Omega \left(|u|^{qq_1} + |v|^{pp_1}\right)dx. \end{aligned} \tag{2.7}$$

Using Young's inequality and letting $j \to \infty$ in (2.7), we conclude that

$$\frac{d}{dt}\left(\|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0}\right) + C_5\left(\|u\|_{p_0+m}^{p_0+m} + \|v\|_{q_0+n}^{q_0+n}\right) \le C \tag{2.8}$$

and

$$\frac{d}{dt}\left(\|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0}\right) + C_6\left(\|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0}\right)^{1+\varrho} \le C \tag{2.9}$$

with $\varrho = \min\{m/p_0, n/q_0\}$. Thus (2.9) implies that $u(t) \in L^\infty(R^+, L^{p_0}(\Omega))$, $v(t) \in L^\infty(R^+, L^{q_0}(\Omega))$ if $u_0 \in L^{p_0}(\Omega)$ and $v_0 \in L^{q_0}(\Omega)$. The proof is completed. $\square$

Lemma 2.2. *Under the assumptions of Lemma 2.1 and for any $T > 0$, the solution $(u(t), v(t))$ also satisfies*

$$\|u\|_\infty \le Ct^{-a}, \quad \|v\|_\infty \le Ct^{-b}, \quad 0 < t \le T, \tag{2.10}$$

$$\|u\|_{m+2}^{m+2} + \|v\|_{n+2}^{n+2} \le C\left(t^{-1-\sigma} + t^{1-2(p+\alpha)\sigma} + t^{1-2(q+\beta)\sigma}\right), \quad 0 < t \le T, \tag{2.11}$$

where the constant C depends on T, $\|u_0\|_{p_0}$, $\|v_0\|_{q_0}$ and $a = N/(p_0(m+2) + mN)$, $b = N/(q_0(n+2) + nN)$, $\sigma = \min\{a, b\}$.

Proof. We only consider $N > \max\{m+2, n+2\}$ and the other cases can be treated in a similar way.

Multiplying the first equation and the second equation in (2.1) by $|u|^{\lambda-2}u$ and $|v|^{\mu-1}v$ respectively, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_\lambda^\lambda + \|v\|_\mu^\mu\right) + C_1\left(\lambda^{-m}\|\nabla u^{\frac{\lambda+m}{m+2}}\|_{m+2}^{m+2} + \mu^{-n}\|\nabla v^{\frac{\mu+n}{n+2}}\|_{n+2}^{n+2}\right) \\ &\quad \le C_2(\lambda+\mu)\left(1 + \int_\Omega |u|^{\alpha+\lambda-1}|v|^p + |u|^q|v|^{\beta+\mu-1}\right)dx. \end{aligned} \tag{2.12}$$

By the Young's inequality, we have

$$|u|^{\gamma_1}|v|^p + |u|^q|v|^{\gamma_2} \le \frac{|v|^{p\varepsilon_1}}{\varepsilon_1} + \frac{|u|^{\gamma_1\varepsilon_2}}{\varepsilon_2} + \frac{|u|^{q\eta_1}}{\eta_1} + \frac{|v|^{\gamma_2\eta_2}}{\eta_2}, \tag{2.13}$$

with $\gamma_1 = \alpha + \lambda - 1$, $\gamma_2 = \beta + \mu - 1$ and $p\varepsilon_1 = \gamma_2\eta_2$, $\gamma_1\varepsilon_2 = q\eta_1$, $\varepsilon_1^{-1} + \varepsilon_2^{-1} = 1$, $\eta_1^{-1} + \eta_2^{-1} = 1$.

The direct computation shows that

$$\eta_1 = \frac{\tau}{q(\gamma_2 - p)}, \ \eta_2 = \frac{\tau}{\gamma_2(\gamma_1 - q)}, \ \varepsilon_1 = \frac{\tau}{p(\gamma_1 - q)}, \ \varepsilon_2 = \frac{\tau}{\gamma_1(\gamma_2 - p)},$$

where $\tau = \gamma_1\gamma_2 - pq > 0$. λ, μ are chosen properly so that $0 < p\varepsilon_1 < \mu + n$ and $0 < q\eta_1 < \lambda + m$. We take two sequences of $\{\lambda_k\}$ and $\{\mu_k\}$ as follows

$$\begin{aligned} \lambda_1 = p_0,\ \lambda = \lambda_k = b_1 + b_{12}R^{k-1}; \\ \mu_1 = q_0,\ \mu = \mu_k = b_2 + b_{22}R^{k-1},\quad k = 2, 3, ... \end{aligned} \tag{2.14}$$

where $b_1 = q + 1 - \alpha$, $b_{12} = (b_1 + m)/s$, $b_2 = p + 1 - \beta$, $b_{22} = (b_2 + n)/s$ and R is chosen so that $R > 1$, $\lambda_2 > p_0$, $\mu_2 > q_0$. Notice that $\lambda_k \backsim \mu_k$ as $k \to \infty$.

We now derive the estimates for the integrals $\int_\Omega |v|^{p\varepsilon_1}dx$ and $\int_\Omega |u|^{q\eta_1}dx$. If $p\varepsilon_1 \le \mu$ and $q\eta_1 \le \lambda$, then we have

$$\int_\Omega |v|^{p\varepsilon_1}dx \le C\left(1 + \int_\Omega |v|^{\mu}dx\right),\quad \int_\Omega |u|^{q\eta_1}dx \le C\left(1 + \int_\Omega |u|^{\lambda}dx\right). \tag{2.15}$$

Without loss of generality, we suppose $\mu < p\varepsilon_1 < \mu + n$, $\lambda < q\eta_1 < \lambda + m$ and $r = \tau/(\gamma_1 - q) - \mu > 0$, $h = \tau/(\gamma_2 - p) - \lambda > 0$. Then from (2.12) and (2.13), we have

$$\begin{aligned} \frac{d}{dt}\left(\|u\|_\lambda^\lambda + \|v\|_\mu^\mu\right) + 2C_1\left(\lambda^{-m}\|\nabla u^{\frac{\lambda+m}{m+2}}\|_{m+2}^{m+2} + \mu^{-n}\|\nabla v^{\frac{\mu+n}{n+2}}\|_{n+2}^{n+2}\right) \\ \le C_2\lambda\left(1 + \|u\|_{\lambda+h}^{\lambda+h}\right) + C_2\mu\left(1 + \|v\|_{\mu+r}^{\mu+r}\right). \end{aligned} \tag{2.16}$$

where the constants C_1, C_2 are independent of λ and μ. Furthermore, we have following by Hölder's and Sobolev's inequalities

$$\begin{aligned} \int_\Omega |u|^{\lambda+h}dx &\le \|u\|_\lambda^{\theta_1}\|u\|_{p_0}^{\theta_2}\|u\|_{\lambda^*}^{\theta_3} \le C\|u\|_\lambda^{\theta_1}\|\nabla u^{\frac{\lambda+m}{m+2}}\|^{\frac{(m+2)\theta_3}{\lambda+m}} \\ &\le C_1C_2^{-1}\lambda^{-1-m}\|\nabla u^{\frac{\lambda+m}{m+2}}\|_{m+2}^{m+2} + C_3\lambda^{\sigma_1}\|u\|_\lambda^\lambda \end{aligned} \tag{2.17}$$

with

$$\lambda^* = \frac{N(\lambda+m)}{N-m-2},\ \theta_1 = \lambda\left(1 - \frac{hN}{p_0(m+2)+mN}\right),\ \theta_2 = \frac{hp_0(m+2)}{p_0(m+2)+mN},$$

$$\theta_3 = \frac{hN(\lambda+m)}{p_0(m+2)+mN},\ \sigma_1 = \frac{(m+1)(p_0(m+2)+N(m-h))}{hN} > 0.$$

Similarly, we can derive that

$$\int_\Omega |v|^{\mu+r}dx \le C_1C_2^{-1}\mu^{-1-n}\|\nabla v^{\frac{\mu+n}{n+2}}\|_{n+2}^{n+2} + C_3\mu^{\sigma_2}\|v\|_\mu^\mu, \tag{2.18}$$

with $\sigma_2 = (n+1)(q_0(n+2) + N(n-r))/(rN) > 0$. Hence it follows from (2.16)-(2.18) that

$$\begin{aligned} \frac{d}{dt}\left(\|u\|_\lambda^\lambda + \|v\|_\mu^\mu\right) + C_1\left(\lambda^{-m}\|\nabla u^{\frac{\lambda+m}{m+2}}\|_{m+2}^{m+2} + \mu^{-n}\|\nabla v^{\frac{\mu+n}{n+2}}\|_{n+2}^{n+2}\right) \\ \le C_3\lambda\left(1 + \lambda^{\sigma_1}\|u\|_\lambda^\lambda\right) + C_3\mu\left(1 + \mu^{\sigma_2}\|v\|_\mu^\mu\right). \end{aligned} \tag{2.19}$$

Now we employ an improved Moser's technique as in[2, 10]. Let $\{\lambda_k\}$, $\{\mu_k\}$be two sequences as defined in (2.14). From Lemma 1.4, we see that

$$\|u\|_{\lambda_k} \le C^{\frac{m+2}{m+\lambda_k}}\|u\|_{\lambda_{k-1}}^{1-\theta_k}\|\nabla u^{\frac{\lambda_k+m}{m+2}}\|_{m+2}^{\frac{(m+2)\theta_k}{\lambda_k+m}}, \tag{2.20}$$

$$\|v\|_{\mu_k} \le C^{\frac{n+2}{n+\mu_k}} \|v\|_{\mu_{k-1}}^{1-\overline{\theta}_k} \|\nabla v^{\frac{\mu_k+n}{n+2}}\|_{n+2}^{\frac{(n+2)\overline{\theta}_k}{\mu_k+n}}, \tag{2.21}$$

where the constant C is independent of λ_k and μ_k, and

$$\theta_k = \frac{\lambda_k+m}{m+2}\left(\frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_k}\right)\left(\frac{1}{N} - \frac{1}{m+2} + \frac{\lambda_k+m}{(m+2)\lambda_{k-1}}\right)^{-1},$$

$$\overline{\theta}_k = \frac{\mu_k+n}{n+2}\left(\frac{1}{\mu_{k-1}} - \frac{1}{\mu_k}\right)\left(\frac{1}{N} - \frac{1}{n+2} + \frac{\mu_k+n}{(n+2)\mu_{k-1}}\right)^{-1}.$$

Let $t_k = \frac{\lambda_k+m}{\theta_k} - \lambda_k$, $s_k = \frac{\mu_k+n}{\overline{\theta}_k} - \mu_k$. Then (2.20) and (2.21) give

$$\lambda_k^{-m}\|\nabla u^{\frac{\lambda_k+m}{m+2}}\|_{m+2}^{m+2} \ge C^{-\frac{m+2}{\theta_k}} \|u\|_{\lambda_k}^{\lambda_k+t_k} \|u\|_{\lambda_{k-1}}^{m-t_k}, \tag{2.22}$$

$$\mu_k^{-n}\|\nabla v^{\frac{\mu_k+n}{n+2}}\|_{n+2}^{n+2} \ge C^{-\frac{n+2}{\overline{\theta}_k}} \|v\|_{\mu_k}^{\mu_k+s_k} \|v\|_{\mu_{k-1}}^{n-s_k}. \tag{2.23}$$

Denote

$$y_k(t) = \|u\|_{\lambda_k}^{\lambda_k} + \|v\|_{\mu_k}^{\mu_k}, \quad t \ge 0.$$

Then inserting (2.22)-(2.23) into (2.19) ($\lambda = \lambda_k, \mu = \mu_k$), we find that

$$\begin{aligned} y_k'(t) + C_1 C^{-\frac{m+2}{\theta_k}} \|u\|_{\lambda_k}^{\lambda_k+t_k} \|u\|_{\lambda_{k-1}}^{m-t_k} + C_1 C^{-\frac{n+2}{\overline{\theta}_k}} \|v\|_{\mu_k}^{\mu_k+s_k} \|v\|_{\mu_{k-1}}^{n-s_k} \\ \le C_3(\lambda_k+\mu_k) + C\lambda_k^{\sigma_1+1}\|u\|_{\lambda_k}^{\lambda_k} + C\mu_k^{\sigma_2+1}\|v\|_{\mu_k}^{\mu_k}. \end{aligned} \tag{2.24}$$

We claim that there exist the bounded sequence $\{\xi_k\}, \{\eta_k\}, \{m_k\}, \{r_k\}$ such that

$$\|u\|_{\lambda_k} \le \xi_k t^{-m_k}, \quad \|v\|_{\mu_k} \le \eta_k t^{-r_k}, \quad 0 < t \le T. \tag{2.25}$$

Without loss of generality, we suppose that $\xi_k, \eta_k \ge 1$. By Lemma 2.1, (2.25) holds for $k = 0$ if we take $m_0 = r_0 = 0$ and $\xi_0 = \sup_{t\ge 0}\|u\|_{p_0}$, $\eta_0 = \sup_{t \ge 0}\|v\|_{q_0}$. If (2.25) is true for $k-1$, then we have from (2.24) that

$$\begin{aligned} y_k'(t) + C_3\|u\|_{\lambda_k}^{\lambda_k+t_k}\left(\xi_{k-1}t^{-m_{k-1}}\right)^{m-t_k} + C_3\|v\|_{\mu_k}^{\mu_k+s_k}\left(\eta_{k-1}t^{-r_{k-1}}\right)^{n-s_k} \\ \le C(\lambda_k+\mu_k)\left(\lambda_k^{\sigma_1}\|u\|_{\lambda_k}^{\lambda_k} + \mu_k^{\sigma_2}\|v\|_{\mu_k}^{\mu_k}\right). \end{aligned} \tag{2.26}$$

We take $\sigma_0 = \max\{\sigma_1, \sigma_2\}$, $\tau_k = \min\{t_k/\lambda_k, s_k/\mu_k\}$, $\alpha_k = \min\{m - t_k, n - s_k\}$ and $A_{k-1} = \max\{\xi_{k-1}, \eta_{k-1}\}$, $\beta_k = \max\{(t_k - m)m_{k-1}, (s_k - n)r_{k-1}\}$. Then we have from (2.26) that

$$y_k'(t) + C_3 A_{k-1}^{\alpha_k} t^{\beta_k} y_k^{t+\tau_k}(t) \le C\lambda_k + C\lambda_k^{\sigma_0+1} y_k(t) + CA_{k-1}^{\alpha_k}T^{\beta_k}, \quad 0 < t < T. \tag{2.27}$$

Applying Lemma 1.5 to (2.27), we get

$$y_k(t) \le B_k t^{-(1+\beta_k)/\tau_k}, \quad 0 < t < T, \tag{2.28}$$

where

$$B_k = 2\left(C_3 A_{k-1}^{\alpha_k}\right)^{-\frac{1}{\tau_k}}\left(C_3\lambda_k^{\sigma_0+1} + \frac{1+\beta_k}{\tau_k}\right)^{\frac{1}{\tau_k}} + 2C\lambda_k\left(C\lambda_k^{\sigma_0+1} + \frac{1+\beta_k}{\tau_k}\right)^{-1}.$$

Moreover, (2.28) implies that

$$\|u\|_{\lambda_k} \le B_k^{\frac{1}{\lambda_k}} t^{-\frac{1+\beta_k}{\lambda_k\tau_k}}, \quad \|v\|_{\mu_k} \le B_k^{\frac{1}{\mu_k}} t^{-\frac{1+\beta_k}{\mu_k\tau_k}}, \quad 0 < t \le T. \tag{2.29}$$

We take

$$\xi_k = B_k^{\frac{1}{\lambda_k}},\ \eta_k = B_k^{\frac{1}{\mu_k}},\ m_k = \frac{1+\beta_k}{\lambda_k \tau_k},\ r_k = \frac{1+\beta_k}{\mu_k}\tau_k.$$

By a similar argument in [2, 10], we know that $\{\xi_k\}$, $\{\eta_k\}$ are bounded and there exist two subsequences $\{m_{kl}\} \subset \{m-k\}$ and $\{r_{kl}\} \subset \{r_k\}$ such that

$$m_{kl} \to a = \frac{N}{p_0(m+2)+mN}, \quad r_{kl} \to b = \frac{N}{q_0(n+2)+nN}, \quad (as\ l \to \infty).$$

Therefore, letting $l \to \infty$ in (2.28), we obtain

$$\|u\|_\infty \le Ct^{-a}, \quad \|v\|_\infty \le Ct^{-b}, \quad 0 < t < T, \tag{2.30}$$

This yields (2.10).

It remains to prove the estimate (2.11). In order to derive (2.11), we use a similar argument in [10]. We first choose $\mu > \max\{\sigma, 2(p+\alpha)\sigma - 2, 2(q+\beta)\sigma - 2\}$ and $h(t) \in C([0,\infty) \cap C^1(0,\infty)$ such that $h(t) = t^\mu,\ 0 \le t \le 1; h(t) = 2, t \ge 2$ and $h(t), h'(t) \ge 0$ in $(0,\infty)$. Then multiplying the first equation by $h(t)u$ and the second equation by $h(t)v$ in (2.1), and letting $j \to \infty$, we obtain

$$\begin{aligned} &\int_0^t h(s)g(s)ds + \frac{1}{2}h(t)\int_\Omega (|u|^2+|v|^2)dx \\ \le\ & \frac{1}{2}\int_0^t\int_\Omega h'(s)(|u|^2+|v|^2)dxds + C\int_0^t\int_\Omega h(s)(|u|^{1+\alpha}|v|^p + |u|^q|v|^{1+\beta})dxds \end{aligned} \tag{2.31}$$

with $g(t) = \|\nabla u\|_{m+2}^{m+2} + \|\nabla v\|_{n+2}^{n+2},\ \ t \ge 0$.

By Young's inequality and the assumption (1.4), we obtain

$$\begin{aligned} &C\int_\Omega (|u|^{1+\alpha}|v|^p + |u|^q|v|^{1+\beta})dx \le \int_\Omega (|u|^{\tau_1} + |v|^{\tau_1})dx \\ \le\ & \varepsilon\int_\Omega (|u|^{m+2} + |v|^{n+2})dx + C_\varepsilon|\Omega| \le C(\|\nabla u\|_{m+2}^{m+2} + \|\nabla v\|_{n+2}^{n+2}) + C_\varepsilon|\Omega| \end{aligned} \tag{2.32}$$

for any $\varepsilon > 0$ and $\tau_1 = ((\alpha+1)(\beta+1) - pq)/(\beta+1-p) < m+2$, $\tau_2 = ((\alpha+1)(\beta+1) - pq)/(\alpha+1-q) < n+2$. Furthermore, we take $\varepsilon = 1/2$. Then (2.31)-(2.32) yields

$$\int_0^t h(s)g(s)ds + h(t)(\|u\|_2^2 + \|v\|_2^2) \le Ct^{\mu-\sigma}. \tag{2.33}$$

Next, let $\rho(t) = \int_0^t h(s)ds,\ t \ge 0$. Similarly, multiplying the first equation in (2.1) by $\rho(t)u_t$ and the second equation by $\rho(t)v_t$, and letting $j \to \infty$, we have from (2.30) and (2.31) that

$$\begin{aligned} \int_0^t \rho(s)(\|u_t\|_2^2 + \|v_t\|_2^2)ds + \rho(t)g(t) &\le C\int_0^t\int_\Omega \rho(s)(|u|^{2\alpha}|v|^{2p} + |u|^{2q}|v|^{2\beta})dxds \\ &+ \int_0^t \rho'(s)g(s)ds \le C\int_0^t \rho(s)\left(s^{-2(\alpha+p)\sigma} + s^{-2(\beta+q)\sigma}\right)ds + Ct^{\mu-\sigma} \\ &\le C\left(t^{\mu-\sigma} + t^{\mu+2-2(p+\alpha)\sigma} + t^{\mu+2-2(q+\beta)\sigma}\right), 0 < t \le T. \end{aligned} \tag{2.34}$$

Thus (2.34) implies

$$g(t) \le C\left(t^{-1-\sigma} + t^{1-2(p+\alpha)\sigma} + t^{1-2(q+\beta)\sigma}\right), \quad 0 < t \le T, \tag{2.35}$$

and (2.11) is proved. The proof is completed. □

Proof of Theorem 1.2. We notice that the estimate constant C in (2.30) and (2.35) is independent of j , we may obtain the desired solution (u, v) as limit of $\{(u_j, v_j)\}$ (or a subsequence) by the standard compact argument as in [6, 8, 9, 10]. The solution (u, v) of problem (1.1) also satisfies (1.5)-(1.6). The proof is completed. □

Remark:

- From the proof of Theorem 1.2, we see that if the assumption (1.3) is replaced by

  $$|f(u,v)| \le K_1(1 + |u|^\alpha |v|^p), \quad |g(u,v)| \le K_2(1 + |u|^q |v|^\beta),$$

  the conclusions in Theorem 1.2 still hold.

3. Proof of Theorem 1.3

By the standard compact argument as in [2, 7, 9, 10], we only consider the estimate (1.8) and show that $(u, v) \in L^{1,m+1}_{loc}\left(R^+, W_0^{1,m+1}(\Omega)\right) \cap L^{1,n+1}_{loc}\left(R^+, W_0^{1,n+1}(\Omega)\right)$ for the solution of (2.1).

Proof of Theorem 1.3. Suppose that $s < 0$ holds. Let

$$p_0 = b_1 + b_{12}\varepsilon > 1, \quad q_0 = b_2 + b_{22}\varepsilon > 1, \tag{3.1}$$

with $b_1 = q+1-\alpha$, $b_2 = p+1-\beta$, $b_{12} = -(q+m+1-\alpha)/s$, $b_{22} = -(p+n+1-\beta)/s$. Since $s < 0$, we can take $\varepsilon > 0$ such that $p_0 \ge \max\{4q, 4\alpha, 2 + 2\alpha\}$, $q_0 \ge \max\{4p, 4\beta, 2+2\beta\}$, $S_0 = (\alpha + p_0 - 1)(\beta + q_0 - 1) - pq > 0$. Then it follows from (2.5) and (2.7) that

$$\begin{aligned}\frac{d}{dt}\left(\|u\|_{p_0}^{q_0} + \|v\|_{q_0}^{q_0}\right) + C_1\left(\|\nabla u^{\frac{p_0+m}{m+2}}\|_{m+2}^{m+2} + \|\nabla v^{\frac{q_0+n}{n+2}}\|_{n+2}^{n+2}\right) \\ \le C\int_\Omega \left(|u|^{qq_1} + |v|^{pp_1}\right)dx,\end{aligned} \tag{3.2}$$

where $qq_1 = S_0/(q_0 + \beta - 1 - p) > p_0 + m, pp_1 = S_0/(\alpha + p_0 - q - 1) > q_0 + n$. We now estimate the right-hand side of (3.2). Let $qq_1 = p_0 + \theta, pp_1 = q_0 + \tau$ and $\theta > m, \tau > n$. Then

$$\int_\Omega |u|^{qq_1}dx = \|u\|_{p_0+\theta}^{p_0+\theta} \le C_2\|u\|_{p_0}^{\theta-m}\|\nabla u^{\frac{p_0+m}{m+2}}\|_{m+2}^{m+2}, \tag{3.3}$$

$$\int_\Omega |v|^{pp_1}dx = \|v\|_{q_0+\tau}^{q_0+\tau} \le C_2\|v\|_{q_0}^{\tau-n}\|\nabla v^{\frac{q_0+n}{n+2}}\|_{n+2}^{n+2}. \tag{3.4}$$

Denote

$$\phi(t) = \|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0}, \quad f(t) = \|\nabla u^{\frac{p_0+m}{m+2}}\|_{m+2}^{m+2} + \|\nabla v^{\frac{q_0+n}{n+2}}\|_{n+2}^{n+2},$$

then (3.2) becomes

$$\begin{aligned}\phi'(t)+C_1f(t) &\le C_2\left(\|u\|_{p_0}^{\theta-m}\|\nabla u^{\frac{p_0+m}{m+2}}\|_{m+2}^{m+2}+\|v\|_{q_0}^{\tau-n}\|\nabla v^{\frac{q_0+n}{n+2}}\|_{n+2}^{n+2}\right) \\ &\le C_2\left(|u\|_{p_0}^{\theta-m}+\|v\|_{q_0}^{\tau-n}\right)f(t)\le C_3\phi^{\alpha_0}(t)f(t),\end{aligned}\tag{3.5}$$

with $\alpha_0=\min\{(\theta-m)/p_0,(\tau-n)/q_0\}>0$.

(3.5) implies that there is $C_0>0$ such that

$$\phi'(t)+C_0f(t)\le 0 \quad \text{if } C_3\phi^{\alpha_0}(0)=C_3\left(\|u_0\|_{p_0}^{p_0}+\|v_0\|_{q_0}^{q_0}\right)^{\alpha_0}<C_1. \tag{3.6}$$

Furthermore, we have from Sobolev embedding theorems that

$$\|\nabla u^{\frac{p_0+m}{m+2}}\|_{m+2}^{m+2}\ge d_1\|u\|_{p_0+m}^{p_0+m}\ge d_2\|u\|_{p_0}^{p_0+m}, \quad \|\nabla v^{\frac{q_0+n}{n+2}}\|_{n+2}^{n+2}\ge d_2\|v\|_{q_0}^{q_0+2},$$

for some $d_2>0$. Hence,

$$f(t)\ge d_2\left(\|u\|_{p_0}^{p_0+m}+\|v\|_{q_0}^{q_0+m}\right)\ge d_2\phi^{1+\vartheta}, \quad \vartheta=\min\{m/p_0,n/q_0\}.$$

Now (3.6) gives

$$\phi'(t)+d_2\phi^{1+\vartheta}\le 0, \quad t\ge 0. \tag{3.7}$$

This implies that

$$\phi(t)\le C(1+t)^{-\frac{1}{\vartheta}}. \tag{3.8}$$

Next, we show that $(u,v)\in L_{loc}^{1,m+1}\left(R^+,W_0^{1,m+1}\right)\cap L_{loc}^{1,n+1}\left(R^+,W_0^{1,n+1}\right)$. By the definition of p_0 and q_0, we have from (3.8) that for any $t\ge 0$,

$$\int_\Omega |u|^{1+\alpha}|v|^p dx\le C\|u\|_{p_0}^{1+\alpha}\|v\|_{q_0}^{p}\le C_1, \quad \int_\Omega |u|^q|v|^{1+\beta}dx\le C\|u\|_{p_0}^{q}\|v\|_{q_0}^{1+\beta}\le C_1.$$

Here C_1 is a constant independent of t. Thus (2.31) yields that

$$\int_0^t h(s)g(s)ds\le C\left(h(t)+\int_0^t g(s)ds\right)\le C(h(t)+\rho(t)), t\ge 0. \tag{3.9}$$

Similarly, we have

$$\int_\Omega |u|^{2\alpha}|v|^{2p}dx\le \|u\|_{p_0}^{2\alpha}\|v\|_{q_0}^{2p}\le C_2, \quad \int_\Omega |u|^{2q}|v|^{2\beta}dx\le \|u\|_{p_0}^{2q}\|v\|_{q_0}^{2\beta}\le C_2.$$

Then from (2.34) and (3.9), we obtain

$$\rho(t)g(t)\le C_3\left(\int_0^t\rho(s)ds+\int_0^t h(s)g(s)ds\right)\le C_3\left(\int_0^t\rho(s)ds+h(t)+\rho(t)\right) \tag{3.10}$$

It implies

$$g(t)\le C_4(t+t^{-1}+1), \quad 0\le t\le T, \tag{3.11}$$

and $(u,v)\in L_{loc}^{1,m+1}\left(R^+,W_0^{1,m+1}\right)\cap L_{loc}^{1,n+1}\left(R^+,W_0^{1,n+1}\right)$. This completes the proof of Theorem 1.2. The proof is completed. □

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